**NATIONAL KAPODISTRIAN UNIVERSITY OF ATHENS**

**Mathematics Department**

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# Generalized Second-Price Auctions under Advertisement Settings

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# Synopsis

At the current M.Sc thesis we study Generalized Second-Price Auctions under advertisement settings. In chapter 1 we make an introduction to the basic concepts of the auctions, presenting several auction models and analyzing their properties. We proceed in chapter 2 studying the equilibria properties of the GSP auction under the advertisement position setting with a presentation based on [1]. We provide several notices and additional proofs regarding the comparison between the pure Nash equilibria and Envy-Free equilibria. In chapter 3 we study the notion of budget and observe the Budgeted Second-Price advertisement auction with a presentation based on [2]. In section 3.6 we display notices and some results from our side, concerning several problems that occur in the original work. Additionally we examine the critical bid notion under the same setting when the items are not divisible. Finally we conclude with chapter 4, introducing two GSP auction models for the advertisement position setting, customized under budget constraints. Our presentation is based on a work currently in progress [3]. We analyze the structure of the two models and provide proofs regarding their equilibria properties.

**Keywords:** Second-Price Auctions, Equilibrium, Budget, Ad Setting, Slot

# Περίληψη

Στην παρούσα διπλωματική εργασία μελετάμε Second-Price δημοπρασίες γενικευμένης μορφής, κάτω από το πρίσμα του περιβάλλοντος των διαφημίσεων. Στο πρώτο κεφάλαιο κάνουμε μια εισαγωγή στις βασικές έννοιες των δημοπρασιών, παρουσιάζοντας μια ποικιλία μοντέλων και αναλύουμε τις ιδιότητες τους. Στο δεύτερο κεφάλαιο μελετάμε τις ιδιότητες ισορροπίας (equilibrium) γενικευμένων Second-Price δημοπρασιών σε περιβάλλον διαφημίσεων με μια παρουσίαση που βασίζεται στο [1]. Παρέχουμε πρόσθετες σημειώσεις καθώς συμπληρωματικά αποτελέσματα που αφορούν στη σύγκριση ανάμεσα στην Nash και την Envy-Free ισορροπία. Στο τρίτο κεφάλαιο μελετάμε την έννοια του budget και το μοντέλο δημοπρασίας διαφημίσεων Budgeted Second-Price με μια παρουσίαση που βασίζεται στο [2]. Στην παράγραφο 3.6 εκθέτουμε κάποια δικά μας αποτελέσματα σχετικά με ορισμένα προβλήματα που υπάρχουν στην αυθεντική εργασία και επιπρόσθετα μελετάμε την έννοια του critical bid στο ίδιο περιβάλλον, χωρίς ωστόσο την υπόθεση των διαιρετών αντικειμένων. Κλείνοντας ολοκληρώνουμε με το τέταρτο κεφάλαιο στο οποίο συστήνουμε δυο καινούργια μοντέλα γενικευμένων Second-Price δημοπρασιών στο περιβάλλον των διαφημίσεων υπό budget περιορισμούς, με μια παρουσίαση που βασίζεται σε μια εργασία που βρίσκεται αυτή τη στιγμή σε εξέλιξη [3]. Αναλύουμε την δομή των δυο μοντέλων και παρέχουμε αποδείξεις σχετικά με τις ιδιότητες ι-

σορροπίας τους. Τέλος κάνουμε μια σύντομη αναφορά σε προτάσεις οι οποίες είναι την παρούσα στιγμή προς απόδειξη.

**Λέξεις Κλειδιά:** Second-Price δημοπρασίες, Ισορροπία, Οικονομική δυνατότητα, Περιβάλλον διαφήμισης, Τοποθέτηση

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# Contents

# Chapter 1

# Auctions and Models

## 1.1 A general view and some basic concepts

### 1.1.1 Auctions History

The word auction comes from the Latin *augeo* which means "I increase" or "I augment". Auctions are a mechanism for selling or exchanging goods and commodities and it seems that have touched almost every century, industry and nationality. Auctions date back so far in history that their origin is uncertain and no one knows exactly who started them or how they started. The earliest record that we have, comes from ancient Greek scribes and Herodotus, containing information that auctions occurring as far back as 500 B.C.. At that time the "items" of the auctions were women for marriage and in fact it was considered illegal for a daughter to be sold outside the auction method. The model of these auctions was a descending method where the auctioneer was setting a starting high price and gradually was decreasing it until the first bid of a potential buyer. It was a single-item kind of model and the auctioneer initially was selling the women that he considered most beautiful and progressed to the least. The buyer could get a refund if he and his wife did not get along. Later on and during the time of Roman Empire after a military success, Roman soldiers would drive a spear into the ground, around the spoils of war were left, to start an auction. Auctions were also popular for selling family estates, for example the roman emperor and philosopher Marcus Aurelius sold family furniture at auctions for months in order to pay off debts. In the year 193 A.C. one of the most historical auctions of all time occurred, when the whole Roman Empire was put on an auction by the Praetorian Guard (a force of bodyguards used by the roman emperors). The Praetorian Guard killed the emperor Pertinax and then put the entire empire into an auction. The highest bidder was Didius Julianus with a bid of 6.250 drachmas per guard. That initiated a civil war and two months later Didius Julianus was beheaded when Septimus Severus conquered Rome. After the end of the Roman Empire, auctions popularity started to fade in Europe. Around 1600 auctions came to America (pilgrims arrival on Americas Eastern shores) and their popularity continued to increase during colonization with the sale of many types of goods, such as crops, tools, tobacco and entire farms. At the American civil war era, spoils of war and surplus were regularly auctioned at public sale by Colonels of the division (thus today some of the auctioneers of the US also referred as Colonels). Back to Europe and at 1674 the oldest auction house in the world, Stockholm Auction House was established in Sweden. During the end of the 18th century, soon after the French Revolution, auctions were held into taverns and coffee houses to sell works of art and auction catalogs that contained the available goods were printed. Today the world's largest auction is Christie's and it was established around 1766. Recently the development of the Internet gave a significant increase in the use of the auctions due to the rise of the electronic markets (stores where you can buy everything you like without the need of your physical presence). Many of these markets use several models of auction mechanisms as the main way for selling goods.

### 1.1.2 Auctions as Formal Models

The origins of the auctions with more specific structures, is said to be the year 1961 with the seminal article of William Vickrey. Although it took many years before his work was followed up by other researchers (including Wilson, Clarke, Groves, Milgrom Weber Myerson, Marskin and Riley), it eventually formed a solid sector of research. Around the 80's there was a widespread sense that the specific research area was almost complete with little remaining to be discovered. This perception however changed at the early 90's due to the occurrence of two major events: the Salomon Brothers scandal in the US Government securities market in 1991 (trader Paul Mozer had been submitting false bids in an attempt to purchase more treasury bonds than permitted by one buyer during the period between December 1990 and May 1991) and the advent of the Federal Communications Commission (FCC) spectrum auctions in 1994. These two events brought up to the surface the serious limitations of the existing theory since the majority of the theorems, models, structures etc that had been developed until then, were based on single-item auctions (auctions where the auctioneer sells only one item to a set of buyers) while the study of multi-item auctions was at an infantile level. So the second wave of research was triggered at the middle of 90's and the main focus was on the study of multi-item auctions with multiple variations (a research that continues until today).

### 1.1.3 Types of single-item Auctions

As we mentioned earlier on, the term single-item auction is used when a seller-auctioneer wants to sell only one item or a single non divisible amount of good to a set of potential buyers. The buyers are also called bidders since they bid, propose or submit a value which technically corresponds to the amount of money they desire to give in order to get the item or the good. They also can be seen as players with different strategies, in a game with certain rules where the winner is the one who manages to get the item. Here we will make a reference to the basic models of single-item auctions.

- *Ascending-bid auctions:* These auctions are also known as English auctions and are carried out in real time. The bidders participate either physically or electronically. The basic idea behind this model is that the auctioneer gradually raises the price of the item that is to be sold and the bidders drop out until only one of them remains. He is the winner bidder and he gets the item at this final price. Examples of this model are oral auctions in which bidders shout out prices (physical participation), or submit them electronically (electronic participation).

- *Descending-bid auctions:* These auctions are also known as Dutch auctions and are carried out in real time as well. The auctioneer, in contrast with the previous model, gradually lowers the price of the item (which has some high initial value) until one of the bidders accepts the item for the first time and pays it at the current price. The "Dutch" characterization comes from the fact that this model was used as way to sell flowers in Netherlands.

- *First-price sealed-bid auctions:* In this kind of auction, bidders submit simultaneous sealed bids to the auctioneer without knowledge of any of their opponents bids. The auctioneer unseals the bids all together and the winner is the one with the highest bid. The price he pays is the value of his bid (hence the name first-price).

- *Second-price sealed-bid auctions:* This model follows the same pattern as the first-price auctions where the bidders submit simultaneously, sealed bids to the auctioneer without knowledge of any of their opponents bids and the auctioneer unseals the bids all together. The winner is once again the bidder with the highest bid, the price he pays however is the second-highest bid (hence the name second-price). These auctions are also known as Vickrey auctions in honor of William Vickrey, who wrote the first game-theoretic analysis of auctions (including the second-price auction) and who won the Nobel Memorial Prize in Economics in 1996 for this body of work.

### 1.1.4 Private-Values Model

It is common fact that in our daily life as consumers, we value the products that we are interested in. When someone goes to a store for example in order to buy a specific model of TV, he checks the specs, the technology, the characteristics in general and comes up with a valuation on how much money this model should cost, based on his own criteria. He asks the seller the price of the TV and then he decides either to wait for a price drop or to buy it depending on his valuation (if the price exceeds his valuation or not respectively). We always value the products we are interested in and this is one of the main ways of how we decide to buy them or not.

The same goes for potential buyers that participate in an auction. An interesting question that we can come up with however is, should these buyers-bidders expose their values to the auctioneer or not?

Generally, each bidder $i$ who participates in an auction have a valuation-value $v_i$, for the item which is sold by the auctioneer. We claim that this type of information must be private, otherwise there is no meaning in forming an auction at all. In order to see this, imagine a case where the auctioneer as well as the bidders, know each other's values. Specifically, consider that an auctioneer is trying to sell an item that he values $x$ and suppose that the highest value among the bidders is some $y = max\{v_i\} > x$. Bellow there is an example of what could happen in a situation like this.

**Example:** If the auctioneer knows the true values of the bidders then he can set the price of the item at a value just below $y$. In that case the bidder with the highest value will buy the item and the auctioneer will have the highest possible profit. In other words there is no need for the auctioneer to form an auction since for the right price he can maximize his utility. From the bidders side of view, exposing the values predetermines the outcome (who gets the item) and leads the winner to get it at the highest possible price. In either case there is no need for the players to bid, so there no meaning in forming an auction at all.

So we can now take a formal look on the private-values model in the auction setting in order to have a basic idea on how such a structure works: An auctioneer wishes to allocate an item or a non divisible amount of good among $N$ bidders ($i = 1, ..., N$). Each bidder's valuation $v_i$ for the item, is private information and depends only on him and not on the adversary bidders. The bidders bid simultaneously and independently. Each one of them wants to get the item, but obviously at a price lower than their own valuation. They also want this price to be the lowest possible. So in order to summarize, each player's goal is:

- Get the item only at a price smaller than your own private valuation
- Get it at the lowest possible price

In a more formal way we can say that all bidders want to maximize their expected *utility* which is defined as:

$$u_i = \begin{cases} (v_i - x) & \text{if } i \text{ gets the item at price x} \\ 0 & \text{if i gets nothing} \end{cases}$$

Notice here that all the models that we will display from now on, follow the basics of the private-values model structure.

## 1.2 Basic Models of single-item Auction

We will now describe two of the main single-item auction models, *second-price auction* and *first-price auction.* In this work we are mainly interested in the second-price model due to its interesting properties, however we will make a brief description of the first-price model in order to have a wider view on the topic. As we said earlier, an auction can be seen as a game where the goal of each player is to maximize his utility. The strategy of each player is the choice of his bid. From now on we will refer to bidders as players.

### 1.2.1 Second-Price Auction

We have $N$ $(i = 1, ..., N)$ players that have private values $v_i$ for the item that is to be sold by the auctioneer. Each player's strategy is to bid an amount $b_i$ which is a function of his true value $v_i$. The utility of player $i$ with value $v_i$ and bid $b_i$ is defined as follows:

*If player i is the winner (i.e. he is the one who gets the item and has the highest bid) then his utility is defined as $v_i - b_j$, where $b_j$ is the second highest bid . Else if player i is a loser (i.e. he does not get the item) then his utility is defined as zero.*

$$u_i = \begin{cases} (v_i - b_j) & \text{if } i \text{ has the highest bid } (b_j \text{ is the second highest bid}) \\ 0 & \text{otherwise} \end{cases}$$

So in order to give a more complete description, we can say that the players announce their bids, the auctioneer ranks them in decreasing order and he sells the item to the first in rank player at a price which is equal to the bid of the second in rank player. There are some additional details that we have to mention here: How the auctioneer handles a situation where two of the people that participate to the auction submit the same bid? A solution that we can come up with, is to order the people with the same bids via their names (with lexicographic order) i.e., if two or more people submit the highest bid, the auctioneer will sell the item to the player with the lower original index (we make this is a hypothesis for all the auction models that are presented in this work). Notice here that at situations like this the winner player gets the item but pays the full value of his own bid.

As we said earlier, second-price auctions have some very interesting properties. Before we proceed it is important to present some definitions:

**Definition 1.2.1.1** Let $S = (s_1, s_2, ..., s_n)$ be a set of strategies of all players where $s_i$ is the strategy of player $i$. As $s_{-i}$ we define the strategies of all players except $i's$ and as $u_i(s_i, s_{-i})$ the utility of $i$ when he chooses to play $s_i$ and the others $s_{-i}$.

**Definition 1.2.1.2** We call *dominant strategy* a strategy that is optimal for a player $i$ (i.e. it maximizes his utility) regardless of what the other players choose to play. In other words, if $s_i$ is a *dominant strategy* then $i$ does not gain something from choosing a different strategy $s'_i$ as well as it is best for him to stick with that strategy no matter what the other players do (i.e. if they go from $s_{-i}$ to $s'_{-i}$). More formally:

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \text{ } and \text{ } u_i(s_i, s'_{-i}) \geq u_i(s'_i, s'_{-i})$$

Lets now see why the second-price auction model is so important.

**Proposition 1.2.1.1** In a second-price auction it is a dominant strategy for every player to bid his true value, i.e. $b_i = v_i$.

*Proof.* We will prove this by contradiction assuming that some player $i$ decides to bid something different from his value. We shall show that with such a deviation, player $i$ will have either the same or lower utility (in other words we will compare his utilities before and after the deviation assuming that the rest players do not change their strategies). Notice here that the bid choice affects only the winning or losing outcome i.e. if a winner alters his strategy he can go to a winner or loser state. However, in his new winner state he will continue to pay the same price as before since the payment amount is determined by the player bellow him (who is the same as before). There are two possible cases that we will examine separately,

**Case 1:** $b_i = v_i$ **vs** $b'_i > v_i$

- If player $i$ was the winner with strategy $s_i$ where $b_i = v_i$ then he remains the winner with strategy $s'_i$ where $b'_i > v_i$ (he is ranked again first since he has the highest bid) and as we said his final utility is the same as before since he pays once again the second highest bid. Therefore $u_i(s_i, s_{-i}) = u_i(s'_i, s_{-i})$.
- If player $i$ was a loser with strategy $s_i$ where $b_i = v_i$ then with strategy $s'_i$ where $b'_i > v_i$ is either a loser or the winner. If he becomes a loser then his utility is zero (the same as before). If he becomes the winner then he gets the item at a price $b_j$. It is obvious however that $b'_i > b_j > b_i = v_i$ (remember that with $b_i = v_i$ he was a loser) so his new utility is $v_i - b_j < 0$. Therefore $u_i(s_i, s_{-i}) = 0 \geq u_i(s'_i, s_{-i})$.

**Case 2:** $b_i = v_i$ **vs** $b'_i < v_i$

- If player $i$ is the winner with strategy $s_i$ where $b_i = v_i$ then with strategy $s'_i$ where $b'_i < v_i$ is either the winner or a loser. If he becomes the winner then his utility is the same as before since he gets the item at the same price. If he becomes a loser then his utility is zero which is equal or less than before. Therefore $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$.
- If player $i$ is a loser with strategy $s_i$ there $b_i = v_i$ then he remains a loser with strategy $s'_i$ where $b'_i < v_i$ so his utility is once again zero. Therefore $u_i(s_i, s_{-i}) = u_i(s'_i, s_{-i}) = 0$.

□

So as we see, in both cases each player $i$ has nothing to gain if he bids something different from his value, thus bidding his true value is a dominant strategy. Concluding, we can make some final comments that summarize the previous analysis:

- In the second-price auction, your bid determines if you are a winner or a loser but not the price of the item that you will get in case you are a winner (since that depends to the bids of the rest players).
- In the second-price auction bidding your true value is the best strategy you can come up with, regardless of what the other players do (whether they also submit their true values, bid higher or lower).

### 1.2.2 First-Price Auction

We have $N$ $(i = 1, ..., N)$ players that have private values $v_i$ for the item that is to be sold by the auctioneer. Each player's strategy is to bid an amount $b_i$ which is a function of his true value $v_i$. The utility of player $i$ with value $v_i$ and bid $b_i$ is defined as follows:

*If player $i$ is the winner (i.e. he is the one who gets the item and has the highest bid) then his utility is defined as $v_i - b_i$. Else if player $i$ is a loser (i.e. he does not get the item) then his utility is defined as zero.*

$$u_i = \begin{cases} (v_i - b_i) & \text{if } i \text{ has the highest bid} \\ 0 & \text{otherwise} \end{cases}$$

As we can see the bid $b_i$ of each player $i$, determines not only the outcome of the auction (if $i$ is a winner or a loser) but also the payment in the winner case scenario (since player $i$ pays his own bid). So this is the first gap between the first and the second-price auctions and is also something that will create many difficulties in the goal of tracing the optimal bid or an optimal strategy. It is obvious that bidding your true value is not the dominant strategy this time around since your utility is zero in both winning or losing state (even if you win, the price you will pay is your full value, so your utility is zero by definition). It is obvious as well, that there is no meaning in bidding higher than your true value because if you are a loser then your utility is zero and if you are a winner your utility is negative by definition (since $p_i = b_i > v_i$). So the only reasonable direction that you can take is bidding lower than your true value. But what is the optimal bid in such a case? If you bid too close to your value then in a winning case scenario you get a very small amount of utility. On the other hand if you bid far bellow your value you increase your utility in a winning case scenario but you also reduce the possibilities for this scenario to happen. As we can see, finding the optimal bid in the first-price auction is a lot more complex than it was in the second-price auction. There are solutions to this problem, however as we mentioned at the beginning of the section we will not present any specific results here.

## 1.3 Basic Models of multi-item Auction

It is obvious that if there are multiple items to be sold, the models that we described are incomplete. Multiple questions also emerge i.e. are these items identical or not? Bellow we present two of the basic models for auctioning multiple items and observe their properties. Multi-item auction models are more general (they can be used for single-items as well) and in a way more realistic since they have many applications in real life (electronic markets and stores use them as mechanisms for selling items).

### 1.3.1 Vickrey-Clarke-Groves Mechanisms

Suppose that we have $N$ players ($i = 1, ..., N$) and $K$ items ($i = 1, ..., K$). Each player $i$ has a vector of private values $V_i = [v_i(1), ..., v_i(K)]$ if the items are non-identical (a value for each item) or a single private value $v_i$ if the items are identical. Additionally each one of them submits a vector of bids (or a single bid) in order to be tagged with the item of his preference (non-identical items) or with an item in general (identical items) respectively.

Suppose also that there is a set $S$ that contains all the possible outcomes (all the possible assignments between the players and the items). Imagine now that there exists a central authority (an auctioneer in our case) who wants to provide an assignment that maximizes $\sum_i b_i(s)$ where $s$ is a possible assignment and $\{b_i(s)\}_{i \epsilon \{1,...,N\}}$ the bids of the players that are tagged with an item in that specific assignment. In other words the center wants to find the best outcome $s$ so that $\sum_i b_i(s)$ has the highest possible value. From now on we denote this best outcome as $\hat{s}$. Finally, the center announces that it will pay each player $i$ that participates in the auction with $\sum_{j \neq i} b_j(\hat{s})$ (the sum of the bids of the players that are tagged to an item at the best outcome except his own) and each player $i$ will pay to the center the amount $max_s \sum_{j \neq i} b_j(s)$ (the sum of the bids of the players that are tagged to an item at the best outcome, when he does not exists-participates in the auction at all). The amount that the center demands from each player to pay, can be seen as the harm that his presence causes on the rest of the players. Notice here that since the center wants to find the best outcome,

$$\sum_{j \neq i} b_j(\hat{s}) \leq max_s \sum_{j \neq i} b_j(s)$$

In order to understand why notice that $\sum_{j \neq i} b_j(\hat{s})$ is the best outcome (the maximum sum) of a $N$ players game minus the bid of player $i$ (so a common outcome of a $N-1$ players game) while $max_s \sum_{j \neq i} b_j(s)$ is the best outcome of a $N-1$ players game. The only case where this inequality is not strict, is when player $i$ does not get any item at the best outcome (so his bid is not computed into the sums). So we can conclude that,

$$max_s \sum_{j \neq i} b_j(s) - \sum_{j \neq i} b_j(\hat{s}) \geq 0$$

which can be seen as the final payment of the player to the center-auctioneer (zero if we have an equality and thus a player who gets no item),

$$p_i = max_s \sum_{j \neq i} b_j(s) - \sum_{j \neq i} b_j(\hat{s})$$

Finally we can define each player's utility as,

$$u_i = \begin{cases} v_i(\hat{s}) - p_i & \text{if } \sum_{j \neq i} b_j(\hat{s}) < max_s \sum_{j \neq i} b_j(s) \\ 0 & \text{if } \sum_{j \neq i} b_j(\hat{s}) = max_s \sum_{j \neq i} b_j(s) \end{cases}$$

where $v_i(\hat{s})$ can be denoted as the value of player $i$ for the item he gets at the best outcome.

**Proposition 1.3.1.1** Bidding your true value is a dominant strategy in the Vickrey-Clarke-Groves Mechanism (VCG).

*Proof.* When player $i$ decides to be truthful, he submits the vector of his values $V_i = [v_i(1), ..., v_i(K)]$. The auctioneer takes this vector as well as the bid vectors of the rest players and tries to form an allocation where $\sum_i b_i(s)$ is maximum. So he computes all the possible $v_i(s) + \sum_{j \neq i} b_j(s)$ for all possible outcomes $s$ and finally he chooses $\hat{s}$ which is the outcome where $v_i(\hat{s}) + \sum_{j \neq i} b_j(\hat{s}) = max_s\{v_i(s) + \sum_{j \neq i} b_j(s)\}$. Player's $i$ utility is,

$$u_i(\hat{s}) = v_i(\hat{s}) + \sum_{j \neq i} b_j(\hat{s}) - max_s \sum_{j \neq i} b_j(s)$$

When player $i$ decides to deviate, bidding a vector different from his valuation vector, his new bids lead him to get either the same or a different item. In any case the auctioneer once again decides the best allocation (which will may be different or the same as before depending on $i's$ bid vector) and player's $i$ utility becomes,

$$u_i(s') = v_i(s') + \sum_{j \neq i} b_j(s') - max_s \sum_{j \neq i} b_j(s)$$

where $s'$ is the best outcome under the new data and $v_i(s')$ is player's $i$ value of the item he is tagged to, at outcome $s'$.

Notice that the amount $max_s \sum_{j \neq i} b_j(s)$ does not change no matter what strategy player $i$ chooses to play. As we said earlier it represents the sum of the bids at the best outcome when player $i$ does not exists at all.

So assume that player $i$ gets a higher utility when he does not bid his true values. Lets see if this is possible:

$$u_i(s') > u_i(\hat{s}) \Rightarrow$$

$$v_i(s') + \sum_{j \neq i} b_j(s') - max_s \sum_{j \neq i} b_j(s) > v_i(\hat{s}) + \sum_{j \neq i} b_j(\hat{s}) - max_s \sum_{j \neq i} b_j(s)$$

$$\Rightarrow v_i(s') + \sum_{j \neq i} b_j(s') > v_i(\hat{s}) + \sum_{j \neq i} b_j(\hat{s})$$

However we know that $v_i(\hat{s}) + \sum_{j \neq i} b_j(\hat{s}) = max_s\{v_i(s) + \sum_{j \neq i} b_j(s)\}$ so we have a contradiction. Notice also that if player $i$ gets no item at the $\hat{s}$ outcome his utility is defined as zero. In such a case and under the same logic, if he deviates from bidding his true values his utility is either zero or some negative amount.

So we can conclude that bidding your true values is a dominant strategy in VCG. □

**Proposition 1.3.1.2** In single-item auctions the second-price and the VCG models are exactly the same.

*Proof.* As we said earlier, there are two possible outcomes at the second-price auction:

- To be the winner (getting the item) with utility $u_i = v_i - b_j$ where $b_j$ is the second highest bid.
- To be a loser with utility $u_i = 0$.

Lets see what happens if we run the same auction via the VCG model,

- **Winner case:** $u_i = v_i + \sum_{j \neq i} b_j(\hat{s}) - max_s \sum_{j \neq i} b_j(s)$. Notice here that $\sum_{j \neq i} b_j(\hat{s})$ is zero since player $i$ got the only item that exists and there is no other player tagged to an item (remember that we take into consideration only the bids of the players that get an item). Lets take a look to $max_s \sum_{j \neq i} b_j(s)$ , if player $i$ does not exist then since the auctioneer wants to maximize the sum, he will give the item to the player with the second highest bid, say $j$ (who is first in this new allocation). So we have that $max_s \sum_{j \neq i} b_j(s) = b_j$ and we can clearly conclude that $i's$ utility is $u_i = v_i - b_j$ which is the same with the second-price run.
- **Loser case:** Since player $i$ is a loser, he does not get any item. We only need to verify that $\sum_{j \neq i} b_j(\hat{s}) = max_s \sum_{j \neq i} b_j(s)$. It is easy to see that the two sums are equal since both represent the highest bid (which does not alter either player $i$ is present or not). We can conclude that $u_i = 0$ which is the same as in the second-price run.

□

**Proposition 1.3.1.3** In an auction with $N$ players and $K < N$ identical items, under the VCG run all the winners pay the $K + 1$ highest bid.

*Proof.* Suppose that player $i$ is a winner and his utility comes as always by the relation $u_i = v_i + \sum_{j \neq i} b_j(\hat{s}) - max_s \sum_{j \neq i} b_j(s)$. Notice that since the items are identical, each player has the one value for every item. Now lets turn again to the amount, $max_s \sum_{j \neq i} b_j(s)$. This amount represents the maximum sum when player $i$ does not exist. It is obvious that in order for the sum to be maximum as well as for the assignment to be complete (no item untagged), player $i's$ place will be covered by the player with the highest bid who previously was not tagged to any item (this player has the $K+1$ highest bid, say $b'$) while the other bids will remain intact. So returning back to the relation of the utility and more specifically to the difference of the sums we can see that, $\sum_{j \neq i} b_j(\hat{s}) - max_s \sum_{j \neq i} b_j(s) = \sum_{j \neq i} b_j(\hat{s}) - \sum_{j \neq i} b_j(\hat{s}) - b' = -b'$ which is the final payment of every player $i$ that is a winner. This is in fact very interesting since we can say that the winners form a class which contains people with the same properties no matter their differences. We can actually see them as one person, the winner player of the second-price auction setting. □

### 1.3.2 VCG Example: Non-identical items

In order to have a better understanding on how the VCG mechanism works as well as why bidding your true values is a dominant strategy, lets take a look at the following example (notice here that the items are non-identical, so the players have different values on each of them depending on their preferences),

Suppose that we have two players and two items with the following properties:

| $Players$ | $Values\ for\ item\ 1$ | $Values\ for\ item\ 2$ |
|---|---|---|
| $A$ | 8 | 4 |
| $B$ | 4 | 2 |

We assume that both players submit their true values since as we saw, this is a dominant strategy. Both players prefer to be tagged with item 1, however the auctioneer must decide an assignment which maximizes the amount $\sum_i b_i(s) = \sum_i v_i(s)$. This is achieved by giving player $A$ item 1 and player $B$ item 2 which gives as a total sum of $v_{A1} + v_{B2} = 8 + 2 = 10$ since the alternative possible assignment gives a total sum of $v_{A2} + v_{B1} = 4 + 4 = 8$. In order to see how the VCG mechanism works, lets take a look on how the utilities of both players are computed,

*Player's A utility:* $u_A = v_{A1} + \sum_{j \neq A} b_j(\hat{s}) - max_s \sum_{j \neq A} b_j(s) = 8 + 2 - max_s \sum_{j \neq A} b_j(s) = 10 - max_s \sum_{j \neq A} b_j(s)$, notice here that if player $A$ didn't exist, the auctioneer would give item 1 to player $B$ while item 2 would remain untagged. So finally we have that $u_A = 10 - 4 = 6$.

*Player's B utility:* $u_B = v_{B2} + \sum_{j \neq B} b_j(\hat{s}) - max_s \sum_{j \neq B} b_j(s) = 2 + 8 - max_s \sum_{j \neq A} b_j(s) = 10 - max_s \sum_{j \neq B} b_j(s)$, notice here that if player $B$ didn't exist, the auctioneer would, once again, give item 1 to player $A$ while item 2 would remain untagged. So finally we have that $u_B = 10 - 8 = 2$.

As we said earlier, both players prefer item 1. So suppose now that player $B$ decides to deviate and does not submit his true value for the first item but instead, he lies and bids $b_{B1} = 9$ in order to get it. In such a case we have the following,

| $Players$ | $Values\ for\ item\ 1$ | $Values\ for\ item\ 2$ | $Bids\ for\ item\ 1$ | $Bids\ for\ item\ 2$ |
|---|---|---|---|---|
| $A$ | 8 | 4 | 8 | 4 |
| $B$ | 4 | 2 | 9 | 2 |

The auctioneer once again wants to maximize the amount $\sum_i b_i(s)$ so he tags player $A$ with item 2 and player $B$ with item 1, an assignment that gives a total sum of $v_{A2} + v_{B1} = 4 + 9 = 13$ (notice that the alternative gives $v_{A1} + v_{B2} = 8 + 2 = 10$). Lets take a look at player's $B$ new utility,

*Player's B utility:* $u_B = v_{B1} + \sum_{j \neq B} b_j(\hat{s}) - max_s \sum_{j \neq B} b_j(s) = 4 + 4 -$

$max_s \sum_{j \neq A} b_j(s) = 8 - max_s \sum_{j \neq B} b_j(s)$, notice here that if player $B$ didn't exist, the auctioneer would give item 1 to player $A$ while item 2 would remain untagged. So finally we have that $u_B = 8 - 8 = 0$. It is clear that although player $B$ prefers to be tagged with the first item, it is best for him to say the truth about his values and get item 2 since he has a better payoff.

### 1.3.3 Generalized Second-Price Auction and the Advertisement Position Setting

Generalized second-price model (GSP) is used mostly at the market of on-line advertisements. It is very popular and its usage was rapidly increased through recent years. Some examples are the Google's total revenue in 2005 (about \$6.14 billion) where over 98 percent of it came from the GSP auctions as well as Yahoo's total revenue in 2005 (about\$5.26 billion) where over the half of it derived from sales via GSP auctions. Lets take a first look on how these auctions actually work. Imagine that you are on-line in the Internet and you want to gather information for something via a search engine. You enter the term that you are interested in and you get in turn relevant links and pages as well as sponsored links like paid advertisements. When you click on a sponsored link (we have to mention here that sponsored links are clearly distinguishable from the original search results) , you are sent to the advertiser's web page. Since you found this particular page via the search engine, the advertiser has to pay an amount of money to the engine for sending you to his page. This -pay per click price- type of payment is known also as "pay-per-click" pricing. But how the auctions and more specifically the GSP comes into play? The advertisements that we mentioned before have different positions in the search engine's web page, and obviously the ads that are placed higher in the page have more possibilities to be clicked than the ones that take place at the bottom or lower in general. Additionally the number of ads that can appear in each search made by a user is limited. So we need a mechanism that somehow can allocate the advertisers in the appropriate positions. GSP comes as a solution to this.

The GSP model sets a keyword to an auction and the advertisers submit their bids. Then they are rearranged at a decreasing bid order i.e. the one with the highest bid is placed first, the one with the second highest bid is placed second etc. When a user enters this keyword in the search engine he gets the relevant web links plus the sponsored links in the (position) order that we described. Now if a user clicks the ad of the advertiser $i$ and visit his web page then $i$ pays to the search engine the bid of the advertiser $i+1$. As we can see the logic behind this mechanism is similar to the on of the second-price auction but in a more general way (every player that gets something i.e. an allocation to the page, pays the bid of the player bellow him) thus the characterization ”generalized” second-price.

Notice here that the GSP and the VCG have similarities on how the payments are formed i.e. in both models, each player's payment does not depend on his own bid but on the bids and the allocations of the rest. However these two mechanisms have differences as well i.e. while bidding your true value is a dominant strategy at the VCG mechanism it is not at the GSP (we will come into these more extensively later on). Finally when we speak about only one single-allocation, second-price, GSP and VCG models are equivalent.

## 1.4 GSP and VCG under the Ad Position Setting

At this final section of chapter 1 we will see the GSP and VCG models under the advertisement position setting. As we said, despite the fact that these two models have similarities, they also have many differences (we will observe them later on at subsection 1.4.3). We need to mention that although the VCG model has better properties than the GSP (truthfulness, smaller payments to the auctioneer), the GSP is used a lot more due to its simpler structure (which gives additionally much more freedom to the players in conducting strategies).

### 1.4.1 GSP under the Ad Position Setting

Let us now formalize the GSP model under the advertisement position setting. Consider that we have $N$ players-advertisers ($i = 1, ..., N$) and that we want to allocate them into $K < N$ positions on a page that comes up when a user enters a specific keyword on the search engine. We can see these positions as slots that are to be bought by the players. Each slot $j$ has an expected clickthrough rate (CTR) $\theta_j$, where $j = 1, ..., K$. We number the slots so that $\theta_1 > \theta_2 >, ..., > \theta_K$ and we assume that all players agree with the ordering. As a parallelism you can imagine that the slot with the highest CTR is the top position sponsored link on the page, which is expected to get the highest number of clicks. Finally we set $\theta_j = 0$ if $j > K$.

Each player $i$ has a value $v_i > 0$ so we can interpret $v_i\theta_j$ as $i's$ expected profit from appearing in slot $j$. Additionally, $i$ will pay a price per click for getting slot $j$ so we can say that his payment to the search engine will be $p_i\theta_j$. Therefore we can set player's $i$ utility to $u_i = \theta_j(v_i - p_i)$.

In order to apply the GSP model, we have that these slots are sold via an auction. Each player $i$ submits a bid $b_i$, the auctioneer ranks the players in decreasing order of bids and renumbers them if necessary so that player 1 has the highest bid, player 2 has the second highest bid etc. The allocation now is made as follows: The player with the highest bid gets the slot with the highest CTR, the player with the second highest bid gets the slot with the second highest CTR and so on. Since the players are renumbered it is clear that player $j$ gets slot $j$. Finally the payments are defined by the GSP model and each player $i$ who gets an item, pays the bid of player $i + 1$, so $p_i = b_{i+1}$ while if he does not get an item he pays zero. The utility function is defined as follows:

$$u_i = \begin{cases} \theta_i(v_i - b_{i+1}) & \text{if } i \text{ is tagged to a slot} \\ 0 & \text{otherwise} \end{cases}$$

**Example:** Suppose that there are 4 players and 3 slots $\theta_1 > \theta_2 > \theta_3$, the ordering and the prices are defined as follows,

| $Position$ | $Values$ | $Bids$ | $Prices$ | $CTRs$ |
|---|---|---|---|---|
| 1 | $v_1$ | $b_1$ | $b_2$ | $\theta_1$ |
| 2 | $v_2$ | $b_2$ | $b_3$ | $\theta_2$ |
| 3 | $v_3$ | $b_3$ | $b_4$ | $\theta_3$ |
| 4 | $v_4$ | $b_4$ | 0 | 0 |

### 1.4.2 VCG under the Ad Position Setting

We will now describe how the VCG mechanism reacts under this setting. There are some rules that are the same in both models such us the positioning: the player with the $i$-th highest bid, gets the slot with the $i$-th highest CTR (or nothing if such a slot does not exist) and is placed at the $i$-th position. The payments however are different. The utility relation (for someone who is tagged with a slot) is reformed as follows (since this setting supports pay-per-click payments),

$$u_i = v_i - (max_s \sum_{j \neq i} b_{j'}(s)\theta_j - \sum_{j \neq i} b_j(\hat{s})\theta_j)$$

where

$$max_s \sum_{j \neq i} b_{j'}(s)\theta_j - \sum_{j \neq i} b_j(\hat{s})\theta_j$$

is the payment of player $i$.

Lets take a look at each sum amount separately:

- $\sum_{j \neq i} b_j(\hat{s})\theta_j = b_1\theta_1 + ... + b_{i-1}\theta_{i-1} + b_{i+1}\theta_{i+1} + ... + b_K\theta_K$ represents the optimal outcome taking into consideration every player that gets a slot except $i$.

- $max_s \sum_{j \neq i} b_{j'}(s)\theta_j = b_1\theta_1 + ... + b_{i-1}\theta_{i-1} + b_{i+1}\theta_i + ... + b_{K+1}\theta_K$ represents the optimal outcome when player $i$ does not exist (thus player $i + 1$ takes $i's$ position and is tagged with $i's$ slot).

So,

$$max_s \sum_{j \neq i} b_{j'}(s)\theta_j - \sum_{j \neq i} b_j(\hat{s})\theta_j =$$

$$(b_1\theta_1 + ... + b_{i-1}\theta_{i-1} + b_{i+1}\theta_i + ... + b_{K+1}\theta_K) - (b_1\theta_1 + ... + b_{i-1}\theta_{i-1} + b_{i+1}\theta_{i+1} + ... + b_K\theta_K) = (b_{i+1}\theta_i - b_{i+1}\theta_{i+1}) + ... + (b_K\theta_{K-1} - b_K\theta_K) + b_{K+1}\theta_K =$$

$$\sum_{j=i}^{K} b_{j+1}(\theta_j - \theta_{j+1})$$

So we can conclude that $p_i = \sum_{j=i}^{K} b_{j+1}(\theta_j - \theta_{j+1})$

### 1.4.3 GSP versus VCG

Although it seems that the two mechanisms have many similarities (actually they are similar when we speak about single-item auctions) they are are not equivalent when we apply them in auctions with multiple items-slots-positions. We will analyze some of their differences in the following propositions.

**Proposition 1.4.3.1** The payments of the advertisers in the GSP model are at least as large as the ones in the VCG model.

*Proof.* Consider that we run the same data on both models and choose a random player $i$. If this player is not tagged to a slot then he pays zero on both mechanisms. If however he is tagged to a slot then he pays $b_{i+1}\theta_i$ in the GSP run and $\sum_{j=i}^{K} b_{j+1}(\theta_j - \theta_{j+1})$ in the VCG run. We have to compare these two amounts,

$$\begin{aligned} VCG_{payment} - GSP_{payment} &= \sum_{j=i}^{K} b_{j+1}(\theta_j - \theta_{j+1}) - b_{i+1}\theta_i \\ &= \sum_{j=i}^{K-1} \theta_{j+1}(b_{j+2} - b_{j+1}) \\ &\leq 0 \end{aligned}$$

since as we said $b_i \leq b_{i-1}$. So we can conclude that $VCG_{payment} \leq GSP_{payment}$ □

**Proposition 1.4.3.2** Bidding your true value is not a dominant strategy under the GSP mechanism.

*Proof.* We can show this with an example. Suppose that we have 3 players and 2 slots with the following properties.

| $Slots$ | $CTRs$ |
|---|---|
| $s_1$ | 5 |
| $s_2$ | 4 |

| $Players$ | $Values$ |
|---|---|
| $A$ | 6 |
| $B$ | 5 |
| $C$ | 2 |

We will examine player's $A$ utility when he bids his true value and when he deviates, bidding something different.

**Case 1:** Players bid their true values

If players bid their true values then the allocation is formed as follows:

| $Players$ | $Values$ | $Bids$ | $Prices$ | $CTRs$ |
|---|---|---|---|---|
| $A$ | 6 | 6 | 5 | 5 |
| $B$ | 5 | 5 | 2 | 4 |
| $C$ | 2 | 2 | 0 | 0 |

Lets take a look at $A's$ utility:

$u_A = \theta_1(v_A - b_B) = 5(6 - 5) = 5$

**Case 2:** Players $B$ and $C$ bid their true values while player $A$ chooses to deviate and bids something else

Suppose that player $A$ wants to deviate in order to get a better utility. So he bids $b_A = 4$. The allocation is formed as follows:

| *Players* | *Values* | *Bids* | *Prices* | *CTRs* |
|---|---|---|---|---|
| $B$ | 5 | 5 | 4 | 5 |
| $A$ | 6 | 4 | 2 | 4 |
| $C$ | 2 | 2 | 0 | 0 |

Lets take a look at $A's$ utility after the deviation:

$$u_A = \theta_2(v_A - b_c) = 4(6 - 2) = 16$$

As we can see player $A$ gets a better utility when he is not bidding his true value. So we can conclude that bidding your true value is not a dominant strategy in the GSP model. □

As we mentioned earlier on, although the VCG model seems to have better properties, the GSP model gives more freedom (players are not bind to bid their true values since truthfulness is not a dominant strategy) and additionally has a much simpler structure (due to the simple definition of each player's payment). From now on and through the following chapters we will stick to the GSP model [1] and its variations.

[1] In the following chapters we observe various GSP models under different setting. Some of these models hold the name second-price despite the fact that they actually use the GSP structure (the term second-price is often used, in general, to describe both singe-item and multi-item auctions).

# Chapter 2

# Equilibrium at the GSP Ad Position Setting

In this chapter we will give definitions of the pure nash equilibrium and envy-free equilibrium at the ad position setting when we use the GSP model and we will examine their properties and their differences. The whole presentation is based on [1]. There is a summary of the basic results and additional notes from our side regarding the comparison between the properties of a pure nash equilibrium set of bids and an envy-free equilibrium set of bids (section 2.3). We also provide more extended proofs at various points (section 2.2: Fact 3 analysis, section 2.7: Fact 6, proof part 2) .

## 2.1 Pure Nash Equilibrium Definiton

In an ad position auction we assume that the goal of each player is to bid such a value so that he can maximize his expected utility. We say that we are in a pure nash equilibrium state if each player prefers his current slot to any alternative slot i.e. his utility at his current position-slot is at least as big as at any other position-slot. Before we well-define the pure nash equilibrium state lets take a look on how some player $i$ can change his position.

**Example:** Suppose that there are 4 players and 3 slots. We know that

$\theta_i > \theta_{i+1}$ by assumption and that $b_i > b_{i+1}$ by the rules of the auction (remember that we renumbered the players) , the ordering and the prices are defined as follows,

| *Position* | *Values* | *Bids* | *Prices* | *CTRs* |
|---|---|---|---|---|
| 1 | $v_1$ | $b_1$ | $b_2$ | $c_1$ |
| 2 | $v_2$ | $b_2$ | $b_3$ | $c_2$ |
| 3 | $v_3$ | $b_3$ | $b_4$ | $c_3$ |
| 4 | $v_4$ | $b_4$ | 0 | 0 |

Now consider that player 2 wants to change his position. He can move either up in order to get a slot with bigger CTR (but more expensive) or down in order to get a less expensive slot (but with lower CTR). Suppose that player 2 wants to move up by one position in order to be placed first. He has to beat player's 1 bid so his new bid must be at least as high as $b_1$. Suppose now that he wants to move down by one position in order to be placed third. In such a case he has to bid lower than player 3 and also at least as high as $b_4 = p_3$ (player's 4 bid). So notice the difference behind the logic of moving up and down: If you want to move up you have to beat the bid of the player who currently occupies the slot you want to get, although if you want to move down you have to beat the price paid by the player who currently occupies the slot you want to get. Lets now formalize the equilibrium state:

**Definition 2.1.1** A pure nash equilibrium (PNE) is a set of prices such that

$$\theta_i(v_i - p_i) \geq \theta_j(v_i - p_j) \text{ for } j > i$$

$$\theta_i(v_i - p_i) \geq \theta_j(v_i - p_{j-1}) \text{ for } j < i$$

where $p_j = b_{j+1}$. In other words, in a PNE every player is "better off" in his current position than in any other position.

## 2.2 Envy-Free Equilibrium Definition and Properties

We will now define a subset of PNE which will simplify the analysis of the ad position auctions. We say that player $i$ envies player $j$ if player $i$ has a bigger utility when he gets $j's$ slot and pays $j's$ price. Notice here that the term "envy" is different from the "better off" term and examines if you have a bigger utility when you are placed in another player's shoes and not when you are forcing your way through his position via changing your bid. As a brief example we can say player 3 of the previous matrix envies player 2 if,

$$\theta_3(v_3 - p_3) < \theta_2(v_3 - p_2) \Rightarrow \theta_3(v_3 - b_4) < \theta_2(v_3 - b_3)$$

while he is better off at player's 1 position if,

$$\theta_3(v_3 - p_3) < \theta_2(v_3 - p_1) \Rightarrow \theta_3(v_3 - b_4) < \theta_2(v_3 - b_2)$$

**Definition 2.2.2** An envy-free equilibrium (EFE) is a set of prices such that

$$\theta_i(v_i - p_i) \geq \theta_j(v_i - p_j) \text{ for all } j \text{ and } i$$

Notice that EFE state unifies the to inequalities of the PNE state (into the first one) so it makes the whole analysis more simple since we do not have to examine the utilities at different positions in cases.

The set of the EFE bids have many good properties. Specifically there are 5 main facts (and an additional 6-th that we will see later on) that hold for the EFE set. These facts show that the EFE form a well-behaved subset of the PNE. Lets take a look at these facts:

**Fact 1 (Non negative surplus)** In an EFE $v_i \geq p_i$.

We can see this as a type of rationality where no player pays his slot more than he values it.

**Fact 2 (Monotone values)** In an EFE $v_{i-1} \geq v_i$ for all $i$.

This is pretty interesting since it implies that in an EFE state, in addition with $b_i$ and $\theta_i$ , values also decrease when their indexes increase.

**Fact 3 (Monotone prices)** In an EFE $p_{i-1}\theta_{i-1} > p_i\theta_{i-1}$ and $p_{i-1} > p_i$ for all $i$.

A property that comes from the first two facts. A question that may come up is, can we have an EFE state if $p_i = p_{i+1}$ for some $i$ ? The answer is no if for every $i$ we have $v_i > v_{i+1}$ and the bidding from the agents is conservative i.e. $v_i \geq b_i$. In order to understand why, take a look at the following matrix which describes a part of an allocation,

| $Position$ | $Values$ | $Bids$ | $Prices$ | $CTRs$ |
|---|---|---|---|---|
| $i$ | $v_i$ | $b_i$ | $b$ | $\theta_i$ |
| $i+1$ | $v_{i+1}$ | $b$ | $b$ | $\theta_{i+1}$ |
| $i+2$ | $v_{i+2}$ | $b$ | $p_{i+2}$ | $\theta_{i+2}$ |

It is clear that in order to have players $i$ and $i+1$ paying the same price, players $i+1$ and $i+2$ have to submit the same bid. Notice that by assumption we know that $\theta_i > \theta_{i+1} > \theta_{i+2}$. In such a situation it is impossible to achieve an EFE state since player $i+1$ envies player $i$, $\theta_{i+1}(v_{i+1} - b) < \theta_i(v_{i+1} - b)$ because at $i's$ position he gets a slot with bigger CTR while he pays it at his previous price. Notice also that $v_{i+1} - b \neq 0$ since $b$ we is at most equal with $v_{i+2} < v_{i+1}$ by assumption.

**Fact 4 (EFE $\subset$ PNE)** If a set of prices is an EFE then it is an PNE as well.

A quick way to see why, is to take a look at the inequalities of the two definitions

$$\theta_i(v_i - p_i) \geq \theta_j(v_i - p_j) \geq \theta_j(v_i - p_{j-1}) \text{ since } p_j < p_{j-1}$$

**Fact 5 (One step solution)** If a set of bids satisfy the EFE inequalities for $i+1$ and $i-1$ then it satisfies these inequalities for all $i$.

In a more formal way, if for all $i$, $\theta_i(v_i - p_i) \geq \theta_{i+1}(v_i - p_{i+1})$ and $\theta_i(v_i - p_i) \geq \theta_{i-1}(v_i - p_{i-1})$ then $\theta_i(v_i - p_i) \geq \theta_j(v_i - p_j)$ for all $i$ and $j$. This means that if all players have utilities that satisfy the EFE inequalities for their neighbors then their utilities satisfy the EFE inequalities in general.

## 2.3 PNE versus EFE

As Fact 4 mentions, EFE is a subset of PNE so we can expect that some of the facts do not hold for the PNE set of bids. Lets take a look to the following propositions.

**Proposition 2.3.1** Fact 2 does not hold for PNE.

*Proof.* Suppose that we have 4 players and 3 slots, bellow we give the properties as well as the allocation

| $Position$ | $Values$ | $Bids$ | $Prices$ | $CTRs$ |
|---|---|---|---|---|
| 1 | 7 | 5 | 4 | 4 |
| 2 | 10 | 4 | 3 | 3 |
| 3 | 6 | 3 | 2 | 2 |
| 4 | 3 | 2 | 0 | 0 |

We claim that under these properties we are in a PNE state so Fact 2 does not hold since $v_2 > v_1$. Lets examine the utilities of each player,

**Utility of player 1:**

- At his current position: $(7-4)4 = 12$
- At position 2: $(7-3)3 = 12$
- At position 3: $(7-2)2 = 10$
- At position 4: $(7-0)0 = 0$

**Utility of player 2:**

- At his current position: $(10-3)3 = 21$
- At position 1: $(10-5)4 = 20$
- At position 3: $(10-2)2 = 16$
- At position 4: $(10-0)0 = 0$

**Utility of player 3:**

- At his current position: $(6-2)2 = 8$
- At position 1: $(6-5)4 = 4$
- At position 2: $(6-4)3 = 6$
- At position 4: $(6-0)0 = 0$

**Utility of player 4:**

- At his current position: $(3-0)0 = 0$
- At position 1: $(3-5)4 = -8$
- At position 2: $(3-4)3 = -3$
- At position 3: $(3-3)2 = 0$

So none of the players is better off in someone else's position therefore we are in a PNE state and Fact 2 does not hold. □

**Proposition 2.3.2** Fact 3 does not hold for PNE.

*Proof.* Suppose that we have 4 players and 3 slots, bellow we give the properties as well as the allocation

| $Position$ | $Values$ | $Bids$ | $Prices$ | $CTRs$ |
|---|---|---|---|---|
| 1 | 7 | 5 | 3 | 4 |
| 2 | 10 | 3 | 3 | 3 |
| 3 | 6 | 3 | 1 | 2 |
| 4 | 3 | 1 | 0 | 0 |

As we can see here $p_1 = p_2$ (we do not have monotone prices). We claim that this set of bids form a PNE.

**Utility of player 1:**

- At his current position: $(7-3)4 = 16$
- At position 2: $(7-3)3 = 12$
- At position 3: $(7-1)2 = 12$
- At position 4: $(7-0)0 = 0$

**Utility of player 2:**

- At his current position: $(10-3)3 = 21$
- At position 1: $(10-5)4 = 20$
- At position 3: $(10-1)2 = 18$
- At position 4: $(10-0)0 = 0$

**Utility of player 3:**

- At his current position: $(6-1)2 = 10$
- At position 1: $(6-5)4 = 4$
- At position 2: $(6-3)3 = 9$
- At position 4: $(6-0)0 = 0$

**Utility of player 4:**

- At his current position: $(3-0)0 = 0$
- At position 1: $(3-5)4 = -8$
- At position 2: $(3-3)3 = 0$
- At position 3: $(3-3)2 = 0$

So we have that although this is a PNE state there is no need for the prices to follow a monotone pattern. □

**Proposition 2.3.3** Fact 5 does not hold for PNE.

*Proof.* Suppose that we have 4 players and 3 slots, bellow we give the properties as well as the allocation

| $Position$ | $Values$ | $Bids$ | $Prices$ | $CTRs$ |
|---|---|---|---|---|
| 1 | 7 | 5 | 4 | 4 |
| 2 | 10 | 4 | 3 | 3 |
| 3 | 6 | 3 | 9/10 | 2 |
| 4 | 3 | 9/10 | 0 | 0 |

We claim that under these properties, for every $i$, the set of bids satisfy the PNE inequalities for $i+1$ and $i-1$, although it does not satisfy them for every $j$ in general.

**Utility of player 1:**

- At his current position: $(7-4)4 = 12$
- At position 2: $(7-3)3 = 12$

**Utility of player 2:**

- At his current position: $(10-3)3 = 21$
- At position 1: $(10-5)4 = 20$
- At position 3: $(10-9/10)2 = 18.2$

**Utility of player 3:**

- At his current position: $(6-9/10)2 = 10.2$
- At position 2: $(6-4)3 = 6$
- At position 4: $(6-0)0 = 0$

**Utility of player 4:**

- At his current position: $(3-0)0 = 0$
- At position 3: $(3-3)2 = 0$

So we have that for every $i$ the set of bids satisfy the PNE inequalities for $i+1$ and $i-1$. However, if we examine the utility of player 1 at position 3 we can see that $(7-9/10)2 = 12.2 > 12$ so he is better off at position 3 than his current one and the inequalities do not hold. □

## 2.4 Bounds of the Bids-Prices at an Equilibrium State

We can now use these facts in order to get more information about the bids and the prices at an equilibrium state. According to the previous definitions, when we are in a EFE state and thus in a PNE state, we have that each player $i$ does not want to move down by one position so,

$$\theta_i(v_i - p_i) \geq \theta_{i+1}(v_i - p_{i+1}) \Rightarrow$$

$$\theta_i p_i \leq v_i(\theta_i - \theta_{i+1}) + \theta_{i+1} p_{i+1}$$

Additionally we have that each player $i+1$ does not want to move up by one position so,

$$\theta_{i+1}(v_{i+1} - p_{i+1}) \geq \theta_i(v_{i+1} - p_i) \Rightarrow$$

$$\theta_i p_i \geq v_{i+1}(\theta_i - \theta_{i+1}) + \theta_{i+1} p_{i+1}$$

Combining these two inequalities we have that,

$$v_i(\theta_i - \theta_{i+1}) + \theta_{i+1} p_{i+1} \geq \theta_i p_i \geq v_{i+1}(\theta_i - \theta_{i+1}) + \theta_{i+1} p_{i+1}$$

which can be seen us the upper and lower bound of what player $i$ totally pays in an EFE state. We can also alter the above inequality into an equivalent form in order to achieve bounds for the bids. Remember here that $p_i = b_{i+1}$ so,

$$v_i(\theta_i - \theta_{i+1}) + \theta_{i+1}b_{i+2} \geq \theta_i b_{i+1} \geq v_{i+1}(\theta_i - \theta_{i+1}) + \theta_{i+1}b_{i+2}$$

and finally if we set $i = i - 1$ we get the limits of player's $i$ bid,

$$v_{i-1}(\theta_{i-1} - \theta_i) + \theta_i b_{i+1} \geq \theta_{i-1}b_i \geq v_i(\theta_{i-1} - \theta_i) + \theta_i b_{i+1}$$

Notice that the lower bound for player's $i$ bid contains his value as well as the bid of the player bellow him while the upper bound contains the value of the player above him as well as the bid of the player bellow him as convex combinations. The interesting part is that now we can use these inequalities recursively in order to find a sequence of PNE or EFE bids. Let us now take a more careful look at each bound:

*Upper Bound:* $\theta_{i-1}b_i^U = v_{i-1}(\theta_{i-1} - \theta_i) + \theta_i b_{i+1}$ (i)
*Lower Bound:* $\theta_{i-1}b_i^L = v_i(\theta_{i-1} - \theta_i) + \theta_i b_{i+1}$ (ii)

Both equations are recursive so finding the solution of the recursions can give us a quick way of computing the bounds. Remember here that in the start of our analysis of the model we described that in a game with $N$ players and $K < N$ slots, $\theta_j = 0$ if $j > K$. Now if we consider player $K + 1$ (the first player who gets no slot) we have that,

$$v_K(\theta_K - \theta_{K+1}) + \theta_{K+1}b_{K+2} \geq \theta_K b_{K+1} \geq v_{K+1}(\theta_K - \theta_{K+1}) + \theta_{K+1}b_{K+2} \Rightarrow$$

$$v_K(\theta_K - 0) + 0 \geq \theta_K b_{K+1} \geq v_{K+1}(\theta_K - 0) + 0 \Rightarrow$$

$$v_K\theta_K \geq \theta_K b_{K+1} \geq v_{K+1}\theta_K \Rightarrow$$

$$v_K \geq b_{K+1} \geq v_{K+1}$$

This describes the base of our recursion for both bounds as well as the potential strategy of the first excluded player (the interesting part comes from the lower bound which says that bidding lower than your value has no meaning if you are the first player who gets no slot). So it is easy to conclude that the solutions to these recursions are,

*Upper Bound:* $\theta_{i-1}b_i^U = \sum_{j \geq i} v_{j-1}(\theta_{j-1} - \theta_j)$
*Lower Bound:* $\theta_{i-1}b_i^L = \sum_{j \geq i} v_j(\theta_{j-1} - \theta_j)$

These equations represent the upper and the lower bound of each player's bids as well as the maximum and the minimum total payment each player has to pay at an EFE state. In other words,

$$\sum_{j\geq i} v_{j-1}(\theta_{j-1}-\theta_j) \geq \theta_{i-1}p_{i-1} = \theta_{i-1}b_i \geq \sum_{j\geq i} v_j(\theta_{j-1}-\theta_j)$$

## 2.5 The meaning behind the Bounds

Suppose that player $i$ is thinking offensively and wants to move up by one position (exceeding the bid of player $i-1$) but in addition, he wants his utility at his new position to be at least as big as in his current position. In other words,

$$\text{worst utility in moving up} = \text{utility in current position} \Rightarrow$$

$$\theta_{i-1}(v_i - b^*) = \theta_i(v_i - b_{i+1}) \Rightarrow$$

$$\theta_{i-1}b^* = v_i(\theta_{i-1}-\theta_i) + \theta_i b_{i+1}$$

So the price he must pay in order for that to happen is exactly the lower bound recursion (ii).

Suppose now that player $i$ is thinking defensively and does not want to bid too high since he is afraid that he will decrease the utility of player $i-1$ so much that he might prefer to move down to his position. Thus he wants to bid an amount so that the least utility player $i-1$ has, is equal to the utility he would make if he was at his position. In other words,

$$i-1's \text{ utility now} = \text{utility if he moves into } i's \text{ position} \Rightarrow$$

$$\theta_{i-1}(v_{i-1} - b^*) = \theta_i(v_{i-1} - b_{i+1}) \Rightarrow$$

$$\theta_{i-1}b^* = v_{i-1}(\theta_{i-1}-\theta_i) + \theta_i b_{i+1}$$

So the amount he has to bid is exactly the upper bound of the recursion (i).

## 2.6 The VCG Payment Resemblance

You may have notice by now that the equations of the bounds we just described have a resemblance with the total payments of the players when we run the VCG model under the same setting. Let us recall the VCG total payment considering player $i-1$,

$$p_{i-1} = \sum_{j=i-1}^{K} b_{j+1}(\theta_j - \theta_{j+1})$$

and lets reform it a little with the proper adjustments into the indexes,

$$p_{i-1} = \sum_{j=i}^{K+1} b_j(\theta_{j-1} - \theta_j)$$

Finally this can be seen us,

$$p_{i-1} = \sum_{j \geq i} b_j(\theta_{j-1} - \theta_j)$$

since the CTR's beyond $K$ are defined as zero. Consider now that bidding your true value is a dominant strategy for the VCG mechanism which gives a PNE state, we can conclude that the equilibrium payment for the VCG is,

$$p_{i-1} = \sum_{j \geq i} v_j(\theta_{j-1} - \theta_j)$$

which is exactly the same as the lower bound of the EFE total payment for player $i - 1$ in the GSP model under the same setting.

## 2.7 Revenues of PNE and EFE

Now that we have all these informations it is our chance to look the things from the auctioneers perspective. The term revenue refers to the total amount of money auctioneer gets. In a more formal way, we can define the revenue in our current setting as,

$$R = \sum_{i=1}^{N} p_i \theta_i$$

The first think we can say is that since we have bounds for the payments $p_i \theta_i$ at an EFE state we can safely assume that there are bounds for the total revenue when we are in an EFE. A question that comes to mind is what is the relation between the bounds of the revenue at PNE and EFE. From Fact 4 we know that $EFE \subset PNE$ so we can speculate that the PNE maximum and minimum revenue is bigger and smaller than the EFE maximum and minimum revenue respectively (both are sets of prices and PNE contains EFE). As we will see, there is a final Fact which claims that this is half right, more specifically:

**Fact 6** The maximum revenue PNE yields is the same as the EFE maximum revenue while the minimum revenue PNE yields is generally less than the EFE minimum revenue.

*Proof.* We will split the proof into two parts (maximum and minimum revenues respectively).

**Part 1:** Maximum revenues are the same

Suppose that $\{p_i^N\}_{i=1,..,K}$ is the set of prices which is associated with the maximum PNE revenue, $maxR^N$ and that $\{p_i^{EF}\}_{i=1,..,K}$ the set of prices associated with the maximum EFE revenue $maxR^{EF}$. From Fact 4 we know that $EFE \subset PNE$ which implies that $maxR^N \geq maxR^{EF}$ (1). We will try to show that $maxR^N \leq maxR^{EF}$:

From the definition of the upper bound of the recursion we have that,

$$p_i^{EF}\theta_i = p_{i+1}^{EF}\theta_{i+1} + v_i(\theta_i - \theta_{i+1})$$

Setting $i = K$ we have,

$$p_K^{EF}\theta_K = p_{K+1}^{EF}\theta_{K+1} + v_K(\theta_K - \theta_{K+1}) \Rightarrow$$

$$p_K^{EF}\theta_K = 0 + v_K(\theta_K - 0) \Rightarrow$$

$$p_K^{EF}\theta_K = v_K\theta_K \Rightarrow$$

$$p_K^{EF} = v_K \ (2)$$

Additionally, according to the PNE definition we have,

$$\theta_i(v_i - p_i^N) \geq \theta_{i+1}(v_i - p_{i+1}^N) \Rightarrow$$

$$\theta_i p_i^N \leq \theta_{i+1}p_{i+1}^N + v_i(\theta_i - \theta_{i+1})$$

Setting $i = K$ we have,

$$\theta_K p_K^N \leq \theta_{K+1}p_{K+1}^N + v_K(\theta_K - \theta_{K+1}) \Rightarrow$$

$$\theta_K p_K^N \leq 0 + v_K(\theta_K - 0) \Rightarrow$$

$$\theta_K p_K^N \leq v_K\theta_K \Rightarrow$$

$$p_K^N \leq v_K \ (3)$$

So using the relations (2) and (3) we can conclude,

$$p_K^N \leq p_K^{EF}$$

With this as a base of the recursion it is easy to see that $p_i^N \leq p_i^{EF}$ for every $i$. So we have that $maxR^N \leq maxR^{EF}$ and using (1) we can conclude that.

$$maxR^N = maxR^{EF}$$

**Part 2:** Minimum PNE revenue is smaller than minimum EFE revenue

We will show this using the example of Proposition 2.3.1 and computing the $minR^{EF}$ and the $R^N$ of a random PNE set of bids. Lets remind the properties of the example:

| $Position$ | $Values$ | $CTRs$ |
|---|---|---|
| 1 | 10 | 4 |
| 2 | 7 | 3 |
| 3 | 6 | 2 |
| 4 | 3 | |

In order to compute the $minR^{EF}$ we need to take into consideration the lower bound of each price $p_i^L$, given by the relation $\theta_i p_i^L = \sum_{j \geq i+1} v_j(\theta_{j-1} - \theta_j)$. So we have that,

$$\begin{aligned} minR^{EF} &= \textstyle\sum_{i=1}^4 p_i^L \theta_i \\ &= p_1^L\theta_1 + p_2^L\theta_2 + p_3^L\theta_3 + 0 \\ &= v_2(\theta_1 - \theta_2) + 2v_3(\theta_2 - \theta_3) + 3v_4\theta_3 \\ &= 7 + 12 + 18 \\ &= 37 \end{aligned}$$

Now for the random PNE revenue we will use the bids and the allocation of the same example as we already know that they form a PNE. So we have the following matrix,

| $Position$ | $Values$ | $Bids$ | $Prices$ | $CTRs$ |
|---|---|---|---|---|
| 1 | 7 | 5 | 4 | 4 |
| 2 | 10 | 4 | 3 | 3 |
| 3 | 6 | 3 | 2 | 2 |
| 4 | 3 | 2 | 0 | 0 |

And the revenue in that case is,

$$R^N = p_1^L\theta_1 + p_2^L\theta_2 + p_3^L\theta_3 + 0 = 16 + 9 + 4 = 29$$

So combining the two results and since this is a random PNE revenue we can conclude that,

$$minR^{EF} > R^N \geq minR^N$$

□

# Chapter 3

# Budgeted Second-Price Ad Auction

In this chapter we present a generalized second-price ad auction model which contains the notion of budgets. The whole presentation is based on [2]. We give a basic description of the model and analyze its properties, with several notices and additional proofs from our side (section 3.3: alternative proof of proposition 3.3.1, section 3.4: introduction of proposition 3.4.1 and proof, section 3.5.1: extended proof of proposition 3.5.1.2). In section 3.6 we describe several problems (regarding the pure nash equilibrium-existence setting of proofs) that occur in the original work and introduce additional results considering the critical bid notion under the assumption of non-divisible items.

## 3.1 Introduction to the new concepts

As we saw in the earlier chapters, players that participate in an auction have a private valuation which represents how they value the item sold by the auctioneer. However in real auctions a potential buyer always has a budget which depends on his economic prosperity. This budget represents how much money a buyer can spent in the auction.

The setting that we will present introduces the notion of budget. The auctioneer sells multiple identical items (such as impressions in an ad auction) and each player is interested in getting more than one items. The true budget $\hat{B}_i$ of a player $i$, limits the number of the items he can get and in addition with his bid, it is a part of his strategy i.e. $s_i(b_i, B_i)$ (he declares to the auctioneer both a bid and a budget). Lets proceed to a more formal description of the model.

## 3.2 The Model

We have a set of $N$ players and $K$ identical divisible items. Each player $i$ has two private values: his budget $\hat{B}_i$ and his valuation for a single item $v_i$. His utility $u_i$ depends on the number of the items he received as well as the price $p_i$ he pays each item. We can interpret $x_i v_i$ as his profit for getting $x_i$ number of items and $x_i p_i$ as his total payment to the auctioneer. So we can now formally define his utility as:

$$u_i = \begin{cases} x_i(v_i - p_i) & \text{if } x_i p_i \leq \hat{B}_i \\ -\infty & \text{if } x_i p_i > \hat{B}_i \end{cases}$$

Notice that $x_i p_i \leq \hat{B}_i$ practically means that player $i$ does not exceed his budget (he does not get more items than he can afford at this price), while he exceeds his budget when $x_i p_i > \hat{B}_i$.

The auction is formed as follows: The auctioneer sets a minimum price $p_{min}$ which is known to the players. Each player $i \in N$ submits two values, his bid $b_i$ and his budget $B_i$ (as we already mentioned, this time each strategy $s_i$ contains not only the bid of player $i$ but also an additional amount, his budget). The auctioneer ranks the players in decreasing order of bids (he sets the one with highest bid first, the one with the second highest bid second etc) and renames them if necessary so that $b_1 \geq b_2 \geq ... \geq b_N$. As the allocation begins, player 1 receives items at price $p_1 = max\{b_2, p_{min}\}$ until he runs out of budget or items, i.e. $x_1 = min\{K, B_1/p_1\}$. Then if there are still items for sale, we get down to player 2 who receives items at price $p_2 = max\{b_3, p_{min}\}$

until his budget or the items run out, so $x_2 = min\{K - x_1, B_2/p_2\}$. Therefore we can say in general that for every player $i$,

**Case 1:** If there are still items left

*The items he receives:* $x_i = min\{K - \sum_{j=1}^{i-1} x_j, B_i/p_i\}$
*The price per item he pays:* $p_i = max\{b_{i+1}, p_{min}\}$

**Case 2:** If there are not items left

*The items he receives:* $x_i = 0$
*The price per item he pays:* $p_i = 0$

The auction stops either when all the items are sold or when all the players exhaust their budget. So we can conclude that the model is a variation of the GSP auction where the input is a vector of bids $\vec{b} = (b_1, ..., b_N)$ and a vector of budgets $\vec{B} = (B_1, ..., B_N)$ while the output is an allocation $\vec{x} = (x_1, ..., x_N)$ such that $\sum_{i \in N} x_i \leq K$ and a vector of prices $\vec{p} = (p_1, ..., p_N)$ such that for every $i = 1, ..., N$, $p_i \in [p_{min}, b_i]$.

Finally we have to mention the basic assumptions of the model:

1. Items are divisible goods and prices are continuous.

2. If there are identical bids the auctioneer ranks the players by lexicographic order i.e. he will first sell items to the player with the lowest original index (this is a very important assumption as we will see later on).

3. Players always bid above the minimum price, $p_{min}$ (set by the auctioneer) i.e. $b_i \geq p_{min}$.

4. For the most part there is the assumption that the bidding is conservative (no player bids above his value) i.e. $b_i \leq v_i$.

Lets now proceed to some definitions regarding the categories-classes, each player can be included considering that we have the outcome of the auction:

**Definition 3.2.1** A player is called *Border* if he is the lowest ranked player who gets a positive allocation i.e. if $h$ is a border player then $h = max\{i : x_i > 0\}$.

**Definition 3.2.2** A player $i$ is called *Winner* if $i < h$ i.e. he is ranked above the border. For winner players we have that $x_i = B_i/p_i$ which means that they exhaust their budgets.

**Definition 3.2.3** A player $i$ is called *Loser* if $i > h$ i.e. he is ranked bellow the border. For loser players we have that $x_i = 0$.

Notice here that according to Definition 3.2.1 a border player is the only player with positive allocation, $x_h > 0$, who may not exhaust his budget (he gets what is left).

## 3.3 Properties of the Model

As we saw in chapter 1, telling the truth (bidding your true value) is not a dominant strategy at the GSP auction under the ad position setting. Now that we have introduced the budget notion it is time to check whether this holds in our current model in this new setting. However as we mentioned before a strategy, $s_i(b_i, B_i)$ in an auction with budgets contains two amounts: the bid and the budget that you declare. So in order to see if telling the truth (in general) is a dominant strategy in the budgeted second-price ad auction, we have to check what happens in two cases: telling the truth about your value i.e. $b_i = v_i$ and telling the truth about your budget i.e. $B_i = \hat{B}_i$.

**Proposition 3.3.1** Bidding your true value is not a dominant strategy at the budgeted second-price ad auction.

*Proof.* We can show this with an example. Consider that we have $K = 7$ items and $N = 3$ players with the following properties:

| $Players$ | $Values$ | $Budgets$ |
|---|---|---|
| $A$ | 5 | 10 |
| $B$ | 4 | 8 |
| $C$ | 2 | 3 |

where $p_{min} = 0$. We suppose that players $B$ and $C$ bid their true values and budgets and we will examine player's $A$ utility when he bids his true value and budget and when he chooses to deviate, bidding his true budget but not his true value.

**Case 1:** Player $A$ bids his true value

When all players say the truth, the allocation is formed as follows:

| *Players* | *Values* | *Budgets* | *Bids* | *Budgets bid* | *Prices* | *Allocation* |
|---|---|---|---|---|---|---|
| $A$ | 5 | 10 | 5 | 10 | 4 | 2 |
| $B$ | 4 | 8 | 4 | 8 | 2 | 4 |
| $C$ | 2 | 3 | 2 | 3 | 0 | 1 |

We mention here that players $A$ and $B$ are winners, player $C$ is a border while there are no losers. Player's $A$ utility is,

$u_A = x_A(v_A - p_A) = 2(5 - 4) = 2$

**Case 2:** Player $A$ bids something different from his value

Suppose that player $A$ chooses to deviate and bids something lower than his value, say $b_A = 3$. The allocation is formed as follows:

| *Players* | *Values* | *Budgets* | *Bids* | *Budgets bid* | *Prices* | *Allocation* |
|---|---|---|---|---|---|---|
| $B$ | 4 | 8 | 4 | 8 | 3 | 2 |
| $A$ | 5 | 10 | 3 | 10 | 2 | 5 |
| $C$ | 2 | 3 | 2 | 3 | 0 | 0 |

We mention here that this time player $B$ is a winner, $A$ is a border and player $C$ is a loser. Player's $A$ utility is,

$u_A = x_A(v_A - p_A) = 5(5 - 2) = 15$

It is clear that player $A$ has a bigger utility when he is not telling the truth about his value so we can conclude that bidding your true value is not a dominant strategy. □

**Proposition 3.3.2** Bidding your true budget is a dominant strategy at the budgeted second-price ad auction.

*Proof.* The key point here to notice is that the budget you submit can affect the number of items you receive but does not change your ranking order (the ranking is based on the bids). Since your ranking does not change no matter what budget you declare, we have that the price you pay for each item also does not change. So we can say that when you are a loser you remain a loser independently of the budget you submit (since your rank does not change) and when you are a winner or a border, in your utility relation $u_i = x_i(v_i - p_i)$, different choices of budgets give different allocations $x_i$, while the rest amounts (the price $p_i$ you pay for instance) remain intact. Thus in order to check the utility of a player in different choices of budget bid, we only need to check the allocations $x_i$ in each case (specifically in the winner and border situation since as a loser your utility is zero no matter what budget you submit).

**Case 1:** $B_i < \hat{B}_i$

Suppose that player $i$ submits a budget $B_i$ which is smaller than his true one $\hat{B}_i$. His allocations and utilities are $x_i = min\{K - \sum_{j=1}^{i-1} x_j, B_i/p_i\}$, $u_i$ and $\hat{x}_i = min\{K - \sum_{j=1}^{i-1} \hat{x}_j, \hat{B}_i/p_i\}$, $\hat{u}_i$ respectively. The allocations of the higher rank players are not effected by the budget reported by player $i$, so $x_j = \hat{x}_j$ for all $j < i$, thus $K - \sum_{j=1}^{i-1} x_j = K - \sum_{j=1}^{i-1} \hat{x}_j$. Since $B_i/p_i < \hat{B}_i/p_i$ we can conclude that $x_i \leq \hat{x}_i$ which implies that $u_i \leq \hat{u}_i$.

**Case 2:** $B_i > \hat{B}_i$

Suppose now that player $i$ submits a budget $B_i$ which is bigger than his true one $\hat{B}_i$. Notice that if $i$ is a loser or a border then $x_i = \hat{x}_i$ since he either gets no items or he gets the items left (so he can not exhaust his real budget regardless of the budget he reports). If he is a winner on the other hand we have either that $x_i = \hat{x}_i$ (if $B_i$ does not give him any more items) or $x_i > \hat{x}_i$ (if $B_i$ gives him more items than his true budget). In the first case the utilities are equal, $u_i = \hat{u}_i$, while in the second case,

$$\hat{x}_i = \hat{B}_i/p_i < x_i \leq B_i/p_i \Rightarrow$$

$$x_i > \hat{x}_i \text{ and } x_i p_i > \hat{B}_i$$

Although $x_i > \hat{x}_i$ which means that player $i$ has a bigger allocation, we have from the definition of the utility that in that case $u_i = -\infty$ since $x_i p_i > \hat{B}_i$. We can conclude that $u_i \leq \hat{u}_i$.

So in any case we can say that bidding your true budget is a dominant strategy at the budgeted second-price ad auction. □

## 3.4 The Market Equilibrium Price

**Definition 3.4.1** The *demand* of a player $i$ at price $p$ is a point or an interval $D_i(p)$ which is defined as follows,

$$D_i(p) = \begin{cases} B_i/p & \text{if } v_i > p \\ 0 & \text{if } v_i < p \\ [0, B_i/p] & \text{if } v_i = p \end{cases}$$

The demand shows how much items of a product, a player wants to buy provided that the product's price is $p$. For example, if he values the product more than the price it is sold he will try to get as much items he can, exhausting his budget, if he values it less he will not buy anything and if his value equals the price, he does not really care since he is not making any profit (so he can go from buying nothing to exhausting his budget).

**Definition 3.4.2** The *aggregated demand* $D(p)$, is a point or an interval that represents the sum of the demands of all $N$ players at price $p$. More formally,

$$D(p) = \sum_{i \in N} D_i(p)$$

Notice that the "interval or point" depends on whether players that their values are equal with the price, exist or not respectively.

**Definition 3.4.3** We call a price $p_{eq}$, *the Market Equilibrium Price*, if at this price the aggregated demand of the players equals or contains the total number of the items that exist. More formally, if there are $K$ items then,

$$K = D(p_{eq}) \text{ or } K \in D(p_{eq})$$

Notice that $p_{eq}$ is unique since the correspondence $D(p)$ is strictly decreasing in $p$. Finally we can say that for $S = \{i : v_i > p_{eq}\}$ which is the set of players with demand $D_i(p_{eq}) = B_i/p_{eq}$ and for $Z = \{i : v_i = p_{eq}\}$ which is the set of players with demand $D_i(p_{eq}) = [0, B_i/p_{eq}]$, we have that $p_{eq} \in [\sum_{i \in S} B_i/K, \sum_{i \in S \cup Z} B_i/K]$

**Proposition 3.4.1** For some random prices $p_1$, $p_2$ we have that if $p_1 < p_2$ then $min(D(p_1)) > max(D(p_2))$.

*Proof.* As we said earlier the correspondence $D(p)$ represents an interval or a point (in the case of the point representation the min, max amounts are equal with the value of the point) and obviously it is not a function. We define the sets,

- $S_p = \{i : v_i > p\}$
- $Z_p = \{i : v_i = p\}$
- $L_p = \{i : v_i < p\}$

Lets take a look on how we can compute the min and the max amounts:

From definition 3.4.2 we know that,

$$D(p) = \sum_{i \in N} D_i(p) = \sum_{i \in S_p} D_i(p) + \sum_{i \in Z_p} D_i(p) + \sum_{i \in L_p} D_i(p)$$

We also know from definition 3.4.1 that,

- $\sum_{i \in S_p} D_i(p) = \sum_{i \in S_p} B_i/p$
- $\sum_{i \in Z_p} D_i(p) = \sum_{i \in Z_p} x_i$, where $x_i \in [0, B_{i \in Z_p}/p]$
- $\sum_{i \in L_p} D_i(p) = 0$

So it is easy to see that in order to compute the min, max points of $D(p)$ we only need to consider the amount $\sum_{i \in Z_p} D_i(p)$. It is obvious that,

$$max(\sum_{i\in Z_p} D_i(p)) = \sum_{i\in Z_p} B_i/p \text{ (setting every } x_i = B_i/p)$$

and

$$min(\sum_{i\in Z_p} D_i(p)) = 0 \text{ (setting every } x_i = 0)$$

So according to that analysis and back to our proof we have that,

$$\begin{aligned} min(D(p_1)) &= \sum_{i\in N} D_i(p_1) \\ &= \sum_{i\in S_{p_1}} D_i(p_1) + 0 + 0 \\ &= \sum_{i\in S_{p_1}} B_i/p_1 \end{aligned}$$

Now it is easy to notice that since $p_2 > p_1$, we are sure that $L_{p_1} \cup Z_{p_1} \subseteq L_{p_2}$ by definition. So we have that,

$$L_{p_1} \cup Z_{p_1} \subseteq L_{p_2} \Rightarrow$$

$$(L_{p_2})^c \subseteq (L_{p_1} \cup Z_{p_1})^c \Rightarrow$$

$$S_{p_2} \cup Z_{p_2} \subseteq S_{p_1}$$

We can now proceed in computing $max(D(p_2))$,

$$\begin{aligned} max(D(p_2)) &= \sum_{i\in N} D_i(p_2) \\ &= \sum_{i\in S_{p_2}} D_i(p_1) + \sum_{i\in Z_{p_2}} D_i(p_2) + 0 \\ &\leq \sum_{i\in S_{p_1}} B_i/p_2 \\ &< \sum_{i\in S_{p_1}} B_i/p_1 \\ &= min(D(p_1)) \end{aligned}$$

So we can conclude that $max(D(p_2)) < min(D(p_1))$. Notice that this proof (in a more simplified way), also holds when $D(p)s$ are points instead of intervals. □

The market equilibrium price is a very important notion as we will see later on since it can be seen as the bound of the price the winner players pay at a PNE state.

## 3.5 Pure Nash Equilibrium

Now that we have described the model as well as its properties we can proceed in analyzing properties and introducing some very important notions regarding the pure nash equilibrium set of bids and budgets. As we saw earlier (proposition 3.3.2), submitting your true budget is a dominant strategy at the budgeted second-price ad auction setting so from now on we assume that all players declare their true budget. Thus, once more the search of a PNE leads to a search of appropriate bids. Typically we say that we are in a PNE if for every player $i$, the utility he has when he bids $b_i$ is at least as much as the utility he would have with a different bid $b_i'$ considering that the rest players do not change their strategies.

### 3.5.1 Pure Nash Equilibrium Properties

The following two propositions analyze the properties of the PNE and give a clear view on how the bids and the prices are formed at such a state.

**Proposition 3.5.1.1** In any PNE, all winner players pay the same price $p$, the border player pays a price $p' \leq p$ and any loser player $j$ (if exists) has a value $v_j \leq p$.

*Proof.* Initially, notice that in order to have the same price $p$ for all the winner players, the winner players as well as the border player have to bid the same value (except from the top rank winner player who can bid higher). The proposition holds trivially if we have only one winner player. Suppose now that there is a PNE with at least two winner players paying different prices. The first rank player pays $p_1$ and let player $j$ be the highest ranked winner player who pays $p_j < p_1$ (notice that $p_j$ can not be bigger than $p_1$ since player $j$ is ranked lower than player 1). Since players 1 to $j$ are all winners, then by definition any player $i \leq j$ has an allocation $x_i = B_i/p_i$ and we can also say that,

$$\sum_{i=1}^{j} x_i \leq K \Rightarrow$$

$$x_1 \leq K - \sum_{i=2}^{j} x_i \text{ (1)}$$

If player 1 chooses to move down to $j's$ position by bidding $b_j - \epsilon$, he is allocated $x_1' = min\{K - \sum_{i=2}^{j} x_i, B_1/p_j\}$. However we know by assumption that $p_1 > p_j$ so $B_1/p_j > B_1/p_1$ and additionally we know that $x_1 \leq K - \sum_{i=2}^{j} x_i$ from (1) thus we can conclude that $x'_1 \geq x_1$. So we have for his utility $u'_1$ at $j's$ position that,

$$u'_1 = x'_1(v_1 - p_j) \geq x_1(v_1 - p_j) > x_1(v_1 - p_j) = u_1 \Rightarrow$$

$$u'_1 > u_1$$

thus this is not a PNE. Now lets take a look on the rest players. The border player is ranked after all winners so he pays a price $p' \leq p$ for each item. The loser players receive no items so they have zero utility. If there was a loser player $i$ with $v_i > p$ then he could bid $p + \epsilon \leq v_i$ becoming a winner player with positive utility, something that contradicts to the PNE definition. Thus at a PNE all loser players must have value at most $p$.

□

**Proposition 3.5.1.2** The price $p$ which all the winner players pay at a PNE is at most the market equilibrium price i.e. $p \leq p_{eq}$.

*Proof.* Suppose that there is a PNE where all the winner players pay an amount $p > p_{eq}$. Since this is a PNE we have that the utility $u_i$ of a winner player $i$ paying price $p > p_{eq}$ is at least as much as his utility $u'_i$ if he was a border player and paid price $p' \leq p$. Thus we have,

$$u_i \geq u'_i \Rightarrow$$

$$(B_i/p)(v_i - p) \geq (K - \textstyle\sum_{j \in S - \{i\}}(B_j/p)(v_j - p') \ (2)$$

where $S = \{j : v_j \geq p\}$. Since $(v_i - p) \leq (v_i - p')$, in order for inequality (2) to hold we have,

$$B_i/p \geq K - \textstyle\sum_{j \in S - \{i\}}(B_j/p) \Rightarrow$$

$$\textstyle\sum_{j \in S}(B_j/p) \geq K \ (3)$$

and from the definition of aggregated demand we also have $\sum_{j \in S}(B_j/p) \in D(p)$ (4). From relations (3) and (4) we have that,

$$K \leq max(D(p))$$

However by assumption we know that $p > p_{eq}$ and according to definition 3.4.3 and proposition 3.4.1 we have that,

$$K \leq max(D(p)) < min(D(p_{eq})) \leq K$$

so finally we have,

$$K < K$$

which is a contradiction. So we can conclude that the price $p$ that all winner players pay at a PNE is at most the market equilibrium price. □

### 3.5.2 Critical Bid

The critical bid of a player is a notion which tries to capture the point where the player is indifferent between being ranked first and being ranked last. Basically we can say that if all players bid the same value, this value is a critical bid if a player has the same utility both at top and the bottom rank. Each player has potentially a different critical bid. As we will see in the process, the critical bid is a very important notion mainly because it makes clear in which position each player prefers to be, something that simplifies the quest of searching the PNE state. But let us give a more formal definition.

**Definition 3.5.2.1** The critical bid, $c_j$, of a player $j$ is defined as follows: Suppose that all players submit the same bid $x \in [p_{min}, v_j]$. We will examine the utilities of player $j$ as functions of $x$ when he is ranked first and when he is ranked last.

**Player $j$ is first in rank**

Since all players submit the same bid, if player $j$ is ranked first he pays $x$ and his utility is defined as[1]

$$f_j(x) = \begin{cases} K(v_j - x) & \text{if } p_{min} \leq x < \frac{B_j}{K} \quad \text{(Player } j \text{ as a border)} \\ \frac{B_j}{x}(v_j - x) & \text{if } \frac{B_j}{K} \leq x \leq v_j \quad \text{(Player } j \text{ as a winner)} \end{cases}$$

[1]Notice that if items were not divisible, $\frac{B_j}{x}$ must be a positive integer since it represents the number of the items that a winner player $j$ gets when the price is $x$. Thus if for some budget and price the quantity $\frac{B_j}{x}$ is a decimal number, we have to round to the nearest lowest positive integer. For example, if $B_j = 3$ and $x = 2$ then $\frac{B_j}{x} = \frac{3}{2} = 1$.

### Player $j$ is last in rank

Since all players submit the same bid, if player $j$ is ranked last he pays either $p_{min}$ or zero and his utility is defined as[2],

$$g_j(x) = \begin{cases} 0 & \text{if } p_{min} \leq x < \frac{\sum_{i \neq j} B_i}{K} & \text{(Player } j \text{ as a loser)} \\ (K - \frac{\sum_{i \neq j} B_i}{x})(v_j - p_{min}) & \text{if } \frac{\sum_{i \neq j} B_i}{K} \leq x < \frac{\sum_{i \neq j} B_i}{K - B_j/p_{min}} & \text{(Player } j \text{ as a border)} \\ \frac{B_j}{p_{min}}(v_j - p_{min}) & \text{if } \frac{\sum_{i \neq j} B_i}{K - B_j/p_{min}} \leq x \leq v_j & \text{(Player } j \text{ as a winner)} \end{cases}$$

It is easy to verify the following properties:

- Both functions are continuous in the range $[p_{min}, v_j]$.
- $f_j$ is strictly decreasing in $x$ while $g_j$ is weakly increasing in $x$
- $f_j(p_{min}) \geq g_j(p_{min})$
- $g_j(v_j) \geq f_j(v_j) = 0$

So we can conclude that functions $f_j$ and $g_j$ must intersect in a unique point in the given range. This point is defined as the critical bid, $c_j$, of player $j$.

Now that the definition is complete we can notify various things as well as several results that come directly from the definition. For instance:

1. For every player $i$ we have the $c_i = [p_{min}, v_i]$. So the critical bid of every player is a quantity between the minimum price set by the auctioneer and his private value.

2. It is impossible for the auctioneer to compute the critical bid of any player $i$ since its computation demands the knowledge of the private value $v_i$.

3. Critical bid of a player $i$ can be seen as a function of the minimum price set by the auctioneer, $p_{min}$ and the number of the players, $N$ i.e. $c_i(N, p_{min})$.

[2]In practice, the amount $\frac{B_j}{p_{min}}$ represents the number of the items that a winner player $j$ gets when the price is $p_{min}$. Thus for a $p_{min} = 0$ we have that $\frac{B_j}{p_{min}} = \frac{B_j}{0} = K$.

4. If $\vec{b} = (x, ..., x)$ then for $x < c_i(N, p_{min})$ player $i$ prefers to be ranked first and for $x > c_i(N, p_{min})$ player $i$ prefers to be ranked last. This comes directly from the definition of the two functions and their monotonicity since for $x < c_i(N, p_{min})$ we have $g_j(x) < f_j(x)$ and for $x > c_i(N, p_{min})$ we have $g_j(x) > f_j(x)$.

### 3.5.3 Incentives

**Proposition 3.5.3.1** For a bid vector $\vec{b} = (b_1, ..., b_N)$ the first in rank player or any winner player $j \in N$, can not improve his utility by bidding higher i.e. $b'_j > b_j$.

*Proof.* If player $j$ is the first in rank player, he can not improve his utility by bidding higher since the price he pays and his allocation do not change. If player $i$ is a winner player in general then his utility is by definition $u_j = \frac{B_j}{p_j}(v_j - p_j)$. By increasing his bid he will get either the same utility (if he does not overbid any player above him, the price he pays and the allocation he gets remain the same) or a smaller utility (if he overbids any player above him, the price $p'_j$ he pays is bigger and thus his allocation $x_j = \frac{B_j}{p'_j}$ is smaller). So in any case his utility does not improve when he bids higher than before. □

**Proposition 3.5.3.2** For a bid vector $\vec{b} = (b_1, ..., b_N)$ the last in rank player or any loser player $j \in N$, can not improve his utility by bidding lower i.e. $b'_j < b_j$.

*Proof.* If player $j$ is the last in rank player he can be either a border player or a loser player. If he is a border, by decreasing his bid he decreases the price the player above him pays. This will may increase the allocation of that player, thus decrease $j's$ allocation (player $j$ is a border and gets the remaining items). Notice that in that case player $j's$ price remains $p_{min}$ (since he is the last in rank) and with a smaller allocation he gets a smaller utility. If he is a loser (as the last in rank or a loser in general) he has a zero utility. Trivially by bidding less he does not gain something since he remains a loser with zero utility. □

**Proposition 3.5.3.3** If every player $i \in N$ bids $c_j$ (the critical bid of player $j$), then $j$ can not improve his utility by changing his bid.

*Proof.* By the definition of critical bid, since all players bid $c_j$ player $j$ is indifferent (has the same utility) between being ranked first and being ranked last. From proposition 3.5.3.1, if $j$ is ranked first or is a winner player in general he does not gain something from bidding higher and from proposition 3.5.3.2, if $j$ is ranked last or is a loser player in general he does not gain something from bidding lower. □

## 3.6 Our Results

### 3.6.1 PNE Existence

The main goal of this section is to show that the proof of one of the main results in [2], the existence of a PNE, is not quite correct. We will present the Theorem, describe the basic idea behind its proof and introduce our notices and results on the whole topic. For a more complete view on the original proof of the Theorem you can look at [2]. Before we proceed we will remind you some assumptions and notices that introduced earlier on:

1. If there are identical bids the auctioneer ranks the players by lexicographic order i.e. he will first sell items to the player with the lower original index.

2. It is impossible for the auctioneer to compute the critical bid of any player $i$ since its computation demands the knowledge of the private value $v_i$.

**Original Theorem:** There exists a PNE for any number of players, where players submit their true budget ($\hat{B}_i = B_i$) and bid at most their value ($b_i \leq v_i$). [3]

The proof is made by induction on the number of players. More specifically:

[3]Theorem 4.13 in the original work.

- *Induction Basis:* There exists a PNE when there are only two players that submit their true budget and bid at most their value.

- *Induction Hypothesis:* We assume that there exists a PNE when there are $N$ players that submit their true budget and bid at most their value.

- We prove that there exists a PNE when there are $N+1$ players that submit their true budget and bid at most their value.

We claim that the induction basis (the PNE existence when there are only two players) does not, in general, hold. In order to prove that, we will present the original proposition of [2] which claims the existence of a PNE at a two players game and show that there are cases where the proposition is not, in general, correct. Specifically:

**Original Proposition:** Assume that we have two players with $c_2 \leq c_1$. Then any bids $b_1 = b_2 \in [c_2, min\{v_2, c_1\}]$ are a PNE and those are the only PNEs where players submit their true budget. [4]

The key point here to notice is that since the bids are equal, we can use the critical bid notion in order to understand in what order each player prefers to be (in which position he gets more utility). Back to the definition of the critical bid we mentioned that if $\vec{b} = (x, ..., x)$ then for $x < c_i(N, p_{min})$ player $i$ prefers to be ranked first (has a bigger utility at the first position) and for $x > c_i(N, p_{min})$ player $i$ prefers to be ranked last (has a bigger utility at the last position). Thus in our case we have that player 1 weakly prefers to be first in rank since $c_1 \geq b_1 = b_2$, so if he is placed first he can not improve his utility by bidding less and player 2 weakly prefers to be last in rank since $c_2 \leq b_1 = b_2$, so if he is placed last he can not improve his utility by bidding more. We also know from proposition 3.5.3.1 that if player 1 is placed first then he can not improve his utility by bidding more and from proposition 3.5.3.2 that if player 2 is placed last he can not improve his utility by bidding less. Therefore if player 1 is placed first an player 2 is placed second then any bids $b_1 = b_2 \in [c_2, min\{v_2, c_1\}]$ lead to PNE. However this specific ranking is not always possible.

[4] Claim 4.7 in the original work.

**Proposition 3.6.1.1** There are cases where we have two players with different critical bids and $[c_2, min\{v_2, c_1\}]$ is an interval and not a point.

*Proof.* We will prove this with an example. Suppose that we have $K = 10$ items, $p_{min} = 0$ and two players with the following properties:

| $Players$ | $Values$ | $Budgets$ |
|---|---|---|
| 1 | 3 | 9 |
| 2 | 1 | 5 |

Lets proceed in finding the critical bids of each player[5].

**Critical bid of player 1**

$$f_1(x) = \begin{cases} 10(3-x) & \text{if } 0 \leq x < \frac{9}{10} & \text{(Player 1 as a border)} \\ \frac{9}{x}(3-x) & \text{if } \frac{9}{10} \leq x \leq 3 & \text{(Player 1 as a winner)} \end{cases}$$

$$g_1(x) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{5}{10} & \text{(Player 1 as a loser)} \\ 3(10-\frac{5}{x}) & \text{if } \frac{5}{10} \leq x < \infty & \text{(Player 1 as a border)} \end{cases}$$

In order to find the intersection point of the two functions, we have to solve the system of the equations and find an amount $x$ which is within the restriction bounds. It is easy to verify that in our example this happens only when,

$$\frac{9}{x}(3-x) = 3(10-\frac{5}{x}) \Rightarrow$$

$$\frac{3}{x}(3-x) = 10-\frac{5}{x} \Rightarrow$$

$$\frac{9}{x} - 3 = 10-\frac{5}{x} \Rightarrow$$

$$\frac{14}{x} = 13 \Rightarrow$$

[5]Notice that is impossible for any player to be a winner when he is ranked last if $p_{min} = 0$, since he can not exhaust his budget.

$$x = \frac{14}{13}$$

Notice that $\frac{9}{10} < \frac{14}{13} < 3$ as well as $\frac{5}{10} < \frac{14}{13} < \infty$.

**Critical bid of player 2**

$$f_2(x) = \begin{cases} 10(1-x) & \text{if } 0 \leq x < \frac{5}{10} \quad \text{(Player 2 as a border)} \\ \frac{5}{x}(1-x) & \text{if } \frac{5}{10} \leq x \leq 1 \quad \text{(Player 2 as a winner)} \end{cases}$$

$$g_2(x) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{9}{10} \quad \text{(Player 2 as a loser)} \\ 1(10 - \frac{9}{x}) & \text{if } \frac{9}{10} \leq x < \infty \quad \text{(Player 2 as a border)} \end{cases}$$

Under the same logic we have,

$$\frac{5}{x}(1-x) = 1(10 - \frac{9}{x}) \Rightarrow$$

$$\frac{5}{x} - 5 = 10 - \frac{9}{x} \Rightarrow$$

$$\frac{14}{x} = 15 \Rightarrow$$

$$x = \frac{14}{15}$$

Notice that $\frac{5}{10} < \frac{14}{15} < 1$ as well as $\frac{9}{10} < \frac{14}{15} < \infty$. So we can conclude that $c_1 = \frac{14}{13}$ and $c_2 = \frac{14}{15}$ thus,

$$[c_2, min\{v_2, c_1\}] =$$

$$[\tfrac{14}{15}, min\{1, \tfrac{14}{13}] =$$

$$[\tfrac{14}{15}, 1]$$

which is an interval. □

**Theorem 3.6.1.1** There are cases where we have two players with $c_2 \leq c_1$ and any bids $b_1 = b_2 \in [c_2, min\{v_2, c_1\}]$ are not a PNE.

*Proof.* Suppose that we have two players, $A$ and $B$ with $c_A < c_B$ so that $[c_A, min\{v_A, c_B\}]$ is an interval and not a point. They bid $b_A = b_B \in [c_A, min\{v_A, c_B\}]$. Since their bids are equal the auctioneer has to rank them lexicographically, so we have that player $A$ is placed first and player $B$ is placed second.

- If $b_A = b_B = c_A$ then we know that player $B$ has a bigger utility at the first position since $b_A = b_B < c_B$.

- If $b_A = b_B = min\{v_A, c_B\}$ then we know that player $A$ has a bigger utility at the second position since $b_A = b_B = min\{v_A, c_B\} > c_A$.

- $c_A < b_A = b_B < min\{v_A, c_B\}$ then we know that player $A$ has a bigger utility at the second position since $c_A < b_A = b_B$ and player $B$ has a bigger utility at the first position since $b_A = b_B < c_B$

So in any case this is not a PNE. The problem is obviously the lexicographic positioning when the bids are equal. If we could rank the players according to their critical bids (i.e. the one with the highest critical bid is placed first, the one with second highest critical bid is placed second etc) then the original proposition is actually correct. However it is impossible for the auctioneer to rank them in that manner since he can not compute the critical bids (he does not know the private values of each player). □

### 3.6.2 Some Additional Notes

There are several other propositions throughout [2] that have similar problems, for example,

*If the lowest critical bid is lower than the value of any agent, i.e. $c_j < v_h$ (where $j$ the player with the lowest critical bid and $h$ the player with the lowest private value), then $\vec{b} = (c_j, ..., c_j)$ is a PNE, where agent $j$ is the border player and other players are winners.*[6]

First of all, player $j$ in such a case will not be the border player if he does not have a proper name (a name that ranks him at the last position). If he is not ranked last then some other player $i$ will be the border player. However since player $j$ has the lowest critical bid then $c_i > c_j$ so player $i$ prefers to be first in rank (he has a higher utility there) and thus this is not a PNE. Notice that we can not let player $j$ to bid $c_j - \epsilon$ for a very small amount $\epsilon$ in order for him to become a border (independently of his name) since according to proposition 3.5.1.1 in a PNE all the winners and the border must bid the

[6] Claim 4.10 in the original work.

same value. A more simple way to see this is that if player $j$ bids $c_j - \epsilon$ then the winner player above him will pay $p_{j-1} = c_j - \epsilon$ which is, by a tiny amount, smaller than the price of every other winner player pays. In such a case, any of the rest winner players will want to get to $j - 1's$ position in order to pay the smaller price an thus improve their utility by a small amount (this is an alternative way to understand why the winner players must pay the same price in a PNE).

So we are led to a contradiction: The critical bid is a notion which helps to understand in which place each player prefers to be, but only when all players bid the same value. So we can use it to understand and approach the properties a of PNE state. However when players bid the same value the ordering is made lexicographically so each player's rank depends only on his name and not on his strategy, thus the critical bid becomes useless.

We conclude that in order for the PNE existence results of [2] to hold, the critical bids must follow the ordering of the names of the players (i.e. the player with the lowest original index must have the highest critical bid, the player with the second lowest original index must have the second highest critical bid etc).

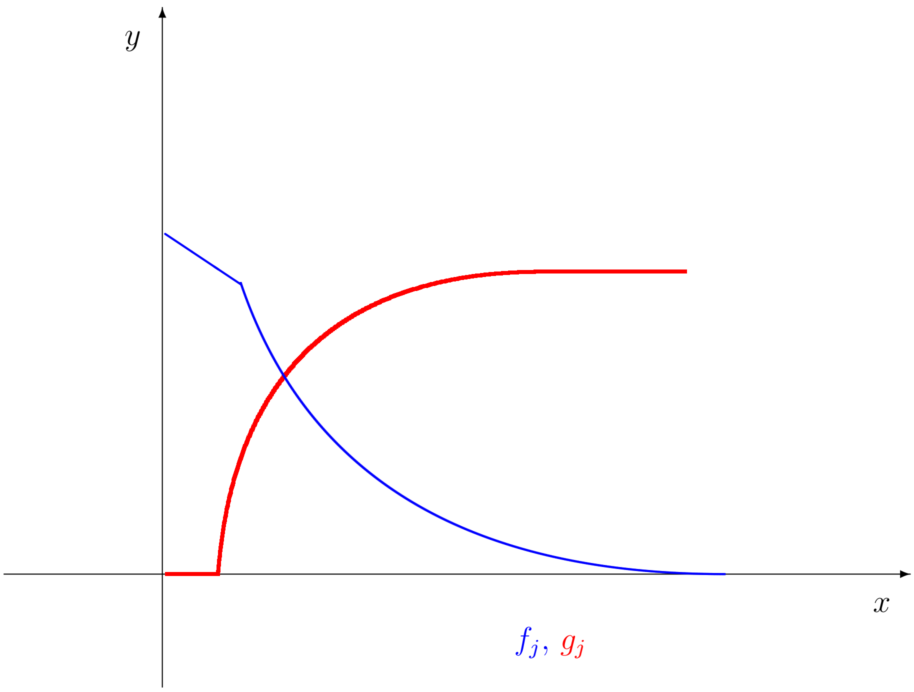
### 3.6.3 Critical Bid under Non-Divisible Items

Earlier on, we defined the critical bid of a player $j$ as the intersection point of functions $f_j, g_j$,

$$f_j(x) = \begin{cases} K(v_j - x) & \text{if } p_{min} \leq x < \frac{B_j}{K} \quad \text{(Player } j \text{ as a border)} \\ \frac{B_j}{x}(v_j - x) & \text{if } \frac{B_j}{K} \leq x \leq v_j \quad \text{(Player } j \text{ as a winner)} \end{cases}$$

$$g_j(x) = \begin{cases} 0 & \text{if } p_{min} \leq x < \frac{\sum_{i \neq j} B_i}{K} \quad \text{(Player } j \text{ as a loser)} \\ (K - \frac{\sum_{i \neq j} B_i}{x})(v_j - p_{min}) & \text{if } \frac{\sum_{i \neq j} B_i}{K} \leq x < \frac{\sum_{i \neq j} B_i}{K - B_j/p_{min}} \quad \text{(Player } j \text{ as a border)} \\ \frac{B_j}{p_{min}}(v_j - p_{min}) & \text{if } \frac{\sum_{i \neq j} B_i}{K - B_j/p_{min}} \leq x \leq v_j \quad \text{(Player } j \text{ as a winner)} \end{cases}$$

Bellow we provide a graphical representation example:



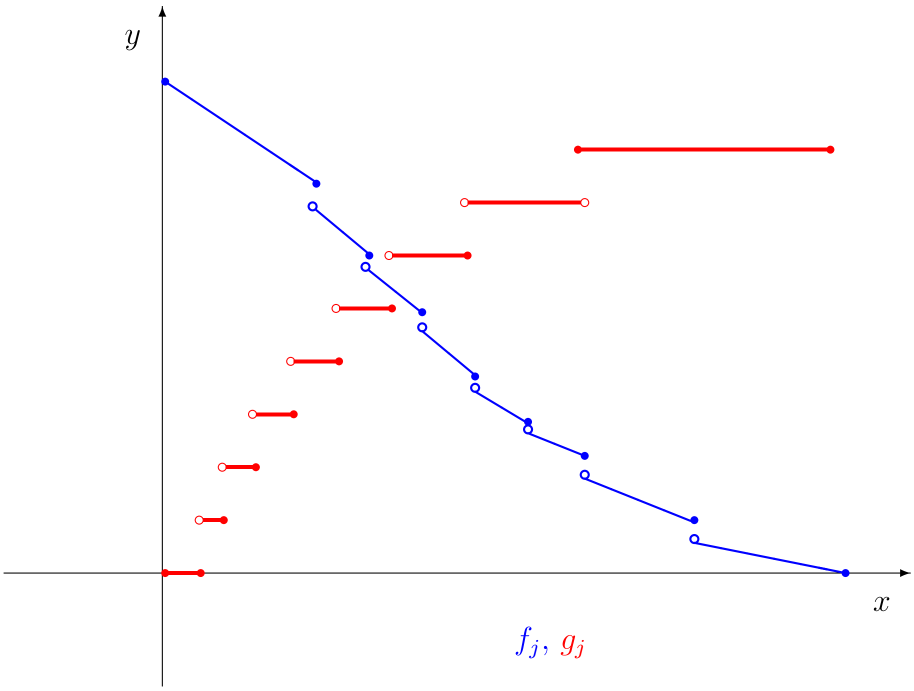

However (as we already mentioned in some of our earlier footnotes), $\frac{B_j}{x}$ represents the number of items player $j$ gets, at price $x$, when he is ranked first and is a winner player and $\frac{\sum_{i \neq j} B_i}{x}$ represents the number of items the rest players get, at price $x$, when player $j$ is ranked last and is a border. So if the items are not divisible and when a player tries to compute how much items he will get at a certain price, even if these two amounts give decimal numbers for some prices, we have to round these numbers to the nearest lowest positive integer (i.e. if $\frac{B_j}{x} = 2.8$, we say that player $j$ gets 2 items at price $x$ since if he gets 3 items he exceeds his budget). So it easy to understand that in such cases, for multiple $x's$ we have the same number of items (which implies the same $\frac{B_j}{x}$ and $\frac{\sum_{i \neq j} B_i}{x}$ for a set of $x's$) thus,

$$f_j(x) = \frac{B_j}{x}(v_j - x), \text{ if } \frac{B_j}{K} \leq x \leq v_j$$

is a set of line functions instead of a hyperbola function and

$$g_j(x) = (K - \frac{\sum_{i \neq j} B_i}{x})(v_j - p_{min}), \text{ if } \frac{\sum_{i \neq j} B_i}{K} \leq x < \frac{\sum_{i \neq j} B_i}{K - B_j/p_{min}}$$

is a set of constant functions instead of a hyperbola function. Bellow we provide a graphical representation example of how $f_j, g_j$ look like when the items are not divisible.




As we can see the functions $f_j, g_j$ are not continuous when the items are not divisible. This implies that there are cases where the critical bid of a player does not exist.

**Theorem 3.6.3.1** In the budgeted second-price ad auction with non-divisible items, there are cases where the critical bid of a player does not exist.

*Proof.* We will prove this using an example. Suppose that we have two players, $K = 4$ items, $B_1 = 3$, $B_2 = 2$, $v_1 = 2$, $v_2 = 1$ and $p_{min} = 0$. We will compute the utility functions of player 1, showing that they do not intersect at some point. We start with function $f_1$,

$$f_1(x) = \begin{cases} 4(2-x) & \text{if } 0 \leq x < \frac{3}{4} \\ \frac{3}{x}(2-x) & \text{if } \frac{3}{4} \leq x \leq 2 \end{cases}$$

Since the items are not divisible, the real form of the function is,

$$f_1(x) = \begin{cases} 4(2-x) & \text{if } 0 \leq x \leq \frac{3}{4} \\ 3(2-x) & \text{if } \frac{3}{4} < x \leq 1 \\ 2(2-x) & \text{if } 1 < x \leq \frac{3}{2} \\ 2-x & \text{if } \frac{3}{2} < x \leq 2 \end{cases}$$

We proceed with function $g_1$,

$$g_1(x) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{2} \\ (4-\frac{2}{x})2 & \text{if } \frac{1}{2} \leq x \leq 2 \end{cases}$$

Under the same logic, the real form of the function is,

$$g_1(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2 & \text{if } \frac{1}{2} < x \leq \frac{2}{3} \\ 4 & \text{if } \frac{2}{3} < x \leq 1 \\ 6 & \text{if } 1 < x \leq 2 \end{cases}$$

As you can notice, function $f_1$ consists of strictly decreasing line functions. We proceed in computing the minimum and the maximum of each line segment:

$$\text{if } x \in [0, \tfrac{3}{4}] \text{ then } f_1(x) \in [5, 8]$$

$$\text{if } x \in (\tfrac{3}{4}, 1] \text{ then } f_1(x) \in [3, 3.75)$$

$$\text{if } x \in (1, \tfrac{3}{2}] \text{ then } f_1(x) \in [1, 2)$$

$$\text{if } x \in (\tfrac{3}{2}, 2] \text{ then } f_1(x) \in [0, 0.5)$$

So it is easy to see and verify that functions $f_1, g_1$ do not intersect at some point, thus player's 1 critical bid does not exist (there is no bid that gives him the same utility at the first-last rank when all players bid the same value).

□

# Chapter 4

# GSP Ad Position Auctions under Budget Constraints

In this chapter we return to the ad position setting, describing models that introduce budgets. Our presentation is based on a work currently in progress [3]. Our goal is to observe how the GSP ad position setting behaves under the introduction of budget constraints. We present two such auction models and additionally we display and prove several of their equilibria properties.

## 4.1 Budget-Conscious Second-Price Auction

The budget-conscious second-price auction (BC-SPA) is basically a model which describes the GSP ad position auction that we saw earlier in chapter 1 but customized under the introduction of budgets.

### 4.1.1 The Model

We have $N$ players and $K$ slots ($N > K$). The slots have expected click-through rates (CTR) $\theta_1 > \theta_2 > ... > \theta_K > 0$, that depend upon their positioning. The players $i = 1, ..., N$ demand at most one slot and have private values $v_1, v_2, ..., v_N > 0$ per click. They also have publicly known budgets $B_1, B_2, ..., B_N$ capping the total payment they are willing to accept.

We assume that there is a minimum price which is always zero, $p_{min} = 0$. The auctioneer announces prices $p_i$ per click and selects $K$ players to be given a unique slot, as follows:

1. The auctioneer orders the players according to their bids and renames them if necessary so that $b_1 \geq b_2 \geq ... \geq b_N$. For players with identical bids, lexicographic ordering is followed i.e. the player with the lower original index is ranked first, the player with the second lower index is ranked second etc.

2. Each player $i$, in his turn as determined by decreasing bid, is assigned the slot with the highest CTR that is currently available (has not being taken so far) and is within his budget i.e. $\theta_{s_i} p_i \leq B_i$ if there is such one, otherwise he gets nothing. In other words, player $i$ gets the highest CTR slot $j = s_i$ which has not be taken by a player $i' < i$ and for which $p_i \theta_j \leq B_i$. A player $i$ who is assigned a slot is called a winner while the remaining players are called losers.

3. The price per click a player $i$ is required to pay in case he is a winner player is defined as follows:

$$p_i = \begin{cases} b_{i+1} & \text{if } i \neq N \\ p_{min} = 0 & \text{if } \ i = N \end{cases}$$

   In other words, if a winner player $i$ is not the player with the lowest bid (he is not the last in rank) he pays the bid of the player bellow him, otherwise if he is a winner player and has the lowest bid (the last in rank) he pays $p_{min}$ (notice that although $K < N$, it is possible for player $N$ to get a slot if for example this slot was not within the budget of the previous players). In case player $i$ is a loser, his price is defined as zero.

4. The amount $v_i \theta_{s_i}$ can be seen as the profit of a winner player $i$ who is assigned to a slot $s_i$ while the amount $p_i \theta_{s_i}$ can be seen as his total payment. If $i$ is a loser player on the other hand he has no profit and

his payment is zero. So we can define the utility of a player $i$ as,

$$u_i = \begin{cases} \theta_{s_i}(v_i - b_{i+1}) & \text{if } i \text{ is a winner and } i \neq N \\ \theta_{s_i} v_i & \text{if } i \text{ is a winner and } i = N \\ 0 & \text{if } i \text{ is a loser} \end{cases}$$

### 4.1.2 Properties

As we said earlier our main goal is to examine the existence of PNE and EFE assignments and examine their properties. Before we proceed we will give some necessary definitions.

**Definition 4.1.2.1** We say that a player $i$ can *afford* a slot $j$ if this slot is within his budget i.e. $\theta_j p_i \leq B_i$.

**Definition 4.1.2.2** We say that a player $j$ envies a winner player $i$, assigned to a slot $s_i$ if the conjunction of the following occurs:

- $p_i \leq v_j$
- $\theta_{s_i} p_i \leq B_j$
- $\theta_{s_j}(v_j - p_j) < \theta_{s_i}(v_j - p_i)$ or $0 < \theta_{s_i}(v_j - p_i)$ (depending on whether player $j$ is a winner or not respectively).

In other words a player envies someone else if can rationally afford his slot and in addition his utility in the other's player position is strictly bigger than his current one. Notice that a player does not envy someone else if one of the conditions does not hold i.e. if $\theta_{s_i} p_i > B_j$ then player $j$ does not envy player $i$.

**Definition 4.1.2.3** We say that the assignment is envy-free, if it is rational and no player who is assigned to a slot, is envied by any other player.

Notice here the *envy* notion is a little different from what we described in the section 2.2 due to the introduction of the budget constraints. But how significant are these differences? As a matter of fact, we saw at section 2.2 that bids that produce an envy-free assignment are also producing a PNE assignment under the GSP ad position setting. Does something similar hold for the excluding-budget second-price auction setting?

**Proposition 4.1.2.1** In the BC-SPA setting, bids that produce an envy-free assignment do not, in general, produce a PNE.

*Proof.* We will prove this using an example. Suppose that we have 3 players and 2 slots with the following properties:

| *Slots* | *CTR* |
|---|---|
| $s_1$ | 3 |
| $s_2$ | 2 |

| *Players* | *Values* | *Budgets* |
|---|---|---|
| $A$ | 10 | 16 |
| $B$ | 8 | 10 |
| $C$ | 6 | 7 |

Let $b_A = 6, b_B = 5, b_C = 4$. We claim that under these bids, the produced assignment is envy-free but not a PNE.

**Envy-Freedom**

| *Players* | *Values* | *Budgets* | *Bids* | *Prices* | *Slots* |
|---|---|---|---|---|---|
| $A$ | 10 | 16 | 6 | 5 | $s_1$ |
| $B$ | 8 | 10 | 5 | 4 | $s_2$ |
| $C$ | 6 | 7 | 4 | 0 | *none* |

The price assignment is obviously rational. Notice that player A gets slot 1 because it's the first in rank slot and is also within his budget since $p_A\theta_{s_1} = 5 \cdot 3 = 15 < 16 = B_A$. Under the same logic, player B gets slot 2 which is the slot with the highest CTR available and is within his budget since $p_B\theta_{s_2} = 4 \cdot 2 = 8 < 10 = B_B$. Let us now proceed to each player separately:

**Player A:** His utility is $u_A = c_{s_1}(v_A - p_A) = 3(10 - 5) = 15$.

- *In B's position:* Slot 2 is within A's budget since $p_B\theta_{s_2} = 4 \cdot 2 = 8 < 16 = B_A$ and his utility is $u_A = \theta_{s_2}(v_A - p_B) = 2(10-4) = 12 < 15$. So A does not envy B.
- *In C's position:* No slot exists in C's position, something that implies $u_A = 0 < 15$. So A does not envy C.

**Player B:** His utility is $u_B = \theta_{s_2}(v_B - p_B) = 2(8-4) = 8$.

- *In A's position:* His utility is $u_B = \theta_{s_1}(v_B - p_A) = 3(8-5) = 9 > 8$ but slot 1 is not within his budget since $p_A\theta_{s_1} = 5 \cdot 3 = 15 > 10 = B_B$ . So B does not envy A.
- *In C's position:* No slot exists in C's position, something that implies $u_B = 0 < 15$. So B does not envy C.

**Player C:** His utility is $u_C = 0$ since he gets no item.

- *In A's position:* His utility is $u_C = \theta_{s_1}(v_C - p_A) = 3(6-5) = 3 > 0$ but slot 1 is not within his budget since $p_A\theta_{s_1} = 5 \cdot 3 = 15 > 7 = B_C$ . So C does not envy A.
- *In B's position:* His utility is $u_C = \theta_{s_2}(v_C - p_B) = 2(6-4) = 4 > 0$ but slot 2 is not within his budget since $p_B\theta_{s_2} = 4 \cdot 2 = 8 > 7 = B_C$ . So C does not envy B.

We can conclude that this is an envy-free assignment.

**PNE**

We claim that under these bids, player A is better off in player's B position so this is not a PNE. Suppose that player A underbids player B by a small amount $\epsilon$. The previous matrix goes as follows:

| *Players* | *Values* | *Budgets* | *Bids* | *Prices* | *Slots* |
|---|---|---|---|---|---|
| $B$ | 8 | 10 | 5 | $5-\epsilon$ | $s_2$ |
| $A$ | 10 | 16 | $5-\epsilon$ | 4 | $s_1$ |
| $C$ | 6 | 7 | 4 | 0 | *none* |

Notice now that player B can not afford slot 1 (the first available slot with the highest CTR) since $p_B\theta_{s_1} = (5-\epsilon)3 = 15-3\epsilon > 10 = B_B$. So he gets slot 2 which is within his budget since $p_B\theta_{s_2} = (5-\epsilon)2 = 10-2\epsilon < 10 = B_B$. It follows that player A will get slot 1 (the slot with the highest CTR available) as he can afford it, $p_A\theta_{s_1} = 4 \cdot 3 = 12 < 16 = B_A$.

His new utility is $u_A = \theta_{s_1}(v_A - p_A) = 3(10-4) = 18 > 15$ (his previous utility). This concludes our proof. □

So as we can see, the introduction of budgets makes the whole model a lot more complex and lessen its properties. With that in mind lets take a look on the next theorem.

**Theorem 4.1.2.1** It is not, in general, possible to find bids that produce a PNE under the BC-SPA setting.

*Proof.* We will prove this using an example. Suppose that we have 3 players and 2 slots with the following properties:

| $Slots$ | $CTR$ |
|---|---|
| $s_1$ | 5 |
| $s_2$ | 2 |

| $Players$ | $Values$ | $Budgets$ |
|---|---|---|
| $A$ | 10 | 50 |
| $B$ | 4 | 5 |
| $C$ | 2 | 2 |

We claim that under this data, there are not bids that produce a PNE. In order to prove that we have to examine every possible ordering of $b_A, b_B, b_C$ as well as every possible slot assignment that might occur. Specifically there are $3! = 6$ ways that we can order the bids and $3!/(3-2)! = 6$ ways to arrange the slots. We shall split our proof into cases and we will examine each case separately in a sketchy and informative way. Before we begin, notice the following inequalities that describe under what restrictions each player can afford each slot:

**Player A:**

- *Slot 1:* $\theta_{s_1} p_A \leq B_A \Rightarrow 5p_A \leq 50 \Rightarrow p_A \leq 10$
- *Slot 2:* $\theta_{s_2} p_A \leq B_A \Rightarrow 2p_A \leq 50 \Rightarrow p_A \leq 25$

**Player B:**

- *Slot 1:* $\theta_{s_1} p_B \leq B_B \Rightarrow 5p_B \leq 5 \Rightarrow p_B \leq 1$
- *Slot 2:* $\theta_{s_2} p_B \leq B_B \Rightarrow 2p_B \leq 5 \Rightarrow p_B \leq 5/2$

**Player C:**

- *Slot 1:* $\theta_{s_1} p_C \leq B_C \Rightarrow 5p_C \leq 2 \Rightarrow p_C \leq 2/5$
- *Slot 2:* $\theta_{s_2} p_C \leq B_C \Rightarrow 2p_C \leq 2 \Rightarrow p_C \leq 1$

We have to mention here that we are only considering cases where both slots are assigned to players. It is easy to see that it is not possible to have an assignment with none of the players tagged to some slot, due to the fact that $p_{min} = 0$. Additionally if only one player is tagged to some slot, he will be the third in rank for the same reason (all the players can buy at $p_{min} = 0$). These cases can be summed up as follows: The first in rank player, underbids the third in rank (the one who is tagged to a slot) and he is better off at this position since he surely gets an item and his utility becomes positive.

**Case I:** $b_A \geq b_B \geq b_C$

| *Slot* 1 | *Slot* 2 | *Change of strategy* |
|---|---|---|
| $A$ | $B$ | It depends on $b_B$. If $1 < b_B$ then Player A bids $b_B - \epsilon$ and gets slot 1 at a smaller price. If $b_B \leq 1$ then Player C $b_B + \epsilon$ bids and gets a slot. |
| $B$ | $A$ | If player A bids $b_B - \epsilon > 10$ then he gets slot 1. |
| $A$ | $C$ | If player B bids $b_C - \epsilon > 5/2$ then he gets slot 2. |
| $C$ | $A$ | If player B bids $b_C - \epsilon > 1$ then he gets slot 1. |
| $B$ | $C$ | If player A bids $b_B - \epsilon > 25$ then he gets slot 1. |
| $C$ | $B$ | If player A bids $b_B - \epsilon > 10$ then he gets slot 1. |

**Case II:** $b_C > b_B > b_A$

| *Slot* 1 | *Slot* 2 | *Change of strategy* |
|---|---|---|
| $C$ | $B$ | If player A bids $b_B + \epsilon$ then he gets a slot. |
| $B$ | $C$ | If player A bids $b_B + \epsilon$ then he gets slot 1. |
| $B$ | $A$ | It depends on $b_B$. If $b_B \leq 5/2$ then Player A bids $b_B + \epsilon$ and gets slot 1. If $b_B > 5/2$ then Player C bids $b_B - \epsilon > 5/2$ and gets a slot. |
| $A$ | $B$ | If player B bids $b_A - \epsilon > 1$ then he gets slot 1. |
| $C$ | $A$ | This outcome is impossible due to the bids ordering. |
| $A$ | $C$ | This outcome is impossible due to the bids ordering. |

**Case III:** $b_A \geq b_C > b_B$

| *Slot* 1 | *Slot* 2 | *Change of strategy* |
|---|---|---|
| $A$ | $C$ | It depends on $b_C$. If $b_C > 2/5$ then Player A bids $b_C - \epsilon > 2/5$ and gets slot 1. If $b_C \leq 2/5$ then Player B bids $b_C + \epsilon$ and gets a slot. |
| $C$ | $A$ | If player A bids $b_C - \epsilon > 10$ then he gets slot 1. |
| $C$ | $B$ | If player A bids $b_C - \epsilon > 25$ then he gets slot 1. |
| $B$ | $C$ | If player A bids $b_C - \epsilon > 25$ then he gets slot 1. |
| $A$ | $B$ | If player A bids $b_C - \epsilon > 1$ then he gets slot 1 at a lower price. |
| $B$ | $A$ | If player A bids $b_C - \epsilon > 10$ then he gets a slot at a lower price. |

**Case IV:** $b_C > b_A \geq b_B$

| *Slot* 1 | *Slot* 2 | *Change of strategy* |
|---|---|---|
| $C$ | $A$ | If player B bids $2/5 < b_A + \epsilon < 1$ then he gets slot 1. |
| $A$ | $C$ | If player C bids $b_A - \epsilon$ then he gets slot 2 at a lower price If $b_A = b_B$ then player B bids $b_A + \epsilon$ and gets a slot. |
| $A$ | $B$ | It depends on $b_B$. If $b_B > 1$ then Player A bids $b_B - \epsilon > 1$ and gets slot 1 at a lower price. If $b_B \leq 1$ then Player C bids $b_A - \epsilon < 10$ and gets slot 2. |
| $B$ | $A$ | If player A bids $b_B - \epsilon > 10$ then he gets slot 1. |
| $C$ | $B$ | This outcome is impossible due to the bids ordering. |
| $B$ | $C$ | This outcome is impossible due to the bids ordering. |

**Case V:** $b_B > b_A \geq b_C$

| *Slot* 1 | *Slot* 2 | *Change of strategy* |
|---|---|---|
| $B$ | $A$ | It depends on $b_B$. If $b_B \leq 5/2$ then Player A bids $b_B + \epsilon$ and gets slot 1. If $b_B > 5/2$ then Player C bids $b_A + \epsilon > 5/2$ and gets slot 2. |
| $A$ | $B$ | If player B bids $b_A - \epsilon > 1$ then he gets slot 2 at a lower price. |
| $B$ | $C$ | This outcome is impossible due to the bids ordering. |
| $C$ | $B$ | This outcome is impossible due to the bids ordering. |
| $A$ | $C$ | It depends on $b_C$. If $b_C \leq 5/2$ then Player B bids $b_A - \epsilon < 10$ and gets slot 2. If $b_C > 5/2$ then Player A bids $b_C - \epsilon > 5/2$ and gets slot 1 at a lower price. |
| $C$ | $A$ | If player A bids $b_C - \epsilon > 10$ then he gets slot 1 at a lower price. |

**Case VI:** $b_B \geq b_C > b_A$

| *Slot* 1 | *Slot* 2 | *Change of strategy* |
|---|---|---|
| $B$ | $C$ | If player A bids $b_C + \epsilon$ then he gets a slot. |
| $C$ | $B$ | If player A bids $b_C + \epsilon > 1$ then he gets slot 1. |
| $B$ | $A$ | This outcome is impossible due to the bids ordering. |
| $A$ | $B$ | If player B bids $b_C - \epsilon > 1$ then he gets a slot at a lower price. |
| $C$ | $A$ | If player B bids $b_C - \epsilon > 5/2$ then he gets slot 1 . |
| $A$ | $C$ | If player B bids $b_C - \epsilon > 5/2$ then he gets slot 1. |

As we can see there is no PNE in this example. An interesting point to mention here is that this example has an EFE. As a matter of fact:

| *Players* | *Values* | *Budgets* | *Bids* | *Prices* | *Slot* |
|---|---|---|---|---|---|
| *A* | 10 | 50 | 9 | 3 | $s_1$ |
| *B* | 4 | 5 | 3 | 2 | $s_2$ |
| *C* | 2 | 2 | 2 | 0 | *none* |

*Player A*: His utility is $\theta_{s_1}(v_A - p_A) = 5(10-3) = 35$. He envies neither player B since $\theta_{s_2}(v_A - p_B) = 2(10-2) = 16$ nor player C since his utility there is zero.

*Player B*: His utility is $\theta_{s_2}(v_B - p_B) = 2(4-2) = 4$. He envies neither player A since he can not afford his slot ($\theta_{s_1} \cdot p_A = 5 \cdot 3 = 15 > 5 = B_B$) nor player C since his utility there is zero.

*Player C*: His utility is zero. He envies neither player A nor player B since he can not afford none of the slots they are tagged to ($\theta_{s_1} \cdot p_A = 5 \cdot 3 = 15 > 2 = B_C$ and $\theta_{s_2} \cdot p_B = 2 \cdot 2 = 4 > 2 = B_C$).

Our proof is now complete. □

**Proposition 4.1.2.2** In the BC-SPA setting it is not, in general, possible to have an EFE if players have identical budgets. [1]

*Proof.* This is in a way trivial, for instance suppose that we have two players with identical budgets and one slot with $\theta$ to be the CTR. We set their values to be bigger from $B/\theta$, so if they can afford the slot, they certainly get a positive utility from it. It is easy to see that no matter the assignment, the player who gets no slot will always envy the other one since at his place he has a positive utility and can afford the slot by assumption. □

[1] It is highly believed however that when players have different budgets, then there exists bids that produce an envy-free assignment. See section 4.3 for more.

## 4.2 Budget-Oblivious Second-Price Auction

The budget-oblivious second-price auction (BO-SPA) is an alternative approach to the same setting. The basic difference is that each player in his turn, determined by decreasing bid, is assigned to the highest CTR available slot, independently of whether he can afford it or not. But lets take a more formal look.

### 4.2.1 The Model

We have N players and K slots ($N > K$). The slots have expected click-through rates (CTR) $\theta_1 > \theta_2 > ... > \theta_K > 0$, that depend upon their positioning. The players $i = 1, ..., N$ demand at most one slot and have private values $v_1, v_2, ..., v_N > 0$ per click. They also have publicly known budgets $B_1, B_2, ..., B_N$ capping the total payment they are willing to accept. The auctioneer announces prices $p_i$ per click and selects $K$ players to be given a unique slot, as follows:

1. The auctioneer orders the players according to their bids and renames them if necessary so that $b_1 \geq b_2 \geq ... \geq b_N$. For players with identical bids, lexicographic ordering is followed i.e. the player with the lower original index is ranked first, the player with the second lower index is ranked second etc.

2. Each player $i$, in his turn as determined by decreasing bid, is assigned the slot with the highest CTR that is currently available (has not being taken so far) if there is such one, independently of whether he can afford it or not, otherwise he gets nothing. A player who is assigned to a slot that he can not afford is called an off-budget player (this slot is assumed to be occupied from then on or to remain available). Notice here that since the number of players $N$ is bigger from the number of slots $K$ there is no need for the auctioneer to set a minimum price (players get the slots in decreasing order of bids, either they can afford them or not, so it is impossible for the last in rank player to be assigned to a slot).

3. The price per click a player $i$ is required to pay in case he is assigned

to a slot is defined as follows:

$$p_i = \begin{cases} b_{i+1} & \text{if he can afford the slot} \\ 0 & \text{if he is an off-budget player} \end{cases}$$

If player $i$ is not assigned to any slot then his payment is defined as zero.

4. The utility of each player $i$ is defined as follows:

$$u_i = \begin{cases} \theta_{s_i}(v_i - b_{i+1}) & \text{if } i \text{ can afford } s_i \\ 0 \quad or -\infty & \text{if } i \text{ is an off-budget player (he can not afford slot } s_i) \\ 0 & \text{if } i \text{ is assigned to no slot} \end{cases}$$

Notice that the utility of an off-budget player is usually assumed to be either zero or $-\infty$.

## 4.2.2 Properties

Lets now take a look on how this model behaves under the search of PNE and EFE assignments. Notice that definitions 4.1.2.1, 4.1.2.2 and 4.1.2.3 can be applied to this model as well.

**Proposition 4.2.2.1** In the BO-SPA there can be no PNE or envy-free assignment with off-budget players, if we assume that their utility is defined as $-\infty$.

*Proof.* Suppose that there is an assignment with off-budget players. By the definition of the model we know that the last in rank player does not get any slot since the number of the players is bigger than the number of the slots. So none of the off-budget players can be last in rank. With that in mind it is easy to say that any off-budget player could underbid the last in rank player an thus get a zero utility which is bigger than his current one ($-\infty$ by assumption). Under the same logic we can say that any off-budget player envies the last in rank player for the same reason. So we can conclude that it is impossible to have a PNE or an envy-free assignment with off-budget players. □

**Theorem 4.2.2.1** It is not, in general, possible to find bids such that the BO-SPA produces an envy-free assignment with no off-budget players.

*Proof.* We will prove this with an example. Suppose that we have 3 players and 2 slots with the following properties:

| $Slots$ | $CTR$ |
|---|---|
| $s_1$ | 1000 |
| $s_2$ | 5 |

| $Players$ | $Values$ | $Budgets$ |
|---|---|---|
| $A$ | 10 | 12 |
| $B$ | 8 | 10 |
| $C$ | 6 | 8 |

We claim that there are not $b_A, b_B, b_C$ that produce an envy-free assignment under the setting of non-excluding-budget second-price auction with no off-budget players. We will examine extensively two cases of bids ordering.

**Case 1:** $b_A \geq b_B \geq b_C$

Suppose that $b_A \geq b_B \geq b_C$, so player $A$ gets slot 1, player $B$ gets slot 2 and player $C$ gets nothing. We have the following:

| $Players$ | $Values$ | $Budgets$ | $Bids$ | $Prices$ | $Slots$ |
|---|---|---|---|---|---|
| $A$ | 10 | 12 | $b_A$ | $b_B$ | $s_1$ |
| $B$ | 8 | 10 | $b_B$ | $b_C$ | $s_2$ |
| $C$ | 6 | 8 | $b_C$ | 0 | *none* |

Since we want no off-budget players, from the definition of the setting we have that bids must restrict to some limits (if the bids go beyond these limits, we have off-budget players and we do not follow the restrictions of Theorem 4.2.2.1). More precisely:

$$p_A \theta_{s_1} \leq B_A \Rightarrow$$

$$b_B \theta_{s_1} \leq B_A \Rightarrow$$

$$b_B \leq 12/1000 \Rightarrow$$

$$b_B \leq 3/250$$

as well as,

$$p_B \theta_{s_2} \leq B_B \Rightarrow$$

$$b_C \theta_{s_2} \leq B_B \Rightarrow$$

$$b_C \leq 10/5 \Rightarrow$$

$$b_C \leq 2$$

So, with A and B to be both no off-budget players, we will show that player C envies player B and therefore this is not an envy-free assignment (notice that due to the bid-limits, our assignment is rational). C's utility is currently zero since he gets no slots. Following definition 4.1.2.2 we proceed to the following (using the previously described no off-budget restriction $b_B \leq 3/250$) :

**Rationality:** $p_B = b_C \leq b_B \leq 3/250 < 6 = v_C$

**Afford-ability:** $p_B \theta_{s_2} = 5b_C \leq 5b_B \leq 15/250 = 3/50 < 8 = B_C$

**Utility:** $u_C = \theta_{s_2}(v_C - p_B) = 5(6 - b_C) = 30 - 5b_C \geq 30 - 5b_B \geq 30 - 15/250 = 30 - 3/50 > 0$

So according to definition 4.1.2.3, this is not an envy-free assignment.

**Case 2:** $b_C > b_B > b_A$

Suppose that $b_C > b_B > b_A$, so player $C$ gets slot 1, player $B$ gets slot 2 and player $A$ gets nothing. We have the following:

| *Players* | *Values* | *Budgets* | *Bids* | *Prices* | *Slots* |
|---|---|---|---|---|---|
| $C$ | 6 | 8 | $b_C$ | $b_B$ | $s_1$ |
| $B$ | 8 | 10 | $b_B$ | $b_A$ | $s_2$ |
| $A$ | 10 | 12 | $b_A$ | 0 | *none* |

As with case 1, in order to have no off-budget players, the bids restrictions are:

$$p_C\theta_{s_1} \leq B_C \Rightarrow$$

$$b_B\theta_{s_1} \leq B_C \Rightarrow$$

$$b_B \leq 8/1000 \Rightarrow$$

$$b_B \leq 1/125$$

as well as,

$$p_B\theta_{s_2} \leq B_B \Rightarrow$$

$$b_A\theta_{s_2} \leq B_B \Rightarrow$$

$$b_A \leq 10/5 \Rightarrow$$

$$b_A \leq 2$$

So, with B and C this time to be both no off-budget players, we will show that player A envies player B and therefore this is not an envy-free assignment (again, the bid-limits set a rational assignment). A's utility is currently zero since he gets no slots. Following definition 4.1.2.2 once more, we proceed to the following (using the previously described no off-budget restriction $b_B \leq 1/125$) :

**Rationality:** $p_B = b_A < b_B \leq 1/125 < 10 = v_A$

**Afford-ability:** $p_B\theta_{s_2} = 5b_A < 5b_B \leq 5/125 = 1/25 < 12 = B_A$

**Utility:** $u_A = \theta_{s_2}(v_A - p_B) = 5(10 - b_A) = 50 - 5b_A > 50 - 5b_B \geq 50 - 5/125 = 50 - 1/25 > 0$

So, according to definition 4.1.2.3, this is not an envy-free assignment.

### Rest Cases

It is easy to see that the same goes for every other possible ordering of bids. Notice that since we want no off-budget players, the following must hold in general, for every ordering $b_1 > b_2 > b_3$ of players A,B and C:

$$p_1\theta_{s_1} \leq B_1 \Rightarrow$$

$$b_2\theta_{s_1} \leq B_1 \Rightarrow$$

$$b_2 \leq B_1/1000 \Rightarrow$$

$$b_3 < B_1/1000 \Rightarrow$$

$$p_2 < B_1/1000$$

This inequality always sets a pretty low upper bound for $p_2$, which assures us (taking into consideration the private values, CTRs and budgets in our example) that in any case, the third in rank player (the one who gets no slot) will always envy the second in rank (the one who gets slot 2). □

## 4.3 Future work

As a conclusion we present some properties of the two models that are currently under consideration or about to be proved:

- There exists bids such that the BC-SPA produces an envy-free assignment (if the budgets of the players are not identical).
- There exists bids such that the BO-SPA with off-budget players, produces an equilibrium (PNE or EFE) if we assume that the utility of the off-budget players is defined as zero.

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