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Graduate Program in Logic and Theory of Algorithms and Computation

MSc Thesis

Generalized Second-Price Ad Auctions
under
Budget Constraints

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Στη μνήμη της γιαγιάς μου,
Γραμματείας Κότσιαλου.

To the memory of my grandmother,
Grammateia Kotsialou.

Contents

Περίληψη

Στην παρούσα διπλωματική εργασία, αναλύουμε γνωστούς, φιλαλήθεις (incentive compatible) και μη, μηχανισμούς δημοπρασιών για διαφημίσεις στο διαδίκτυο, δίνοντας έμφαση στο Γενικευμένο Μηχανισμό Δεύτερης Τιμής (Generalized Second-Price Mechanism). Αρχικά, δείχνουμε με αντιπαραδείγματα ότι κάποιες από τις ιδιότητες που ισχύουν για τις αναθέσεις απαλλαγμένες - φθόνου (envy-free assignments) και παρουσιάζονται στο 'Position Auctions' του Hal R. Varian [1], δεν ικανοποιούνται από σύνολα ισορροπιών Nash.

Στη συνέχεια, βρίσκουμε αντιπαραδείγματα για το κύριο θεώρημα στο 'Repeated Budgeted Second Price Ad Auction' των A. Arnon και Y. Mansour [2], το οποίο αφορά την ύπαρξη ισορροπίας Nash. Επιπλέον, παρουσιάζουμε μία συνθήκη κάτω από την οποία υπάρχει ισορροπία Nash για δύο πράκτορες (agents) με διαφορετικές, συντηρητικές προσφορές (conservative bids).

Τέλος, αναφερόμαστε στο μοντέλο που θα παρουσιαστεί στο 'On the Stability of Generalized Second Price Auctions with Budgets" των J. Diaz, I. Γιώτης, E. Κυρούσης, E. Μαρκάκης και M. Serna [4]. Συγκεκριμένα, εξετάζουμε τη σχέση ανάμεσα σε ισορροπίες Nash και σε αναθέσεις απαλλαγμένες-φθόνου στην περίπτωση όπου κάθε παίκτης έχει ένα συγκεκριμένο ποσό που μπορεί να ξοδέψει (budget constraint), και παρουσιάζουμε αναφορικά τα κύρια αποτελέσματα της δουλειάς αυτής.

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My years in Athens would be totally different/boring if I hadn't met Stelios Despotakis, Haris Angelidakis[1], Dimitra Stamatopoulou, George Askalidis and Andreas Göbel. I would like to thank them for their friendship, the "Bang[2]" and "Batman[3]" nights. I hope one day that our paths will cross again. My thanks are extended to all my Master's colleauges (mpla - corelab) who have been inspirational to me, to Nathan Collins for introducing me to the english language world, and, to Iosif Salem for his help on last moment technical Latex problems.

Special acknowledgements go to Harris Hanialidis for his continuous tremendous support and all the laughter he brings to my life. Moreover, to Eirini-Danai Vragali and Hara Toloumi, just for their existence and for reminding me how internal beauty is.

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[1] Haris Angelidakis also helped me to correctly translate parts of my thesis, so a double 'thank you' goes to him.

[2] Bang is an addictive board game.

[3] Batman is a bar with authentic greek rebetika.

Chapter 1

Introduction

"Advertisers constantly invent cures to which there is no disease."

-Unknown

An auction[1] is a traditional method of buying and selling goods or services by offering them up for a bid. The auctioneer ranks the participants according to their bid and the product is being sold to the person who bids the highest.

Auctions have a long history, having been recorded as early as 500 B.C. when Herodotus reported the use of an auction. For most of history, they have been a relatively uncommon way to negotiate the exchange of goods and commodities, in contrast with haggling and sale by set-price. Before the 17th century, auctions were quite infrequent and were usually used to sell women for marriage, slaves, spoils of war or even to liquidate property and estate goods (atrium auctionarium). One of the most bizarre auctions recorded in ancient times, took place in the year 193 A.D. when the entire Roman Empire was put up for auction after being sacked. Also in China, there is evidence that Buddhist monks were using auctions in order to fund the creation of temples. It is unknown whether these auctions were ascending or not, but according to the word origin, someone could deduce that they were ascending.

The earliest modern era records of auctions appeared in the Oxford English Dictionary in 1595. Following that, in the late 17th century, the London Gazette reported auctioning for selling art at coffeehouses and taverns while in the early 18th century, the first auction houses were created. The oldest in the world is Stockholm Auction House in Sweden, established in 1674. Nowadays, the world's largest auction house is Christie's, established around

[1] The word "auction" is derived from the Latin augeō which means 'I increase' or 'I augment'.

1766, followed by Sotheby's, which held its first auction in 1744. Reportedly, back to 1887 in Netherlands, a grower named Jongerling, discovered a strong demand for his produce. Instead of selling his products to a specific dealer, he decided to allow buyers to compete with each other by using an auction. Similarly, during the same year, a fisherman in Germany decided to use auctions every time he was arriving in port, in order to rapidly liquidate his catch. Having been recognised as an efficient business tool that meets the needs of the public, over the 20th century, auctions had a tremendous success in marketing real estate and personal property.

But why are auctions so preferable and when are they appropriate? Converting property, possessions or inventory to cash fast is a major benefit of the auction method of marketing. In addition, the majority of sellers prefer auction method to sell goods due to following benefits. First, competitive bidding brings higher prices while auction's terms, conditions, day and time will be seller's decision. Second, carrying costs, which are usually high, are limited and the products are sold at current market value. Furthermore, auctions are event oriented and high impact marketing tools that draw attention to the seller's property, thus providing maximum exposure and visibility to the market. Regarding buyers, they are the ones who actually determine final prices and market values of the purchased items. They walk away knowing that they bought an item only one bid higher than someone else was willing to pay. Therefore, everyone is satisfied. In the final analysis, the auction method is generally used in situations where sellers do not have a good estimate of the buyers' true values and where buyers do not know the values of the other participants.

In recent years, the development of the Internet has led to a significant rise in the use of auctions. Online auctions broke down and removed physical limitations of traditional auctions such as geography, presence, time, space and small target audience. As a result, auctioneers can solicit bids via the Internet from a wide range of buyers in a much wider range of commodities than was previously feasible. This environment became ideal for selling sponsored search advertising space. By now, online ads via auction mechanisms are responsible for billions of dollars in annual revenue for many Internet companies and this major source of income explains why they are a booming industry. Online ads are essential to monetize valuable internet services, offered free to the general public, like search engines, blogs, and social networking sites. As a result, Ad Auctions have effectively created a giant virtual marketplace where people can gather to buy, sell, trade and check out the goods of the day. The auction site that leads the online auction industry is eBay and according to "Nielsen Ratings"[2], it is among the top ten most-trafficked sites on the Internet.

[2]Nielsen Ratings is the primary source of audience measurement information. The system was developed by Nielsen Company.

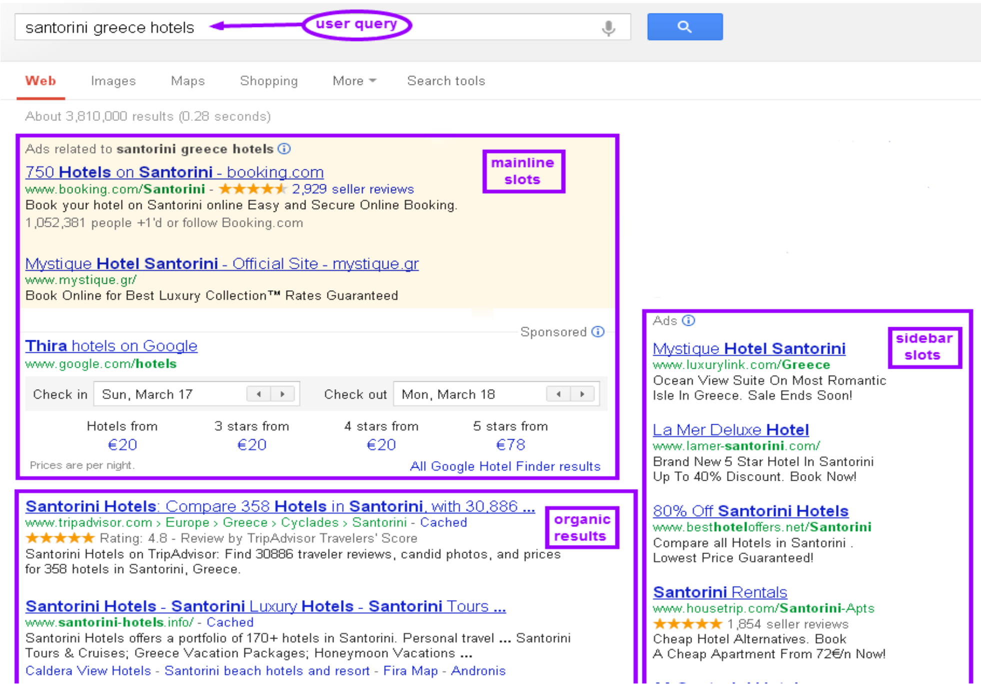


Figure 1.1: An example of sponsored search results

In this massive virtual marketplace, the benefits of online auctions are great for everyone who takes part in the auction (advertiser, user, search engine). It is an ideal place to capitalize on readily available, widespread exposure, selling products almost immediately with no overhead costs, no website of your own and no initial investments (sellers pay between 2 and 5 percent of their final sale price to the auction site). Moreover, online advertisers are able to observe the results of their campaign within days or even hours. The rapid increase of advertisers in such auctions creates new channels for new products offering buyers an extremely desirable search experience and favorable purchasing conditions. Last, Ad Auctions are extremely beneficial for search engines. Whenever a user performs a search with commercial interest e.g. on Google, a position auction takes place and then, the winning ads appear next to the search results that Google outputs. For every such auction, Google and other search engines like Yahoo! and Bing, earn several cents and these amounts add up to billions of dollars every year. For these reasons, the last decade, there is an urgency to study and deeper analyze the game theoretical side of Ad Auctions. Weighting all the information above, it could be said that any slight theoretical improvement of their mechanisms would be a step of significant importance in real-life applications.

Before continuing to the thesis outline, we should first describe in detail the actual idea and process of sponsored search. When a user enters a query (search) into a search engine, he gets back a page with results. This page contains the most relevant to the query links, named organic results, and the paid advertisements. The ads are clearly distinguishable and search engines like Google, Yahoo! and Bing allow ads to be displayed on the top or on the side of the organic results. Typically, search engines allow up to three ad links on the top, named mainline slots, and up to eight alongside, named sidebar slots. Figure 1.1 depicts an example. The main advantage of such ads, is that they are relevant to the keyword (query) that the Internet user enters and for this reason, it is more likely for a transaction to occur. When the user clicks on a certain ad link, he is sent to the advertiser's web page and the advertiser pays the search engine for sending the potential buyer to his web site. Except for the previous and most popular charging model, the"pay-per-click" (PPC) payment, there are also two more charging schemes for the selected ads. The "Pay-Per-Impression" (PPI) model, where each advertiser is charged every time his ad is displayed and the "Pay-Per-Transaction" (PPT) model, where each advertiser is charged only when a transaction occurs.

1.1 Thesis Outline

In chapter 2, we describe the following known auction mechanisms: First-Price Auction, Second-Price Auction and VCG mechanism. Afterwards, we present the Generalised Second-Price Ad Auction and the VCG mechanism implemented to that setting. For each mechanism, we show its truthfulness or non-truthfulness giving proofs or examples, respectively.

In chapter 3, we present the Ad Auction model used by Google and Yahoo, that was introduced in [1]. We focus to the equilibrium analysis of position auction. More precisely, in [1], a new set of equilibria is introduced, the *symmetric* or *envy-free* equilibria, which is a subset[3] of pure Nash equilibria. Moreover, the envy-free equilibria sets have some properties given in [1]. At this point, we found counterexamples proving that some of these properties are not satisfied under pure Nash equilibria sets.

In chapter 4, I introduce the main field of my thesis which refers to Ad Auction models under budget constraints focusing on the work of [2]. Initially, the model is described in details giving examples of how it works for two players and then, we present the preliminaries and properties of pure Nash equilibrium existence, according to [2]. Secondly, we construct a counterexample 1 for its main theorem. This counterexample is a result under my collaboration with my colleauge George Mpirmpas. The main theorem is proved using an induction method. The base of the induction is a claim which refers to the PNE existence for two players. Our counterexample 1

[3]This property is not satisfied under budget constraints

concerns this claim. However, the problem we observed is also extended to the other induction steps, the general case of any number of agents. Moreover, I present a second counterexample 2 for the second part of theorem 3.1 in [3] and prove a new theorem which is a similar version to the previous. Last, I present a second theorem about a special case of PNE existence for two players and non-identical, conservative bids. The condition of this theorem is simple and does not involve the critical bids.

In chapter 5, we present the work of an on-going paper [4] of J. Diaz, Y. Giotis, L. M. Kirousis, E. Markakis and M. Serna which concerns second-price auctions for the allocation of advertisement space under budget contraints. The definition of envy-free assignment slightly differs from that one in chapter 3, since there also exists a budget constraint for each player. We examine the relation between envy-free assignments and pure Nash equilibria, and present some of the main results of this work.

Chapter 2

Auction Mechanisms

> "Plato is dear to me, but dearer still is truth."
>
> -Aristotle

Mechanism design is a subfield of economic theory which is interested in designing economic mechanisms in the terms of social choice. The latter can be expressed as an aggregation of preferences towards a single joint decision. A function that maps various individual preferences to a single decision is called *social choice function*.

In the context of auctions, we can say that a social choice function maps the players' preferences into a single outcome which will be the identity of the winner and it depends on the auction rules. The goal in auctions is to optimise an objective, such as money and social welfare (to give the item to the player who desires it more than any other player, e.g. to the highest bidder). Social welfare is given by the aggregation of all players' values for a certain allocation outcome. Formally, there are the following:

- Set of players $I = \{1, \ldots, n\}$.

- Set of alternative outcomes A.

- For every player i, a set of possible values (strategies), $V_i = \{v_i : A \longrightarrow \mathbb{R}\}$, where $v_i(\alpha)$ expresses how much player i values an outcome $\alpha \in A$. In fact, it expresses how much he values the item he gets in an outcome α. The preference profile over the set A for a player i is given by V_i, since it contains his values for all the feasible outcomes. The Social Welfare (SW) for an $\alpha \in A$ is given by $\sum_i v_i(\alpha)$.

In the context of social choice, as mentioned above, every player has his own preferences that reports them into a mechanism. The mechanism outputs a single joint decision which is to be implemented.

Definition 2.0.1. *[8]. A* mechanism (f, p) *is a social choice function* $f : V_1 \times V_2 \times \cdots \times V_n \rightarrow A$ *and a vector of payment functions* $p = (p_1, \ldots, p_n)$, *where* $p_i : V_1 \times V_2 \cdots \times V_n \rightarrow \mathbb{R}$ *is the amount that player i pays.*

Due to the need of measuring the level of every player's satisfaction, the term *utility* was introduced and defined by $u_i(v_1, \ldots, v_n) = v_i(\alpha) - p_i(v_1, \ldots, v_n)$, where $a = f(v_1, \ldots, v_n) \in A$. The value functions' arguments are outcomes while the payment functions' arguments are valuations. Also, a player i may lie, meaning that he reports a different amount $v_i^{'}$ from his value v_i. The reported values are called *bids*. Even if he lies ($b_i \neq v_i$), his utility (satisfaction) is going to be computed by the same function:

$$u_i(v_{-i}, v_i^{'}) = v_i(\alpha^{'}) - p_i(v_{-i}, v_i^{'}), \quad \text{where } \alpha^{'} = f(v_{-i}, v_i^{'}).$$

Definition 2.0.2. *[8]. A mechanism is called* incentive compatible (or truthful or strategy-proof) *if for every player participating in the auction,* $\forall v_i \in V_i$ *and* $\forall v_i^{'} \in V_i$, *his utility when he lies (bidding* $v_i^{'}$, $v_i^{'} \neq v_i$*),is at most his utility when he bids his value* v_i*. That is,* $v_i(a^{'}) - p_i(v_i^{'}, v_{-i}) \leq v_i(a) - p_i(v_i, v_{-i})$, *where* $a = f(v_i, v_{-i})$ *and* $a^{'} = f(v_i^{'}, v_{-i})$.

Intuitively, the above definition expresses the preference of a player to bid his value (truthfully bidding), since this option gives him a (weakly) higher utility than any other, different from his value, bidding option.

Definition 2.0.3. *A mechanism can be* strategically manipulated *by a player i if, for an outcome* α, *his utility becomes greater if he bids untruthfully than his utility if he bids his value.*

It turns out that a mechanism is incentive-compatible when it cannot be strategically manipulated.

2.1 First-Price (sealed-bid) Auction (FPA)

A First Price sealed-bid Auction is a type of auction in which bidders simultaneously submit sealed bids competing for a single item. The one who will submit the highest amount is awarded the object being sold and pays equal to the bid amount.

However, under such rules, the bidder will prefer to submit a different amount instead of his true value because, when a player with value v_i wins an item and pays v_i, gets a total profit-utility of zero. Due to this fact, submitting a lower bid than his value v_i, it is possible to win the item obtaining a positive utility, $u_i = v_i - v_i^{'} > 0$. Therefore, this form of auction is not incentive-compatible and a bidder bids an amount equal to $(1 - \frac{1}{\#\text{bidders}}) \cdot v_i$ which means, the more players participate, the more close to his value a player bids. The payment rule of this mechanism was gradually recognised to be unstable because it led to cycling bidding patterns and low revenue [9].

From the game-theoretic point of view, the results and strategies of First Price sealed-bid Auctions are equivalent to those of the Dutch[1] auction.

2.2 Second-Price Auction (SPA)

A Second Price Auction refers to a single item and it is also known as Vickrey's Second Price Auction, as it was first described by Professor William Vickrey in 1961. It is a truthful and socially efficient mechanism which means that it maximizes social welfare. The bidder who submits the highest price is awarded the object being sold, but instead of his bid amount, he pays the second highest bid. In this way, the winner can never affect the price that he is going to pay and the mechanism cannot be strategically manipulated. Consequently, bidders have no incentive to misreport their true values, thus we have an incentive-compatible mechanism.

In practice, second-price auctions are either sealed-bid[2], in which bidders submit their bids simultaneously, or English auctions, in which bidders continue to raise their bids until only one bidder remains, the winner.

2.3 Vickrey-Clarke-Groves mechanism (VCG)

Based on Vickrey's auction, Edward H. Clarke and Theodore Groves devised a new mechanism in order to treat public goods problems. It was first introduced by Clarke in 1971 [13] and later by Groves in 1973 [14]. This mechanism is a generalisation of Vickrey's Auction which concerns multiple items and works not only for homogenous but also, for heterogenous items. VCG assigns the items with an efficient socially way while still ensures bidders that reporting their true valuation is a dominant strategy. The dominant strategy property provides reliability to the efficiency prediction, because every participant knows that the result does not depend on their potential assumptions about the others' values and strategies. In addition, VCG mechanism is the only one who obtains all the previous properties.

Formal setting of VCG

Consider a set of n players $I = \{1, \dots, n\}$ and A a set of alternative outcomes. Let $v_i(\alpha)$ be player's i value for any outcome $\alpha \in A$. Then, each bidder submits a bid (reported value) $b_i(\alpha)$. In the next part of this section,

[1] In the Dutch auction, the auctioneer begins the process denoting a high asking price which is lowered until a bidder is able to accept the current price, or a reserve price is reached.

[2] Sealed-bid type: Bidders simultaneously submit written bids without knowing the bids of the other people participating in the auction

we show that it is optimal for everyone to bid his value, $b_i = v_i$. The mechanism runs a computation and chooses the outcome that maximizes social welfare (SW):

$$f(b_1, \ldots, b_n) \in \arg\max_{\alpha \in A} \sum_{i \in I} b_i(\alpha),$$

and charge prices p_i given by:

$$p_i(v_1, \ldots, v_n) = \max_{\beta \in A} \sum_{j \neq i} b_j(\beta) - \sum_{j \neq i} b_j(\alpha), \quad \text{where } \alpha = f(b_1, \ldots, b_n),$$

and this amount can be indicated as the impact, that his presence causes on the other participants, or else, his social cost. That is the difference between what they would get if bidder i did not participate in the auction and what they get when he is present. Moreover, the first term of the charge form: $\max_{\beta \in A} \sum_{j \neq i} b_j(\beta)$ is known as *Clark pivot rule*. It has some very good properties such as

individual rationality: for every player i, $v_i - p_i \geq 0$. All participants are getting a non-negative utility. No one is forced to bid.

no positive tranfers: $p_i \geq 0$. The mechanism does not pay anything to the bidders.

It also represents the maximum social welfare when player i is absent and has no strategic importance for him, since this amount will be the same regardless on what he says. Thus, from player i's point of view, $\max_{\beta \in A} \sum_{j \neq i} b_j(\beta)$ is a constant. Also, notice that his payment depends only on the other participant's reported values and not on what player i has reported. His final utility equals to $u_i(v_1, \ldots, v_n) = v_i(\alpha) - p_i(v_1, \ldots, v_n)$.

►*Example.*

To illustrate how VCG mechanism works, lets describe here an example. Assume that we have two players, $i = 1, 2$ and two items, A and B. Each player submit three bids, one bid $b_i(A)$ for item A, one bid $b_i(B)$ for item B and one bid $b_i(AB)$ for both items. Suppose now that $b_1(AB) \geq b_2(AB)$ and $b_1(A) + b_2(B) \geq b_1(B) + b_2(A)$.

If $b_1(AB) \geq b_1(A) + b_2(B)$, then the efficient mechanism will assign both items to player 1. His payment will be the amount that player 2 would get if 1 were absent minus what player 2 gets when player 1 participates. That is, $p_1 = b_2(AB) - 0 = b_2(AB)$.

If $b_1(AB) \leq b_1(A) + b_2(B)$, then the efficient allocation would be the following. Player 1 wins item A paying $p_1 = b_2(AB) - b_2(B)$ while player 2 wins item B paying $p_2 = b_1(AB) - b_1(A)$. ◄

Optimality of truthfulness using VCG

Proof. Assume we have a set $I = \{1, \ldots, n\}$ of players and a set M of items. Let player's i value of an outcome α be denoted by $v_i(\alpha)$. Suppose that player 1 bids truthfully ($b_i(\alpha) = v_i(\alpha)$) and VCG chooses the outcome α_κ that maximises SW. This outcome gives to player 1 the item $t \in M$. In this case of truthfully bidding, the utility of player 1 becomes

$$u_{1\_truth} = v_1(\alpha_\kappa) - \left(\sum_{i \neq 1}^{M} b_i(\alpha_\lambda) - \sum_{i \neq 1}^{M - \{t\}} b_i(\alpha_\kappa) \right) \tag{2.1}$$

On the other case, suppose that player 1 bids untruthfully and the mechanism chooses another outcome μ in which he gets item $s \in M$. Thus, his utility becomes

$$u_{1\_untruth} = v_1(\alpha_\mu) - \left(\sum_{i \neq 1}^{M} b_i(\alpha_\lambda) - \sum_{i \neq 1}^{M - \{s\}} b_i(\alpha_\mu) \right) \tag{2.2}$$

By abstracting these two utilities, we have

$$u_{1\_truth} - u_{1\_untruth} = \left(v_1(\alpha_\kappa) + \sum_{i \neq 1}^{M - \{t\}} b_i(\alpha_\kappa) \right) - \left(v_1(\alpha_\mu) + \sum_{i \neq 1}^{M - \{s\}} b_i(\alpha_\mu) \right) \tag{2.3}$$

The first term at (2.3) represents the maximum total social value when b_1 wins item t and the second term represents the maximum total social value, when b_1 wins item s. However, outcome α_κ maximises social welfare over all alternatives in A, which means that the first term is weakly greater:

$$u_{1\_truth} - u_{1\_untruth} \geq 0.$$

That means, player's 1 utility when bidding truthfully is at least equal with his utility when he misreports his value. □

2.4 Comparing VCG with Second Price Auction

In this section, we show that the VCG mechanism regarding a single item for sale and the Second Price Auction are equivalent.

Consider two players $I = 1, 2$ with values v_1, v_2 and bids b_1, b_2 competing for an item t. Without loss of generality, assume that $b_1 > b_2$. The players are ranked according to their bids and then, VCG mechanism, maximising social welfare, chooses outcome α that gives the item to player 1. According to (2.3), his payment will be:

$$p_1 = \max \sum_{j \neq 1} b_j(\beta) - \sum_{j \neq 1} b_j(\alpha).$$

If player 1 were not participating in the auction, player 2 would get the item. Consequently, in the outcome that player 2 gets the item, we have: $\max \sum_{j \neq 1} b_j(\beta) = b_2$. If player 1 is present, he will be the one who gets the item, and there is no item left for player 2 to buy, so $\sum_{j \neq 1} b_j(\alpha) = 0$. As a result,

$$p_1 = b_2 - 0 = b_2$$

which is player 1's payment if Second Price Auction, 2.2, was used ($p_i = b_{i+1}$).

According to VCG setting, payment of player 2 will be:

$$p_2 = \max \sum_{j \neq 2} b_j(\beta) - \sum_{j \neq 2} b_j(\alpha)$$

However, in this case, if player 2 was absent, there would be no impact on the allocation outcome, since $b_1 > b_2$, and player 1 would get the item regardless of player's 2 presence. So $\max \sum_{j \neq 2} b_j(\beta) = b_1$ and $\sum_{j \neq 2} b_j(\alpha) = b_1$. As a result,

$$p_2 = b_1 - b_1 = 0$$

since he gets no item. Similarly, in Second Price Auction, the players who get no item pay $p_i = 0$.

We observe that the ranking and pricing rules are the same, so the two mechanisms are equal.

2.5 Mechanisms for Ad Auctions

A mechanism for the Internet advertising settings has the following form. Consider a set of advertisers (players) $I = \{1, \dots, n\}$ who compete for a set $S = \{1, \dots, s\}$ of advertising spaces in a web page, named slots. Typically, we have that $n > s$ and every advertiser has his own private value v_i for every slot. Since v_i is private, every player submits a bid b_i (reported value) and we write $b = (b_1, \dots, b_n)$ for the bid vector. The bids can be viewed as the maximum amount that an advertiser is willing to pay per click. The mechanism allocates the slots to the players through a function $\pi : [s] \to [n]$.

Each slot s is associated with a CTR, θ_s, which denotes the probabilty of an ad on slot s being clicked by a user. The ads on the slots which are on the top of the web page have a larger probability of being clicked, so: $\theta_1 \geq \theta_2 \cdots \geq \theta_k$[3]. At this point we have to define a ranking rule for the players, a rule that will compute a score for each player, in order for them to be ranked in a decreasing order, according to that score. The most frequent ranking rules are the following, however, in this thesis we will use only the first one. According to [5], we have:

the *rank-by-bid*, in which every player submits a bid and the auctioneer ranks and renames the players according to their bids. The advertiser who bids the highest price gets the slot on the top, the second highest bidder gets the slot at the second highest position, etc., and

the *rank-by-revenue*, in which every player submits a bid b_i, however, every bid is associated with a quality score q_i. This score denotes how much related is the ad of advertiser i to the keywords that the Internet user has entered for search. In other words, q_i expresses the probabilitty that the Internet user will click on advertiser's i ad. Obtaining slot s, the probability of advertiser i to receive a click is $q_i \cdot \theta_s$. The advertisers are ranked according to the amount $q_i \cdot b_i$, and the first highest bidder wins the highest slot, etc.

In order for a mechanism to be completed, a payment rule must be also defined. Consider an advertiser i who, after the end of an auction, obtains a slot for his ad. This rule will declare the amount p_i he must pay when he receives a click by a user. We write $p = (p_1, \ldots, p_n)$ for the price vector. Thus, an auction mechanism can be defined as the joint of a ranking and a payment rule.

In the context of Ad Auctions, the final utility of a player i is defined by $u_i = \theta_i(v_i - p_i)$, where θ_i is the CTR of the slot that have been assigned to player i by the mechanism. The social welfare (SW) generated by the mechanism is given by $SW(v, \pi) = \sum_i \theta_i v_i$ and the total revenue is given by $\mathcal{R}(b) = \sum_i \theta_i p_i$.

2.5.1 Implementing VCG to Ad Auctions

We will implement here the VCG mechanism [2.3] to the Ad Auction setting which is, assigning positions (slots) to players (advertisers). The players are ranked according to their bids and, as I mention above, the total social welfare is defined by $\sum_i v_i \theta_i$. The total payment of every player is $\theta_i \cdot p_i$ and consequently, the total revenue for the mechanism is given by $\sum_i \theta_i \cdot p_i$. Assume now that agent $s-1$ is missing. Then, all agents below him will

[3]Assuming that s_1 is at the highest position, s_2 at the second highest and goes on.

be transfered one position above, while the agents above him stay in the position they were. That means that only agents below agent $s-1$ are affected by his absence. Acording to VCG setting, 2.3, the first term of the agent's $s-1$ payment is the impact[4] of his absence which, in this case, is

$$\sum_{t\geq s} b_t\theta_{t-1} + \sum_{t<s-1} b_t\theta_t.$$

To all the players t for $t \geq s$, VCG will now assign a slot one position higher than before and thus, we have $\sum_{t\geq s} b_t\theta_{t-1}$. The rest t players, for $t < s-1$, will be stay assigned to their original slot, so $\sum_{t<s-1} b_t\theta_t$.

The total payment of player $s-1$ will be:

$$\begin{aligned}
p_{s-1}\cdot\theta_{s-1} &= \big(\sum_{t\geq s} b_t\theta_{t-1} + \sum_{t<s-1} b_t\theta_t\big) - \sum_{t\neq s-1} b_t\theta_t \\
&= \sum_{t\geq s} b_t\theta_{t-1} - \sum_{t>s-1} b_t\theta_t = \sum_{t\geq s} b_t(\theta_{t-1}-\theta_t) \Leftrightarrow \\
p_{s-1}\cdot\theta_{s-1} &= \sum_{t\geq s} b_t(\theta_{t-1}-\theta_t) \Leftrightarrow^{\text{VCG is truthful}} \\
p_{s-1}\cdot\theta_{s-1} &= \sum_{t\geq s} v_t(\theta_{t-1}-\theta_t) \Leftrightarrow \qquad (2.4) \\
p_{s-1} &= \frac{1}{\theta_{s-1}}\sum_{t\geq s} v_t(\theta_{t-1}-\theta_t), \qquad (2.5)
\end{aligned}$$

which is the price per click of player $s-1$.

$\star$ In chapter 3, it is shown that there is a Pure Nash equilibrium which is the same as the VCG payment, equation (2.4) on page 24.

The total revenue generated by the mechanism is:

$$\mathcal{R}^{\text{VCG}} = \sum_s\sum_{t\geq s} v_t(\theta_{t-1}-\theta_t) = \sum_{t=2}^{n}(t-1)\cdot v_t(\theta_{t-1}-\theta_t).$$

Even though VCG mechanism has very good properties, it is rarely used in practice. However, it is worth mentioning that VCG was recently adopted by Facebook for its AdAuction system. In addition, Google also considered to switch its advertising system to VCG some years ago, but eventually decided against it and switched to the following mechanism.

.

[4] Since all the players below $s-1$ will move up one position, they will also be assigned to one step higher slot than before.

2.5.2 Generalised Second-Price mechanism (GSP)

In 2002, starting with Google, search engines switched to the so-called Generalised Second Price mechanism. Currently, it is the premier method by which sponsored search advertising space is sold and is employed by big internet companies such as Bing, Google and Yahoo!. It is worth mentioning that in 2005, over 98 percent of Google's total revenue came from GSP auctions while in 2008 its total advertising revenues were \$28 billions. Similarly, over half of Yahoo!'s profit was derived from GSP auctions while in May 2006, the joint capitalization of both previous internet companies exceeded \$150 billion.

Even though generalised second price auctions generalise the truthful Vickrey's Second Price Auction, section 2.2, they are known neither to be incentive-compatible nor to maximise social welfare and this fact comes in sharp contrast to their wide success.

Formal setting of GSP

Each advertiser has a value υ_i which expresses his personal evaluation for each slot. To participate in the auction, advertiser i submits a bid b_i which may differ from his true value. The GSP mechanism charges each player the minimum amount that would be necessary to bid in order to keep his current position. That is, the bid that have submitted by the player who gets the slot below him, plus a very small amount ϵ[5]. We will assume that $\epsilon = 0$, as this is not an important parameter. The pricing rule per click of GSP depends on the ranking rule. Remember that in this thesis we will use only the rank-by-bid rule and thus, only the first pricing rule below. According to [5], we have the following.

> In *rank-by-bid*, the player who wins slot s pays per click the bid of the player who is exactly below him and wins slot $s+1$, which is $p_s = b_{s+1}$.
>
> In *rank-by-revenue*, in order for the winner of slot s to keep his current position, his payment should satisfy the inequality $q_i \cdot p_s \geq q_{s+1} \cdot b_{s+1}$. The minimum price which satisfies the inequality is $p_s = \frac{q_{s+1} \cdot b_{s+1}}{q_s}$.

The ranking only by bid may lead to non-profitable results. For instance, ads with low quality score (in which Internet user is less interested), may be assigned to the highest slots. The fact that such ads have very low probability of being clicked, significantly lowers the total revenue of the search engine. Note, for example, that Yahoo! was originally using the First Price Auction system 2.1, then switced to GSP using the rank-by-bid ranking rule and finally, to GSP using the rank-by-revenue ranking rule.

[5] It usually equals $\epsilon = 0.01$ and is provided by every search engine for every currency.

Untruthfulness of GSP

We will describe here an example which shows that bidding truthfully is not a dominant strategy using GSP mechanism.

▶ *Example.*

Assume that we have three players, A, B, C who compete for two slots, s_1, s_2. We need to show that at least one player will be "happier" if he lies and we focus on player's A utility. In the first case, they bid their true value and the utility of player A, getting slot s_1, is: $u_A = \theta_{s_1}(\upsilon_A - p_A) = 1 \cdot (7-4) = 3$.

Players	Values	Bids	Prices	CTRs
A	$\upsilon_A = 7$	$b_A = 7$	$p_A = 4$	$\theta_{s_1} = 1$
B	$\upsilon_B = 4$	$b_B = 4$	$p_B = 3$	$\theta_{s_2} = \frac{4}{5}$
C	$\upsilon_C = 3$	$b_C = 3$	$p_C = 0$	

Assume now that player A decides to bid an amount lower than his true valuation, which ranks him in the second place.

Players	Values	Bids	Prices	CTRs
B	$\upsilon_B = 4$	$b_B = 4$	$p_A = \frac{7}{2}$	$\theta_{s_1} = 1$
A	$\upsilon_B = 7$	$b_B = \frac{7}{2}$	$p_B = 3$	$\theta_{s_2} = \frac{4}{5}$
C	$\upsilon_C = 3$	$b_C = 3$	$p_C = 0$	

Then, being ranked in the second position, he wins slot s_2 and his utility becomes: $u_A = \theta_{s_2}(\upsilon_A - p_A) = \frac{4}{5} \cdot (7-3) = \frac{16}{5}$ which is greater than his utility in the first truthful case.

◀

Chapter 3

Equilibrium Analysis of Position Auctions

> "Greek crisis is not economics, it's game theory."
>
> -Matthew Lynn
> *Matthew Lynn's London Eye*

In this chapter, we will show a theoritical analysis of equilibria of the ad auction used by Google and Yahoo! under GSP mechanism, introduced in [1]. It is shown that position auctions represent reasonably accurately the Google's ads. Moreover, the full information game always has a Pure Nash equilibrium and there is a Pure Nash equilibrium which has same outcome and payments as VCG mechanism.

Auction Model

The mechanism which is being used here is the GSP mechanism which we have already analysed in section 2.5.2 on page 25. Consider an assignment problem of agents (advertisers) $\alpha = 1, \ldots, A$ to slots (positions on a web page) $s = 1, \ldots, S$. Each slot has a different click-through-rate (CTR) depending on the slot's position in the web page. Higher positions receive more clicks so denoting s_1 as the highest slot, s_2 as the second highest and go on, CTRs' ordering can be expressed by $\theta_1 > \theta_2 > \cdots > \theta_S$ so that all agents agree. To avoid the assignment of the lowest position slot to an agent who bids very low, assume that $A \geq S + 1$, the number of agents is at least greater than the number of slots plus one. This ensures that even for the last slot, the last ranked players will bid competitively. Last, for every $s > S$, set $x_s = 0$.

This problem was motivated by the actual ad auctions used by Google and Yahoo. Every agent (advertiser) α has a value υ_α that expresses his expected profit per click and therefore, his expected profit obtaining slot s is given by $u_{\alpha s} = v_a \cdot \theta_s$. The slots are allocated via an auction that uses the Generalised Second Price mechanism. Each player submits a bid b_α and, if necessary, the players are renumbered according to their bids. The highest slot, the one with the best click-through-rate, is assigned to the agent who bids the highest price, the second highest slot to the second highest bidder and goes on. Every agent pays the bid of the agent below him, $p_s = b_{s+1}$ so, when an agent is assigned to slot s, his total profit (utility) equals to $u_{\alpha s} = \theta_s(v_\alpha - p_s)$.

3.1 Nash Equilibrium Analysis

Consider the table below which depicts in ranking the positions, values, bids, payments of 5 agents and the slots assigned to them. Notice that there is no slot left to be assigned to the last ranked agent. Assume now that agent 3 desires to move up one position. In order to do this, we can see that he must bid slightly above agent's 2 bid, b_2. Assume now that agent 2 wants to move down by one position. Then, according to the table, he must bid an amount at least slightly higher than b_3 and lower than b_4.

Position	Value	Bid	Price	CTRs
1	v_1	b_1	$p_1 = b_2$	θ_1
2	v_2	b_2	$p_2 = b_3$	θ_2
3	v_3	b_3	$p_3 = b_4$	θ_3
4	v_4	b_4	$p_4 = b_5$	θ_4
5	v_5	b_5	$p_5 = 0$	no slot

It can be seen from the above analysis that the actual rule for an agent A who wants to move in another position is the following. When an agent who obtains slot s desires to move down, in a position below him, say at $s+i$, then he must bid an amount higher than the payment of $s+i$'s current owner and lower than the current owner's bid. That is, agent A's bid should be in the interval (p_{s+i}, b_{s+i}). If agent A wants to move up, in a position (slot) above him, say $s-j$, then he must bid an amount higher than the bid $s-j$'s current owner is making, but lower than the bid of $s-j-1$'s owner.

In this point, we will introduce a definition that expresses a particular set of bids in which all agents are satisfied with their obtained slot.

Definition 3.1.1. *A Nash Equilibrium (NE) is a set of prices such that*

$$\theta_s(\upsilon_s - p_s) \geq \theta_t(\upsilon_s - p_t), \quad for \ \ t > s \qquad and \tag{3.1}$$

$$\theta_s(\upsilon_s - p_s) \geq \theta_t(\upsilon_s - p_{t-1}), \quad for \ \ t < s \tag{3.2}$$

where $p_t = b_{t+1}$.

In other words, when a set of bids is in an Nash equilibrium, every agent prefers his current slot to any alternative slot or he has nothing to gain if he changes his position. Also notice that, given values υ_s and click-through-rates θ_s, the above inequalities comprise a system and we can solve for the maximum and minimum equilibrium revenue, attainable by the auction, using a linear program.

However, there is another set of bids which is a subset of Nash Equilibria. It was first presented by Prof. Hal. Varian in [1] as Symmetric Nash equilibria but the early years surveys are reffered to as Envy-Free equilibria. For this reason, we start defining when a player envies another player.

Definition 3.1.2. *A player i assigned to slot s_i* envies *a player j assigned to a slot s_j, if i has strictly higher utility in j's place than his current slot. That is, $\theta_{s_i}(v_i - p_i) < \theta_{s_j}(v_i - p_j)$.*

$\star$ In chapter 5, we remodel this definition adding one more condition due to an additional parameter, a budget restristion for each player (definition (5.3.2), page 61).

Definition 3.1.3. *An assignment of goods (slots) associated with particular prices is* envy-free (or is in envy-free equilibrium) *if no winner*[1] *is envied by any other player. That is, $\theta_{s_i}(v_i - p_i) \geq \theta_{s_j}(v_i - p_j)$ for all i, j.*

Definition 3.1.4. *A mechanism is called* envy-free mechanism *if there are bids which lead to an envy-free assignment.*

But what is actually the main difference between envy-free and Nash equilibria?

3.1.1 Envy-Free versus Nash equilibrium

As I mention above, in Nash equilibria, moving in a lower slot, an agent will pay the price that the current agent of the new slot is paying. However, moving in a higher slot, the moving agent will pay the bid amount of the current owner of that slot.

In contrast, the definition of envy-free equilibria implies that when an agent moves to an another slot, either higher or lower, he pays the price of the

[1] A player with positive allocation is called winner.

agent who currenlty obtains the new slot. This means that in an envy-free assignment, in case of moving higher, a player pays less[2] comparatively to Nash equilibrium. However, he still prefers his original slot. This description helps us to understand that an envy-free assignment is a more restricted set than a Nash equilibrium, which motivates the following property.

Property 3.1.5 (EF $\subset$ NE)**.** *Assume that we have an envy-free assignment, then those prices represent also a Nash equilibrium.*

$\star$ We must mention here that this property is not satisfied when budget constraints exist, section 5.4 on page 62.

Proof. Assume that we have an Envy-Free equilibrium with k players and $k-1$ slots. After agents are ranked, we rename them according to their bids, $b_1 > b_2 > \dots b_n$.

Players	*Bids*	*Prices*	*Values*	*CTR*
1	b_1	$p_1 = b_2$	υ_1	θ_{s_1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	b_i	$p_i = b_{i+1}$	υ_i	θ_{s_i}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-1$	b_{k-1}	$p_{k-1} = b_k$	υ_{k-1}	$\theta_{s_{k-1}}$
k	b_k	$p_k = 0$	υ_k	no slot

By definition (3.1.3) of envy-free equilibrium we have: $\forall i \; \forall j, \quad \theta_{s_i}(\upsilon_i - p_i) \geq \theta_{s_j}(\upsilon_i - p_j)$.

We split the proof in two parts:

- If player i goes to a position below him, we have

$$\forall i \;\; \forall j (j > i), \quad \theta_{s_i}(\upsilon_i - p_i) \geq \theta_{s_j}(\upsilon_i - p_j). \tag{3.3}$$

  In order to have a PNE, we need to show that if agent i underbid agent j, he will prefer his initial position. Notice that in this case, agent i will pay agent's $j+1$ bid, $b_{j+1} = p_j$. That means, the inequality we must show is the same with (1), which is already true.

[2]Lower payment implies higher utility, $u_{\alpha s} = \theta_s(\upsilon_\alpha - p_\alpha)$

- If player i goes to a position above him, then we have

$$\forall i \ \ \forall j (j < i), \quad \theta_{s_i}(\upsilon_i - p_i) \geq \theta_{s_j}(\upsilon_i - p_j) \geq \theta_{s_j}(\upsilon_i - b_{j+1}) \tag{3.4}$$

In order to have a PNE, we need to show that if agent i overbid agent j, he will prefer his initial position. Notice that in this case, agent i will pay agent's j bid, b_j. That is,

$$\forall i \ \ \forall j (j < i), \quad \theta_{s_i}(\upsilon_i - p_i) \geq \theta_{s_j}(\upsilon_i - b_j) \tag{3.5}$$

By (3.4) and since $b_j > b_{j+1}$, we have that (3.5) is true.

By (3.3), (3.6), it is implied that this is also a Nash equilibrium. □

Property 3.1.6. *In an envy-free (Symmetric) equilibrium, $\upsilon_i \geq p_i$.*

Proof. By (3.1.3) and by the fact that $\theta_s = 0, \forall s > S$

$$\theta_{s_i}(\upsilon_i - p_i) \geq \theta_{s_{S+1}}(\upsilon_i - p_{S+1}) = 0$$

□

The following properties of Envy-Free equilibria are not satisfied by all Nash equilibria sets. In order to prove this argument, we found counterexamples which are analytically presented in appendix, page 65.

Property 3.1.7. *In an envy-free equilibrium, we have $\upsilon_{s-1} \geq \upsilon_s \ \ \forall s$.*

Proof. By definition of E-F we have

$$\begin{aligned} \theta_{s_i}(\upsilon_i - p_i) \ &\geq \ \theta_{s_j}(\upsilon_i - p_j) \Rightarrow \quad \upsilon_i(\theta_{s_i} - \theta_{s_j}) \geq p_i\theta_{s_i} - p_j\theta_{s_j} \\ \theta_{s_j}(\upsilon_j - p_j) \ &\geq \ \theta_{s_i}(\upsilon_j - p_i) \Rightarrow \quad \upsilon_j(\theta_j - \theta_i) \geq p_j\theta_{s_j} - p_i\theta_i \end{aligned}$$

Adding the two above inequalities, we have

$$(\upsilon_j - \upsilon_i)(\theta_{s_j} - \theta_{s_i}) \geq 0, \tag{3.6}$$

which means that υ_j and θ_{s_j} will have the same ordering. Assume that $j < i$, then $\theta_{s_j} > \theta_{s_i}$ by the auction model rules and $\upsilon_j > \upsilon_i$ by (3.6). □

However, this property is not satisfied over Nash equilibria sets.

▶ *Counterexample. [Appendix, page 65]*

The following table depicts the values, bids, prices of four agents in ranking and also, the CTRs of the slots assigned to each one of them.

Positions	Values	Bids	Prices	CTRs
1	$v_1 = 10$	$b_1 = 8$	$p_1 = 7$	$\theta_{s_1} = 0.4$
2	$v_2 = 11$	$b_2 = 7$	$p_2 = 4$	$\theta_{s_2} = 0.2$
3	$v_3 = 5$	$b_3 = 4$	$p_3 = 2$	$\theta_{s_3} = 0.15$
4	$v_4 = 4$	$b_4 = 2$	$p_4 = 0$	no slot

The next table shows that even if $v_2 > v_1$, the above set of bids represent a Nash equiliblium. The numbers in the first column represent the starting position, for instance, if the player in position 2 moves to position 3, his new utility will become 1.35 and if player in position 3 moves to 2, his utility will become -0.4. The (i, i) boxes represent the utilities in their original positions.

Positions	1	2	3	4
1	1.2	1.2	1.2	0
2	1.2	1.4	1.35	0
3	-1.2	-0.4	0.45	0
4	-1.6	-0.6	0	0

We observe that for every row i, we have $(i, i) \geq (i, j) \quad \forall j (j \neq i)$. This means that every player prefers his slot to any alternative one.

◀

Property 3.1.8. *(One step solution). If each player at his slot, say s, does not envy his neighbors at slots $s+1$ and $s-1$, then no player envies any other player.*

Proof.
Suppose that the inequality in defintion (3.1.3), on page 29, holds for slots $s-1$ and s, and for slots s and $s+1$. Then we show that it holds also for $s-1$ and $s+1$. By definition (3.1.3) of E-F equibrium, we have

$$
\begin{aligned}
s-1 \rightsquigarrow s: \quad & v_{s-1}(\theta_{s-1} - \theta_s) \geq p_{s-1}\theta_{s-1} - p_s\theta_s && (3.7)\\
s \rightsquigarrow s+1: \quad & v_s(\theta_s - \theta_s) \geq p_s\theta_s - p_{s+1}\theta_{s+1} && \Rightarrow^{v_{s-1} > v_s}\\
& v_{s-1}(\theta_s - \theta_s) \geq p_s\theta_s - p_{s+1}\theta_{s+1} && (3.8)
\end{aligned}
$$

Adding (3.7) and (3.8), we have

$$
s-1 \rightsquigarrow s+1: \quad v_{s-1}(\theta_{s-1} - \theta_{s+1}) \geq p_{s-1}\theta_{s-1} - p_{s+1}\theta_{s+1}
$$

We show the opposite direction following the same way.

$$
\begin{aligned}
s+1 \rightsquigarrow s: \qquad & \upsilon_{s+1}(\theta_{s+1}-\theta_s) \geq \ p_{s+1}\theta_{s+1} - p_s\theta_s \qquad (3.9)\\
s \rightsquigarrow s-1: \qquad & \upsilon_s(\theta_s-\theta_{s-1}) \geq \ p_s\theta_s - p_{s-1}\theta_{s-1} \quad \Rightarrow^{\upsilon_s > \upsilon_{s+1}}_{\theta_s-\theta_{s-1}<0}\\
& \upsilon_{s+1}(\theta_s-\theta_{s-1}) \geq \ p_s\theta_s - p_{s-1}\theta_{s-1} \qquad (3.10)
\end{aligned}
$$

Adding (3.9) and (3.10), we have

$$
s+1 \rightsquigarrow s-1: \qquad \upsilon_s(\theta_{s+1}-\theta_{s-1}) \geq \ p_{s+1}\theta_{s+1} - p_{s-1}\theta_{s-1}
$$

□

We show again by a counterexample that this property is not satisfied by all Nash equilibria.

▶ *Counterexample. [Appendix, page 67]*

The following table depicts the values, bids, prices of four players in ranking and also, the CTRs of the slots assigned to each one of them.

Positions	Values	Bids	Prices	CTRs
1	$\upsilon_1 = 10$	$b_1 = 8$	$p_1 = 7$	$\theta_{s_1} = 0.4$
2	$\upsilon_2 = 11$	$b_2 = 7$	$p_2 = 4$	$\theta_{s_2} = 0.2$
3	$\upsilon_3 = 5$	$b_3 = 4$	$p_3 = 1.9$	$\theta_{s_3} = 0.15$
4	$\upsilon_4 = 4$	$b_4 = 1.9$	$p_4 = 0$	no slot

The next table shows that even if the inequalities of Nash equilibria hold for every slot and its neighbor slots, they are not satisfied by slots who are more than one step away. In this example, they are not satisfied for slots 1 and 3.

Positions	1	2	3	4
1	1.2	1.2	1.215	0
2	1.2	1.4	1.365	0
3	-1.2	-0.4	0.45	0
4	-1.6	-0.6	0	0

We observe that if every player moves to a neighbor slot, either lower or higher, he does not prefer the new position since his utility at the original slot is at least equal to his utility at any of its neighbor slots. However, this is not true if agent 1 moves to position 3 in which his utility becomes 1.215 from 1.2 that was before.

◀

Property 3.1.9. *In an envy-free equilibrium, we have* $p_{i-1}\theta_{s_{i-1}} > p_i\theta_{s_i}$ *and* $p_{i-1} > p_i, \quad \forall i$.

Proof.
By definition (3.1.3) we have

$$(\upsilon_i - p_i)\theta_{s_i} \geq (\upsilon_i - p_{i-1})\theta_{s_{i-1}} \Rightarrow$$
$$p_{i-1}\theta_{s_{i-1}} \geq p_i\theta_i + \upsilon_i(\theta_{s_{i-1}} - \theta_{s_i}) > p_i\theta_{s_i}.$$

Applying $\upsilon_i \geq p_i$ in (3.1) we have

$$p_{i-1}\theta_{s_{i-1}} > p_i\theta_{s_i} + \upsilon_i(\theta_{s_{i-1}} - \theta_{s_i}) \geq p_i\theta_{s_i} + p_i(\theta_{s_{i-1}} - \theta_{s_i}) = p_s\theta_{s_{i-1}}.$$

□

However, this property is not satisfied by all Nash equilibria sets.

▶ *Counterexample.* *[Appendix, page 68]*

The following table depicts the values, bids, prices of four players in ranking and also, the CTRs of the slots assigned to each one of them.

Positions	Values	Bids	Prices	CTRs
1	$\upsilon_1 = 10$	$b_1 = 8$	$p_1 = 7$	$\theta_{s_1} = 0.4$
2	$\upsilon_2 = 11$	$b_2 = 7$	$p_2 = 4$	$\theta_{s_2} = 0.2$
3	$\upsilon_3 = 5$	$b_3 = 4$	$p_3 = 4$	$\theta_{s_3} = 0.15$
4	$\upsilon_4 = 4$	$b_4 = 4$	$p_4 = 0$	no slot

In contrast to envy free equilibria , the next table shows that in a set of Nash equilibrium, it is possible for equal[3] prices to exist.

Positions	1	2	3	4
1	1.2	1.2	0.9	0
2	1.2	1.4	1.05	0
3	-1.2	-0.4	0.15	0
4	-1.6	-0.6	0	0

◀

Hal Varian proved in [1] that pure strategy Nash equilibria can be found by the recursive forms below. Actually, they represent intervals, upper and

[3] In case of equal biddings, the agents are usually ranked by lexicographic order.

lower bounds, in which sequences of bids leads to envy free assignments and therefore, to Nash equilibria sets.

$$b_s^U \theta_{s-1} = \upsilon_{s-1}(\theta_{s-1} - \theta_s) + b_{s+1}\theta_s$$
$$b_s^L \theta_{s-1} = \upsilon_s(\theta_{s-1} - \theta_s) + b_{s+1}\theta_s$$

The solutions to the above recursions are:

$$b_s^U \theta_{s-1} = \sum_{t \geq s} \upsilon_{t-1}(\theta_{t-1} - \theta_t)$$
$$b_s^L \theta_{s-1} = \sum_{t \geq s} \upsilon_t(\theta_{t-1} - \theta_t) \tag{3.11}$$

The starting point of the recursive forms follow from the fact that $\theta_s = 0, \quad \forall s > S$. Then $b_{S+1}^L \theta_S = \upsilon_{S+1}(\theta_S - \theta_{S+1}) = \upsilon_{S+1}\theta_S \Rightarrow b_{S+1}^L = \upsilon_{S+1}$ which meams that it is optimal for the first excluded player to bid his value. If you are excluded, it is pointless to bid lower than your value and, especially for the first excluded player, there is a chance for him to join again the auction due to a possible dropping out of another higher player.

At this point, we observe the following result. The solution of the lower bound recursion (3.11) equals to the payment of VCG mechanism (2.4), on page 24. In other words, there is a Pure Nash Equilibrium in Ad Auctions game with the same outcome and payment as VCG.

Equally important is that the upper recursive solution for the EF (envy-free) revenue is the same as the maximum revenue for the NE, while the minimum revenue for NE is less than the solution to the lower recursion of EF equilibria. Therefore,

$$\text{max revenue NE} = \text{value of upper recursion of EF } \geq$$
$$\text{value of EF lower recursion} \geq \text{min NE revenue.}$$

3.1.2 Equilibrium Hierarchy in GSP

Consider a function $\pi : \mathbb{N} \longrightarrow \mathbb{N}$ that assigns slot i to player $\pi(i)$. According to [11], we have the following.

Definition 3.1.10. *An equilibrium is* efficient *if it maximises social welfare, which occurs when $\pi(i) = i$ for all i.*

Even though all envy-free equilibria are efficient, there are also efficient equilibria which are not envy-free, as well as inefficient equilibria. Therefore, we have the following hierarchy [11]:

$$\left\{ \begin{array}{c} \text{VCG} \\ \text{outcome} \end{array} \right\} \subseteq \left\{ \begin{array}{c} \text{envy-free} \\ \text{equilibria} \end{array} \right\} \subseteq \left\{ \begin{array}{c} \text{efficient} \\ \text{Nash} \\ \text{equilibria} \end{array} \right\} \subseteq \left\{ \begin{array}{c} \text{all Nash} \\ \text{equilibria} \end{array} \right\}.$$

Chapter 4

Generalized Second Price Ad Auctions under Budget Constraints

> "Don't tell me what you value, show me your budget and I' ll tell you what you value."
>
> -Joe Biden

An interesting research line arises if we add one more parameter to the Ad Auction models we previously analysed, which is a budget constraint for each player. Even though budgets are a main feature of all actual Ad Auctions, the most of the works completely ignore this issue. In this chapter, firstly, we present the budget auction model in [2] along with the results about the existence and properties of its pure Nash equillibria. Secondly, based on an omission we observed in the induction process of the main theorem's proof in [2], we described a counterexample for the claim that was used as base in the induction. For this purpose, we also slightly remodeled the definition of critical bid. Moreover, we also found a counterexample for an additional theorem in [3] and our positive result is a condition under which pure Nash equilibria exist. This condition is simpler and does not involve the critical bids.

4.1 The Budget Auction Model

Consider k agents, $K = \{1, \ldots, k\}$ who bid $\vec{b} = (b_1, b_2, \ldots, b_k)$ to buy N identical divisible items. Each agent $i \in K$ has two private values: (ı) a *value* v_i, which is his personal evaluation of item i and (ıı) his true *budget* $\hat{B}_i$, which is the restriction on the amount of money he can use each day. We use x_i to define the amount of items he receives and p_i the amount he pays per item. The utility of each agent is given by $u_i = x_i(\upsilon_i - p_i)$, subject to $x_i p_i \leq \hat{B}_i$ which means that he can afford x_i items. In case of exceeding his budget ($x_i p_i > \hat{B}_i$), his utility becomes $u_i = -\infty$.

Ranking

Initially, the auctioneer sets a minimum price p_{min}. After all agents submit their bid b_i ($b_i \in [p_{min}, \upsilon_i]$)[1] and their budget B_i, he renames the agents according to their bid, such that $b_1 \geq b_2 \geq \cdots \geq b_k$. In the case of equal bids, he first sell items to agent with the lower original index (lexicographic order).

Charge

The auctioneer runs a sequence of Second Price Auction (section 2.2), which means that each agent pays per item the bid of the agent exactly below him. Due to the minimum price p_{min} that auctioneer sets, we have $p_i = max\{b_{i+1}, p_{min}\}, \forall i \in K$ per item. The total price agent i has to pay is $p_i x_i$ (for the x_i items he receives). If he gets no items, then $p_i = 0$.

Allocation

The items agent i will receive depends on (ı) his budget, (ıı) his price per item and (ııı) on how many items are left by the first buyers. Thus, the allocation of agent i is given by

$$x_i = \min(N - \sum_{j=1}^{i-1} x_j, \frac{B_i}{p_i}). \tag{4.1}$$

The actual allocation process is the following. Initially, agent 1 starts to buy items paying $p_1 = \max\{b_2, p_{min}\}$ and he stops when he runs out of budget or items. In case of running out of his budget and there are still items for sale, agent 2 starts to buy paying $p_2 = \max\{b_3, p_{min}\}$. This procedure is continued until all items all sold or all agents exhaust their budget. If all items are sold before agent's i turn to buy, then $x_i = 0,\ \forall i(i \geq k)$.

$\star$ The aggregate allocation of items to agents will never exceed the total items ($\sum_{i \in K} x_i \leq N$).

[1] According to [2], bids are not always conservative. It is mentioned that, theoretically, an agent i can bid above his value making the agent above him pay more and taking more items for himself. Indeed such a choice may lead to a profit for agent i. However, it is a risk since the agent above him may underbid i. If this happens, agent i pays a price higher than his value which yields to an unsatisfactory transaction.

Categories of Agents

Definition 4.1.1 (winner)**.** *An agent i is called* winner *when $x_i > 0$ and he is not the last one with positive allocation.*

A winner agent always exhausts his budget since at least one more agent ranked below him has a positive allocation. This means that for every winner we have

$$x_i \cdot p_i = \hat{B}_i.$$

Definition 4.1.2 (border)**.** *An agent i is called* border *when $x_i > 0$ and he is the last agent with positive allocation.*

If agent i is a border agent, no player ranked below him (in case he is not the last ranked agent) gets any items, so $\forall j > i, x_j = 0$. Moreover, a border agent can either exhausts his budget or not, having that

$$x_i \cdot p_i \leq \hat{B}_i.$$

Definition 4.1.3 (loser)**.** *An agent i is called* loser *when he gets no item, $x_i = 0$.*

Every agent who is ranked below the border agent, is a loser.

How it works for 2 agents

In the following example, I describe in details how this budget auction model works for two players.

►*Example.*

Consider two agents, $K = \{1, 2\}$, a minimun price p_{min}, N identical divisible items and a vector $\vec{b} = (b_1, b_2)$. Assume that $b_1 > b_2$ then, according to the previous model, the agents' ranking and payments are:

1	b_1	v_1	$p_1 = b_2$	B_1
2	b_2	v_2	$p_2 = p_{min}$ or 0	B_2

Depending on the combination of agents' types, we have the following cases.

- *agent 1: winner*
  agent 2: border
  In this case, agent 1 will first start buying items. According to equation

(4.1), his allocation will be $x_1 = min(N, \frac{B_1}{p_1}) = \frac{B_1}{p_1} = \frac{B_1}{b_2}$ and his utility will be $u_1 = x_1(v_1 - p_1) = \frac{B_1}{b_2}(v_1 - b_2)$.

When agent 1 exhausts his budget, agent 2 will start buying items. As a border, he will buy all the remaining items. So, his allocation will be $x_2 = N - \frac{B_1}{p_1} = N - \frac{B_1}{b_2}$ and his utility $u_2 = x_2(v_2 - p_2) = (N - \frac{B_1}{b_2})(v_2 - p_{min})$.

⋆ Notice that $N - \frac{B_1}{b_2}$ may be equal to $\frac{B_2}{p_{min}}$ which means that the items are left by the first ranked player may be equal to the items that the second ranked player can buy with his whole budget. In this case, he also exhausts his budget.

- *agent 1: border*
  agent 2: loser
  In this case, similarly agent 1 will first start buying items. As a border, he is the last one with positive allocation so, he will buy all the items. Thus, his allocation will be $x_1 = min(N, \frac{B_1}{p_1}) = N$ and his utility will be $u_1 = x_1(v_1 - p_1) = N(v_1 - b_2)$. Since all items are sold before agent's 2 turn to buy, we have that $x_2 = 0$ which means that also his utility and payment are zerow, $u_2 = 0$ and $p_i = 0$.

◀

4.2 Preliminaries

In a Pure Nash Equilibrium, no agent can increase his utility by changing unilaterally his bid b_i and budget B_i. According to [2], we present the definition of a dominant strategy. We use symbolizations b_{-i} and B_{-i} for all the bids and budgets, respectively, except for agent's i.

Definition 4.2.1. *(In [2]) Submitting budget y is a* dominant strategy *for agent i if for any bid vector $\vec{b}$, any alternative budget y' and B_{-i}, we have that $u(x_i, p_i) \geq u(x'_i, p'_i)$, where x_i and p_i (x'_i and p'_i, respectively) are the allocation and price under bids $\vec{b}$ and budgets (B_{-i}, y) ((B_{-i}, y') respectively). Submitting bid z is a* dominant strategy *for agent i if for any submitted budgets $\vec{B}$, for any alternative bid z' and b_{-i}, we have that $u(x_i, p_i) \geq u(x'_i, p'_i)$, where x_i and p_i (x'_i and p'_i, respectively) are the allocation and price under budgets $\vec{B}$ and bids (b_{-i}, y) ((b_{-i}, y') respectively).*

Next, we present the definitions of the demand of a player and the Market Equilibrium price.

Definition 4.2.2. *(In [2]). The* demand *of agent i at price p is an interval*

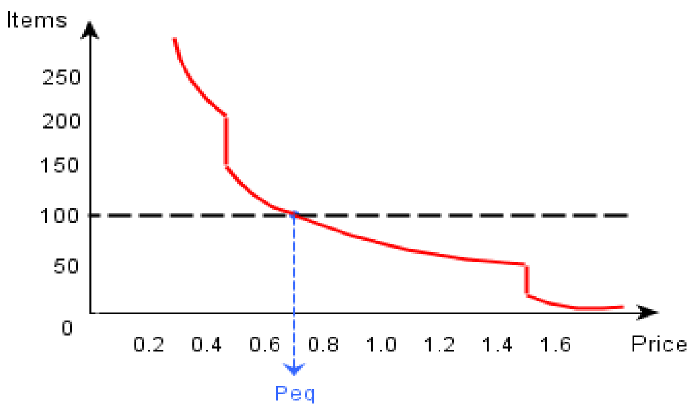


Figure 4.1: Market Equilibrium Price

$\mathcal{D}(p)$ *or a point, as follows,*

$$\mathcal{D}_i(p) = \begin{cases} \frac{B_i}{p} & \text{if } v_i > p, \\ 0 & \text{if } v_i < p, \\ [0, \frac{B_i}{p}] & \text{if } v_i = p. \end{cases}$$

In the second case, the price he is asked to pay is strictly higher than his value, hence, he does not demand any item. In third case, the agent does not care about how many items he will buy because, for any amount of items his profit will be zero, $u_i = x_i(v_i - p_i)$. Thus, his "demand" is the interval $[0, \frac{B_i}{p}]$. We use $\mathcal{D}(p)$ for the aggregated demand of all agents at price p, such that $\mathcal{D}(p) = \sum_{i \in K} \mathcal{D}_i(p)$.

Definition 4.2.3. *The* Market Equilibrium Price p_{eq} *is the price point where the total supply equals to the aggregated demand of items.*

Notice that if $N \in \mathcal{D}_(p)$ then p is the Market Equilibrium Price p_{eq}. Moreover, we observe that function $\mathcal{D}(p)$ is decreasing in p which means that p_{eq} is a unique point.

►*Example.*

In figure 4.1 we observe that the Market Equilibrium price is 0.75 which is the point that the total supply equals the aggregated demand, as we previously said in definition 4.2.3. We also observe some intervals in which the drops are vertical, i.e. for price 0.5. This happens when the values equals to prices and, thus, agents are indifferent to any allocation amount between $[150, 200]$ for price 0.5, as we have explained in the third case of definition 4.2.2 on page 40.

◀

4.3 Properties of the budget auction model

This mechamisn does not optimise the Social Welfare and thus, it is not efficient. The following properties concerns actions which are either dominant strategies or not. In section 2.5.2, we had proved that in GSP mechanism is not a dominant strategy for an agent to bid his true value. In this chapter, even if we have the additional budget constraints, it is proven again.

Property 4.3.1. *For an agent i, bidding his true value v_i is not a dominant strategy.*

Proof.
(By an example, in [2]).

Consider four agents competing for $N = 100$ items. We focus in agent B showing that submitting his true value is not a dominant strategy. The table below depicts their budgets B_i, values v_i, submitted budgets B_i, bids b_i, allocations x_i, prices per item p_i, utilities u_i and last, the type of each agent. The items being on sale are and the minimum price is $p_{min} = 0$.

Agents	$\hat{B}_i$	v_i	B_i	b_i	p_i	x_i	u_i	agent type
A	20	2	20	1	1	20	20	winner
B	25	1.5	25	1	0.5	50	50	winner
C	30	1.5	20	0.5	0.3	30	36	border
D	20	0.5	20	0.3	0	0	0	loser

If agent B bids his value $b_B = 1.5$, he will be ranked first as we see in the next table.

Agents	$\hat{B}_i$	v_i	B_i	b_i	p_i	x_i	u_i	agent type
B	25	1.5	25	1.5	1	20	20	winner
A	20	2	20	1	0.5	50	50	winner
C	30	1.5	20	0.5	0.3	30	36	border
D	20	0.5	20	0.3	0	0	0	loser

His allocation now is $x_B = \frac{B_B}{p_B} = \frac{25}{1} = 25$. Therefore, his utility is: $u_B = x_B(v_B - p_B) = 25(1.5 - 0.5) = 25$ which is lower than before ($u_B = 50$). □

Property 4.3.2. *For an agent i, bidding his true budget $\hat{B}_i$ is a dominant strategy.*

Proof.
Each agent i reports his bid and budget to participate in the auction and his utility becomes $x_i(v_i - p_i)$ when $x_i \cdot p_i \leq B_i$. We see that utility depends on values v_i, prices p_i and allocation x_i. The budgets does not affect the ranking order and thus, neither the prices. However, they affect the allocation of agent i. Assume now that agent i wants to change his budget from $\hat{B}_i$ to B_i. Let $\hat{x}_i$, $\hat{u}_i$ and x_i, u_i, be the allocation and utility when he submits B_i and $\hat{B}_i$, respectively. Then, we have the following cases:

- $B_i < \hat{B}_i$.
  Assume that agent i submits a lower budget B_i than his true one $\hat{B}_i$ and receives x_i items. By the allocation form in section 4.1, we have

$$x_i = min(N - \sum_{j=1}^{i-1} x_j, \frac{B_i}{p_i}) \qquad \text{and} \qquad \hat{x}_i = min(N - \sum_{j=1}^{i-1} \hat{x}_j, \frac{\hat{B}_i}{p_i}). \tag{4.2}$$

  All the agents who are ranked above agent i are not effected by his budget misreport. Therefore, $x_j = \hat{x}_j$ for $j < i$ which implies that $N - \sum_{j=1}^{i-1} x_j = N - \sum_{j=1}^{i-1} \hat{x}_j$. Due to the inequality $B_i < \hat{B}_i$, we have that $\frac{B_i}{p_i} < \frac{\hat{B}_i}{p_i}$ or better, $x_i \leq \hat{x}_i$. This means that also $u_i \leq \hat{u}_i$.

- $B_i > \hat{B}_i$.
  Assume that agent i submits a higher budget B_i than his true one $\hat{B}_i$. Then, there are two cases. Either he will receive the same number of items or a higher number.

  - $x_i = \hat{x}_i$. In this case, he submits a higher budget but still he gets the same items. This happens if he is a border or a loser. Since $\hat{x}_i \cdot p_i \leq \hat{B}_i$, we have also that $x_i \cdot p_i \leq \hat{B}_i$ which means that he does not even exhaust his real budget, implying that $u_i \leq \hat{u}_i$.

  - $x_i > \hat{x}_i$. In this case, he sumbits a higher budget and he gets more items. This means that he had exhausted his true budget. That is, $\hat{x}_i \cdot p_i = \hat{B}_i$ which implies that $x_i \cdot p_i > \hat{B}_i$. Thus, $u_i = -\infty < \hat{u}_i$.

□

4.4 Properties of a Pure Nash Equilibrium

According to Arnon's work in [2], any PNE must have some properties. One of them is that all winner agents must pay the same price p, which implies that every winner and the border agent must bid the same amount. We observe here that the first winner may bid higher than the others since his bid has no impact to his price ($p_1 = b_2$). More precisely,

Claim 4.4.1. *In any PNE, all winner agents pay the same price p, the border agent pays a price $p' \leq p$, and any loser agent j (if it exists) has value $v_j \leq p$. In addition, p is at most the Market Equilibrium price, $p \leq p_{eq}$.*

Proof.
Assume that there is a PNE with two[2] winners who pay different prices. Say p_1 the price paid by the first ranked winner agent 1 and, say p_j, the first different price paid by a lower in ranking winner agent j, $p_j < p_i$. it is proven that agent 1 can increase his utility which contradicts the fact that we have a PNE. Since all agents above agent j are winners, we have that for any $i \geq j$, $x_i = \frac{B_i}{p_i}$ and $\sum_{i=1}^{i=j} x_i \leq N \Rightarrow x_1 \leq N - \sum_{i=2}^{j} x_i$. Assume that agent 1 drops down in order to be ranked in position j (by bidding $b_j - \epsilon$). His allocation becomes $x'_i = min(N - \sum_{j=2}^{j} x_j, \frac{B_1}{p_j})$. However, $\frac{B_1}{p_j} > \frac{B_1}{p_1} = x_1$ and $N - \sum_{i=2}^{j} x_i > x_1$, so $x'_1 \geq x_1$. Being ranked in a lower position, agent i pays less. In addition, he receives at least the same items as before. Hence, he strictly increases his utility which contradicts that this is a PNE. Thus, all the winners pay the same price p.

The border agent is ranked after all the winners, so he pays a price $p' \leq p$. Regarding the losers, assume that a loser had a value such that $v_i > p$. Then he could bid an amount $p + \epsilon$ and be a winner having a positive utility. But, as a loser, his utility is zero and this contradicts the fact that it is a PNE. So, for a loser agent, we have that $v_i \leq p$.

Regarding the Market Equilibrium Price p_{eq}, assume that there is a $p > p_{eq}$ which yields in a PNE. Since the prices are in PNE, the utility of the winner j who pays $p > p_{eq}$, is at least equal if he was the border agent paying $p' \leq p$. Thus,

$$\frac{B_j}{p}(v_j - p) \geq (N - \sum_{i \in S - \{j\}} \frac{B_i}{p})(v_j - p'), \quad \text{where} \quad S = \{i : v_i \geq p\}.$$

Since $(v_j - p) < (v_j - p')$, we have that

$$\frac{B_j}{p} \geq (N - \sum_{i \in S - \{j\}} \frac{B_i}{p}) \quad \Rightarrow \quad \sum_{i \in S} \frac{B_i}{p} \geq N \tag{4.3}$$

[2]For one winner the claim holds trivially

Since $p > p_{eq}$ and the function of demand $\mathcal{D}$ is decreasing in p, we have that $\mathcal{D}(p) < \mathcal{D}(p_{eq})$. By the fact that $N \in \mathcal{D}(p_{eq})$ (note under definition 4.2.3 pg.) and $\sum_{i \in S} \frac{B_i}{p} \in \mathcal{D}(p)$, we have $\sum_{i \in S} \frac{B_i}{p} < N$ which contradicts the inequality (4.3). □

The next claim concerns the strategy of loser agents.

Claim 4.4.2. *If loser agents are restricted to bid their true value and budget then there exists a budget auction in which there is no PNE.*

Proof. (Example in [2])

In this example, we saw that the loser agent D bids his true value. However, the border agent C has higher utility if he underbids agent D, as we can see at the second table. Thus, this is not a PNE.

Agents	$\hat{B}_i$	v_i	B_i	b_i	p_i	x_i	u_i	agent type
A	40	2	40	1.143	1.143	35	30	winner
B	40	2	40	1.143	1.143	35	30	winner
C	40	2	40	1.143	1	30	30	border
D	8	1	8	1	0	0	0	loser

Agents	$\hat{B}_i$	v_i	B_i	b_i	p_i	x_i	u_i	agent type
A	40	2	40	1.143	1.143	35	30	winner
B	40	2	40	1.143	1	40	40	winner
D	8	1	8	1	$1-\epsilon$	8	0	winner
C	40	2	40	$1-\epsilon$	0	17	34	border

□

4.5 Critical Bid

Let the function $f_j(x)$ be agent's j utility if he bids x, is ranked first and pays x:

$$f_j(x) = \begin{cases} N(\upsilon_j - x) & \text{if } p_{min} \leq x \leq \frac{B_j}{N} \quad \text{(j: border)} \\ \frac{B_j}{x}(\upsilon_j - x) & \text{if } \frac{B_j}{N} < x \quad \text{(j: winner)} \end{cases}$$

Let the function $g_j(x)$ be agent's j utility if he bids x, ranks last and pays p_{min}:

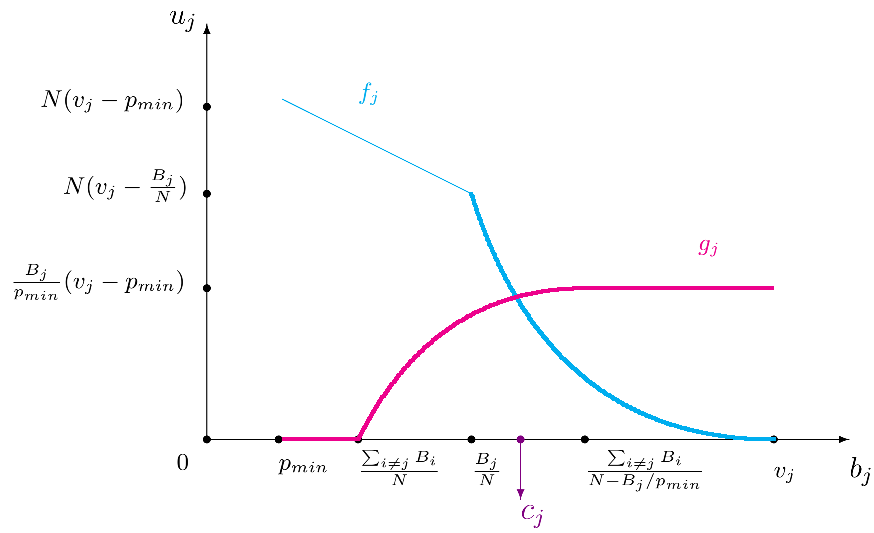
$$g_j(x) = \begin{cases} 0 & \text{if } p_{min} \leq x \leq \frac{\sum_{i \neq j} B_i}{N} \quad \text{(loser)} \\ (N - \frac{\sum_{i \neq j} B_i}{x})(\upsilon_j - p_{min}) & \text{if } \frac{\sum_{i \neq j} B_i}{N} < x < \frac{\sum_{i \neq j} B_i}{N-(B_j/p_{min})} \quad \text{(border)} \\ \frac{B_j}{p_{min}}(\upsilon_j - p_{min}) & \text{if } \frac{\sum_{i \neq j} B_i}{N-(B_j/p_{min})} \leq x \quad (\text{border} \mid N - \frac{\sum_{i \neq j} B_i}{x} \leq \frac{B_j}{p_{min}}) \end{cases}$$

The following properties are satisfied:

(i) Both functions are continuous in the range $[p_{min}, \upsilon_j]$.
(ii) Function f_j is (strictly) decreasing and g_j is (weakly) increasing in x.
(iii) $f_j(p_{min}) \geq g_j(p_{min})$.
(iv) $g_j(\upsilon_j) \geq f_j(\upsilon_j) = 0$.

The conclusion is that these two functions intersect in a unique point, as the following picture shows, which motivates the following definition.

Definition 4.5.1. *Consider an auction with k agents and a minimum price* p_{min}*. The* critical bid *for agent* j*,* $x = c_j(k, p_{min})$*, is the intersection of the two above functions,* f_j *and* g_j*. Therefore, when all agents bid* $\vec{b} = (x, x, \ldots, x)$ *, then agent j is indifferent between the top rank (being a winner or a border) and the bottom rank (being a border or a loser).*
($\star$) In addition, $f_j(x) > g_j(x)$ *for* $x < c_j$ *and* $f_j(x) < g_j(x)$ *for* $x > c_j$*. In other words, for* $x < c_j$*, agent* j *prefers top rank and, for* $x > c_j$*, agent* j *prefers bottom rank.*



Critical bid for all cases

According to the interval in which the two utility functions, $g_j(x)$ and $f_j(x)$, intersect, we have the cases of bottom and top rank, right below. For each case, we present the critical bid, $c_j(k, p_{min}) = x$.

Cases:

- (i) *top rank:* 'border'
  bottom rank: 'loser'

$$N(v_j - x) = 0 \quad \Leftrightarrow$$
$$x = v_j.$$

- (ii) *top rank:* 'border'
  bottom rank: 'border'

$$N(v_j - x) = (N - \frac{\sum_{i \neq j} B_i}{x})(v_j - p_{min}) \quad \Leftrightarrow$$
$$Nx^2 - Np_{min}x - \sum_{i \neq j} B_i(v_j - p_{min}) = 0.$$

$$\text{Since} \quad \Delta = N^2 p_{min}^2 + 4N \sum_{i \neq j} B_i(v_j - p_{min}), \tag{4.4}$$

$$\text{we have} \quad x = \frac{Np_{min} + \sqrt{\Delta}}{2N}, \; x \in [p_{min}, v_j] \tag{4.5}$$

- (iii) *top rank:* 'border'
  bottom rank: 'border who exhausts his budget $(N - \frac{\sum_{i \neq j} B_i}{x} = \frac{B_j}{p_{min}})$'

$$N(v_j - x) = \frac{B_j}{p_{min}}(v_j - p_{min}) \overset{N \neq \frac{B_j}{p_{min}}}{\iff}$$
$$x = v_j - \frac{B_j}{N \cdot p_{min}}(v_j - p_{min}) \tag{4.6}$$

- (iv) *top rank:* 'winner'
  bottom rank: 'loser'

$$\frac{B_j}{x}(v_j - x) = 0 \quad \Leftrightarrow \text{(The allocation of the first ranked agent is always positive.)}$$
$$x = v_j.$$

- (v) *top rank:* 'winner'
  bottom rank: 'border'

$$\frac{B_j}{x}(v_j - x) = (N - \frac{\sum_{i \neq j} B_i}{x})(v_j - p_{min}) \quad \Leftrightarrow$$
$$x = \frac{\sum_{i \neq j} B_i(v_j - p_{min}) + B_j \cdot v_j}{N(v_j - p_{min})}.$$

- (vi) *top rank:* 'winner'
  bottom rank: 'border who exhausts his budget $(N - \frac{\sum_{i \neq j} B_i}{x} = \frac{B_j}{p_{min}})$'

$$\frac{B_j}{x}(v_j - x) = \frac{B_j}{p_{min}}(v_j - p_{min}) \quad \Leftrightarrow$$
$$x = p_{min}.$$

⋆ Notice that in every case except for (vi), the critical bid $c_j(k, p_{min}) = x$ depends on agent's private value v_j.

4.6 Existence of Pure Nash Equilibrium

In this section, firstly, we present the main result of [2] which concerns the existence of PNE in the general budget auction model for any number of agents. Secondly, we present our counterexample for this theorem. More specifically, the main theorem is proven by an induction and our counterexample concerns the induction base. In addition, our observation has impacts on the next steps of the proof induction, the general case. The actual main theorem is the following.

Theorem 4.6.1. *There exists a PNE for any number of agents, where agents submit their true budgets ($\hat{B}_i = B_i$) and bid at most their value $b_i \leq v_i$.*

To prove this theorem, they use a specific induction kind which goes as follows. They start by proving the existence of PNE for two agents and for multiple agents with identical budgets. Then, they prove that even if we add one more agent, there is still a PNE (induction step). Last, using all the previous steps, they prove the PNE existence for any number of agents (general case). The actual process is the following.

- *Base.*
  (Two agents)
  Assume that we have two agents with $c_2 \leq c_1$. Then any bids $b_1 = b_2 \in [c_2, min\{v_2, c_1\}]$ are a PNE, and those are the only PNEs where agents submit their true budget.
  (Claim [4.7], page17 in [2])

  (Multiple agents with Identical Budgets.)
  There exists a PNE for any number of agents with identical budgets, where each agent bids $b_i \in [pmin, v_i]$, loser agents bid $b_i = v_i$, and agent submit their true budget ($B_i = \hat{B}_i$). (Theorem [4.9], page18 in [2])

- *Induction Step.*
  (Increasing the numbers of agents by one.)
  Let $\vec{b_1}$ be a PNE with $h$agents and minimum price p_min, such that all winner agents pay price p. If there is a new agent $h+1$ such that (a) $v_h \geq vh + 1$, and (b) For every $i \in S_h$ the new critical bid $c_i(S_{h+1}, p_{min}) \geq v_{h+1}$, then we can define a $\vec{b_2}$ which is a PNE for S_{h+1} with the same minimum price p_{min}, where agent $h+1$ is a loser agent.
  (Claim [4.12], page 20 in [2])

- *Result.*
  (General case)
  Using all the previous steps and additional lemmas, they prove the general case of a budget auction for any numbers of agents (Theorem 5.5.1).
  (Theorem [4.13], page 22 in [2])

4.7 Counterexample 1

We recall here the claim that was proved as the induction base of the main theorem's proof.

Claim 4.7.1. *[4.7. p.17 in [2]] Assume that we have two agents with* $c_2 \leq c_1$*. Then any bids* $b_1 = b_2 \in [c_2, min\{v_2, c_1\}]$ *are a PNE, and those are the only PNEs where agents submit their true budget.*

Our first observation in the proof of this claim was the following. It is assumed that the two agents are ranked by the auctioneer according to their critical bid, instead of their names. However, this is impossible since the critical bid is a private information. It depends on the private value v_i[3], as we have noticed at section 4.5, pg 47-48. According to the original budget auction rules, in the case of identical bids, the tie is broken using the lexicographic order. This means that the agents, after their bidding, they are renamed according to their name. The agent with the lower original index will first start buying items. Therefore, we conclude that the order of critical bids respects the order of agents' names which does not seem to be legitimate.

▶ *Counterexample 1.*

Assume that we have two agents, A and B, with $c_A \leq c_B$ and $b_A = b_B \in [c_A, min\{v_A, c_B\}]$. Since $b_A = b_B$, the auctioneer uses the lexicographic order. Thus, agent A is ranked first and agent B is ranked second.

A	$b_A = b_B$
B	$b_B = b_A$

According to claim 4.7.1, any identical bids in the interval $[c_A, min\{v_A, c_B\}]$ are a PNE. We choose $b_A = b_B \in \big(c_A, min\{v_A, c_B\}\big)$. Therefore, $c_A < b_A$ which implies, by definition 4.5.1($\star$), that agent A prefers bottom. Moreover, since $b_B < min\{v_A, c_B\} \leq c_B$, we have also that $b_B < c_B$ which implies, by definition 4.5.1($\star$), that agent B prefers top rank. So, both agents want to change their positions which contradicts the claim that this is a PNE.

What if their bid is on the boundaries of the interval $[c_A, min\{v_A, c_B\}]$? In these cases, only one of the agents prefers to change his position which still implies that there is no PNE. Assume that:

- $b_A = b_B = c_A$. Then, $b_B < c_B$ and, as previously, agent B prefers top rank.
- $b_A = b_B = min\{v_A, c_B\}$

[3] Only in case v_i, the critical bid does not depend on the private value v_i.

- $b_A = b_B = v_A < c_B$. Similarly, agent B prefers top rank.
- $b_A = b_B = c_B$. Then, $b_A > c_A$ and agent A prefers bottom.

Therefore, in the case that agent with the lower original index has the minimum critical bid, there is no PNE.

◀

We should mention here that the only case PNE exists in claim 4.7.1, is when $c_A = min\{v_A, c_B\}$ and $b_A = b_B = c_A = c_B$. This implies that both agents bid their critical bid so, according to critical bid's definition, they are both indifferent between top and bottom rank which means that this is always a PNE, regardless to their order.

This problem is also extended to the next induction steps for any number of agents. For the general case the following claim is being used.

Claim 4.7.2 (4.10. p.19 in [2])**.** *If the lowest critical bid is lower than the value of any agent, i.e.,* $c_j = \phi_h(p_{min}) < v_h$*, then* $\vec{b} = (c_j, \ldots, c_j)$ *is a PNE, where agent* j *is the border agent and other agents are winner agents.*

Similarly, in this claim, it is assumed that there is a connection between the order of critical bids and the order of names. More precisely, the agent with the highest original index must have the lower critical bid, c_j. However, there is a footnote in which they assume that agent j slightly underbids c_j, in order for the previous problem to be avoided.

If agent j slightly underbids c_j, say $b_j = c_j - \epsilon$, then the winner agent who is exactly above him (the last winner) will pay $c_j - \epsilon$ per item, which is a different price than c_j that the other winners pay. This fact contradicts claim (4.4.1) on page 44 which says that one of the PNE properties is that all winners pay the same price.

4.8 Counterexample 2

In this section, we present one more counterexample for theorem 3.1 in [3] which consists of two parts. The first part concerns the PNE existence for equal bids in $[c_2, min\{v_2, c_1\}]$ for which we previously showed counterexample 1. The second part concerns the PNE existence for non-identical bids.

Theorem 4.8.1 (Theorem 3.1 in [3], second part)**.** *Assume that we have two agents with* $c_2 \leq c_1$*. If* $\frac{B_1}{v_2} \geq N$*, then any bids* $b_2 \in [p_{min}, v_2]$ *and* $b_1 \in [v_2, v_1]$ *are a PNE.*
(c_i, B_i, v_i *is the critical bid, budget and value of agent* i*, respectively.)*

▶ *Counterexample 2.*

We show that in the case of $b_1 = b_2 = v_2$, the previous theorem is not true. The following table depicts the data (values, budgets) of two agents with $N = 2$ (total items) and $p_{min} = 3$.

Agents	Values	Budgets
1	$v_1 = 4$	$B_1 = 9$
2	$v_2 = 3$	$B_2 = 8$

Let both agents bid $b_1 = b_2 = v_2 = 3$ and assume that agent 1 has a higher original index than agent 2. Then the ranking is the following.

Agents	Bids	Prices	Values	Budgets	Allocation
2 (A)	$b_2 = v_2 = 3$	$p_2 = b_1 = 3$	$v_2 = 3$	$B_2 = 8$	$x_2 = min\{\frac{8}{3}, N\} = N$
1 (B)	$b_1 = v_2 = 3$	$p_1 = p_{min} = 3$	$v_1 = 4$	$B_1 = 9$	$x_1 = 0$

The conditions are satisfied:

- - $b_2 = v_2 \in [p_{min}, v_2]$.
  - $b_1 = v_2 \in [v_2, v_1]$.
- - $\frac{B_1}{v_2} = \frac{9}{3} = 3 > 2 = N$.
- - $c_2 \leq c_1 \Leftrightarrow 3 \leq 4$.

Even though all the conditions are satisfied, this is not a PNE.

Proof. We focus on agent 1. The utility of agent 1 is : $u_1 = 0$ (loser).
If he overbids agent 2 then we have the following.

Agents	Bids	Prices	Values	Budgets	Allocation
1	$b_1 > b_2 = v_2$	$p_1 = b_2 = 3$	$v_1 = 4$	$B_1 = 9$	$x_1 = min\{\frac{9}{3}, N\} = N$
2	$b_2 = v_2 = 3$	$p_2 = p_{min} = 3$	$v_2 = p_{min} = 3$	$B_2 = 8$	$x_2 = 0$

Agent's 1 new utility is $u_1 = x_1(v_1 - p_1) = N(4 - 3) \overset{N=2}{=} 2$ which is higher than his previous utility (zero). Therefore there is at least one agent who wants to change his position. Hence there is not a PNE, which contradicts theorem 3.1. □

Checking the critical bids

By definition 4.5.1, the critical bid c_j of an agent j is the bid amount x such that, if both agents bid that amount, i.e. $\vec{b} = (x, x)$, agent j is indifferent between top and bottom rank. We will check this for both agents.

Firstly, we assume that both agents bid $b_1 = b_2 = 3$ and we show that agent 2 is indifferent between top and bottom.
$u_{2-top} = x_2(v_2 - p_2) \overset{v_2=p_2}{=} 0.$

Agents	Bids	Prices	Values	Budgets	Allocation
2	$b_2 = 3$	$p_2 = b_1 = 3$	$v_2 = p_{min} = 3$	$B_2 = 8$	$x_2 = min\{\frac{8}{3}, N\} = N$
1	$b_1 = 3$	$p_1 = p_{min} = 3$	$v_1 = 4$	$B_1 = 9$	$x_1 = 0$

$u_{2-bottom} = x_2(v_2 - p_2) \overset{x_2=0}{=} 0.$

Agents	Bids	Prices	Values	Budgets	Allocation
1	$b_1 = c_2 = 3$	$p_1 = b_2 = 3$	$v_1 = 4$	$B_1 = 9$	$x_1 = min\{\frac{9}{4}, N\} = N$
2	$b_2 = c_2 = 3$	$p_2 = p_{min} = 3$	$v_2 = p_{min} = 3$	$B_2 = 8$	$x_2 = 0$

Indeed, $c_2 = 3$.

Secondly, we do the same for agent 1. Assume that both agents bid $b_1 = b_2 = c_1 = 4$ and then we show that $u_{1-top} = u_{1-bottom}$.
$u_{1-top} = x_1(v_1 - p_1) \overset{v_1=p_1}{=} 0.$

Agents	Bids	Prices	Values	Budgets	Allocation
1	$b_1 = c_1 = 4$	$p_1 = b_2 = 4$	$v_1 = 4$	$B_1 = 9$	$x_1 = min\{\frac{9}{4}, N\} = N$
2	$b_2 = c_1 = 4$	$p_2 = p_{min} = 3$	$v_2 = p_{min} = 3$	$B_2 = 8$	$x_2 = 0$

$u_{1-bottom} = x_1(v_1 - p_1) \overset{x_1=0}{=} 0.$

Agents	Bids	Prices	Values	Budgets	Allocation
2	$b_2 = c_1 = 4$	$p_2 = b_1 = 4$	$v_2 = p_{min} = 3$	$B_2 = 8$	$x_2 = min\{\frac{8}{3}, N\} = N$
1	$b_1 = c_1 = 4$	$p_1 = p_{min} = 3$	$v_1 = 4$	$B_1 = 9$	$x_1 = 0$

◀

4.8.1 Correctness

At this section, we present a theorem which is a remodeled version of the previous theorem 4.8.1 on page 51.

Theorem 4.8.2. *[New version] Assume that we have two agents with $c_2 \leq c_1$. If $\frac{B_1}{v_2} \geq N$, then any conservative bids such that $b_1 \geq c_2$, are a PNE.*

Proof of Theorem 4.8.2.

Claim 4.8.3. *If $B_1/v_2 \geq N$, then $c_2 = v_2$.*

Proof. Assume that both agents bid $x = v_2$. According to the auction rules, in case of equal bidding, the agents are ranked according to their original index, name.

- If agent 2 is ranked first, then he is either a border with utility $N(v_2 - v_2) = 0$ or a winner with utility $\frac{B_2}{b_2}(v_2 - v_2) = 0$.
- If agent 2 is ranked second, then by $\frac{B_1}{v_2} \geq N^4$ we observe that we are on the first branch of function g_j on page 46. Hence, agent 2 is a loser with $u_2 = 0$.

Consequently, agent 2 has the same utility being either first or second ranked. This means that he is indifferent between top and bottom rank. By the critical bid definition 4.5.1, $c_2 = v_2$.

□

Claim 4.8.4. *Assume that we have two agents, 1 and 2, who bid $b_2 \leq c_2 < b_1$ such that $b_2 \leq \frac{B_1}{N}$. If agent 2 changes his bid such that $b_2 \leq \frac{B_1}{N}$, he will have no additional profit.*

Proof.
(A) Firstly, we find agent's 2 utility for the given bids. By $b_2 \leq c_2 < b_1$, agent 1 is first and agent 2 is ranked second. By $b_2 \leq \frac{B_1}{N} \Rightarrow N \leq \frac{B_1}{b_2}$, which means that if agent 1 is first, he gets all the items and agent 2 is a loser. So $u_2 = 0$.

(B) Secondly, we find agent's 2 utility for any other different b_2 and compare with the original utility.

- Assume agent 2 changes his bid, either higher or lower, but he is still second ranked. Then, by $N \leq \frac{B_1}{b_2}$, agent 2 will be always a loser regardless of his bid. So, $u_2 = 0$ and he has no additional profit.
- Assume agent 2 changes his bid and he becomes first. He can be ranked first either with $b_1 = b_2$ (in case he has a lower original index), or with $b_1 > b_2$.
  - $c_2 < b_1 = b_2$. By 4.5.1($\star$), agent 2 prefers bottom. If he was in bottom, his utility would be zero (by $N \leq \frac{B_1}{b_2}$). Since he prefers to be bottom ranked with zero utility, his utlity in top is negative, $u_2 < 0$.
  - $c_2 < b_1 < b_2$. In the previous case of $c_2 < b_1 = b_2$, agent 2 has negative utility. For any higher b_2, his utlity will still be negative. Thus, $u_2 < 0$.

[4]Since agent 1 is ranked first, he can buy at least all the items ($p_1 = b_2 = v_2$). So, agent 2 is a loser.

In all cases, agent 2 has either equal or lower profit than his original one. Hence, he has no additional profit. □

Claim 4.8.5. *Assume that $b_2 < b_1$ such that $b_2 \leq c_1$. Then if agent 1 changes his bid, he will have no additional profit.*

Proof.

- Assume that agent 1 changes his bid remaining first (even for $b_1 = b_2$ if he has lower original index). Then his utility remains the same as before, $u_1* = u_1$.

- Assume that he changes his bid and becomes second ranked. He could be ranked in bottom either with $b_1 = b_2$, or with $b_1 < b_2$.

  - $b_1 = b_2 < c_1$. By 4.5.1($\star$), agent 1 prefers top. Thus, his new utility u_1* is lower than his utility in top for $b_1 = b_2$ which equals to his original utility u_1 for $b_1 > b_2$. So, $u_1* < u_1$.
  - $b_1 = b_2 = c_1$. By critical bid definition 4.5, he is indifferent between top and bottom, thus $u_1* = u_1$.
  - $b_1 < b_2 \leq c_1$. If agent 1 lowers his bid, the only he could succeed is to lower his utility. Lowering his bid, he also lowers the price of agent 2 (second-price: $p_2 = b_1$). Hence, agent 2 can buy more items so the items are left for agent 1 are less[5] than before. So $u_1 ** \leq u_1*$ and $u_1* \leq u_1$ (previous cases).

In any case, agent 1 has no additional profit changing his bid. □

Claim 4.8.6. *If $c_2 \leq c_1$, then for bids $b_2 \leq c_2 < b_1$, no agent has a net gain if he changes his bid (under the condition that agent 2 changes conservatively his bid such that $b_2 \leq \frac{B_1}{N}$).*

Proof. By claim 4.8.4 and claim 4.8.5. □

Moreover, we have the following observation.

$$\frac{B_1}{v_2} \geq N \Rightarrow \frac{B_1}{b_2} \geq N \text{ for any conservative } b_2. \tag{4.7}$$

Finally, by claim 4.8.3, claim 4.8.6 and the previous observation 4.7, the theorem is proven.

□

[5] If agent 2 had bought all the items, then even if agent 1 lowers more his bid, his allocation would be the same as before, $x_2* = x_2 = 0$.

In Appendix A.4 on page 69, we give a specific numerical counterexample showing that if bids can be changed non-conservatively, then a pair of bids as in theorem 4.8.2 is not necessarily PNE.

4.9 Special case of PNE existence

Our contribution in this work is a condition under which PNE exists for two agents. This condition is simple, does not involve the critical bids and concerns non-identical bids.

Theorem 4.9.1. *Consider two agents with non-identical conservative bids. Let agent 1 be the first and 2 the second ranked agent. If agent 2 has a budget such that* $\frac{B_2}{v_2} \geq N$*, then any bids* $b_2 = v_2$ *and* $b_1 \in (v_2, v_1]$ *are a PNE.*
(B_i, v_i *are the budget and value of agent* i*, respectively.)*

Proof.
Suppose that we have two agents with bids $b_1 > b_2$ such that $b_2 = v_2$ and $b_1 \in (v_2, v_1]$. Then

Agents	Bids	Prices	Values	Budgets	Allocation
1	$b_1 > v_2$	$p_1 = v_2$	v_1	B_1	$x_1 = \min(N, \frac{B_1}{p_1})$
2	$b_2 = v_2$	$p_{\min}$	v_2	B_2	$x_2 = \min(N - \frac{B_1}{p_1}, \frac{B_2}{p_{\min}})$

Agent's 1 utility:

$$u_1 = x_1(v_1 - p_1) = x_1(v_1 - v_2) > 0, \tag{4.8}$$

since $v_1 \geq b_1 > b_2 = v_2$.

Agent's 2 utility:

$$u_2 = x_2(v_2 - p_2) = x_2(v_2 - p_{\min}). \tag{4.9}$$

In the worst case $u_2 = 0$ (either because of he is a loser, or $v_2 = p_{\min}$).

We will show that this is always a PNE.

- If agent 2 changes his bid.

  Agent 2 cannot bid higher than v_2 due to the conservatively bidding. If he lowers his bid, the only he couls succeed is to lower his utility. More precisely, he lowers the price of agent 1, so agent 1 can buy more items. Thus, less items are left for agent 2. So agent 2 has no profit if he changes his bid.

- If agent 1 underbids agent 2.

Agents	Bids	Prices	Values	Budgets	Allocation
2	$b_2 = v_2$	$p_2 = b_1$	v_2	B_2	$x_2 = \min(N, \frac{B_2}{p_2})$
1	$b_1 \leq v_2$	$p_1 = p_{\min}$	v_1	B_1	$x_1 = \min(N - x_2, \frac{B_1}{p_{\min}})$

If agent 1 has higher original index (name), he could be ranked last even by bidding $b_1 = b_2 = v_2$. The following explanation satisfies this case too. By our assumption we have that $\frac{B_2}{v_2} \geq N$. This means that agent 2 gets all the items if he pays v_2. Consequently, agent 2 gets also all the items for any lower price than v_2. Since we have a second price auction, the payment of agent 2 is agent's 1 bid, $p_2 = b_1$. Therefore, for any $b_1 < v_2$, agent 2 gets all the items and agent 1 is a loser ($x_1 = 0$). So $u_1 = 0$, which is lower than his original utility (4.8).

($\star$) In case of $b_2 = v_2 = p_{\min}$, if agent 1 has lower original index than agent 2, the only way that he could be ranked last, is by bidding $b_1 < b_2 = p_{\min}$.

Agents	Bids	Prices	Values	Budgets	Allocation
2 (B)	$b_2 = p_{\min}$	$p_2 = b_1 = p_{\min}$	$v_2 = p_{\min}$	B_2	$x_2 = N$
1 (A)	$b_1 < p_{\min}$	$p_1 = 0$	v_1	B_1	$x_1 = 0$

However, bidding under the minimum price he cannot participate in the auction according to the auction rules. Thus, his utility would be $u_1 = 0$ which is lower than his original one (4.8).

($\star$) In case of $b_2 = v_2 = p_{\min}$, if agent 1 has higher original index than agent 2, he could be ranked last even if he bids the same amount with agent 2, $b_1 = b_2 = v_2 = p_{\min}$. This happens due to the lexicographic order.

Agents	Bids	Prices	Values	Budgets	Allocation
2 (A)	$b_2 = p_{\min}$	$p_2 = b_1 = p_{\min}$	$v_2 = p_{\min}$	B_2	$x_2 = N$
1 (B)	$b_1 = b_2 = p_{\min}$	$p_1 = p_{\min}$	v_1	B_1	$x_1 = 0$

As we mentioned before, due to the condition $B_2/v_2 \geq N$, agent 2 gets all the items. Therefore, agent 1 is a loser, $x_1 = 0$. His utility ($u_1 = 0$) is lower that his original utility.

In any case, no agent has a profit if he changes his position, so this is a PNE. □

4.9.1 Non-existence of PNE without the condition

At this point, we give an example which shows that for the same bids as before, $b_2 = v_2$ and $b_1 \in (v_2, v_1]$, but without the previous condition $\frac{B_2}{v_2} \geq N$, there exist settings with no balance.

▶*Example.*

$N = 100$, $p_{\min} = 1$

Agents	Values	*Budgets*
1	$v_1 = 7$	$B_1 = 60$
2	$v_2 = 4$	$B_2 = 30$

- If $b_1 > b_2$

Players	*Bids*	*Prices*	*Values*	*Budgets*	*Allocation*
1	b_1	$p_1 = b_2 = 4$	$v_1 = 7$	$B_1 = 60$	$x_1 = \frac{60}{4} = 15$
2	$b_2 = 4$	$p_2 = p_{\min} = 1$	$v_2 = 4$	$B_2 = 30$	$x_2 = \min(N - 15, \frac{30}{1}) = 30$

The utility of agent 1 is

$$u_1 = x_1(v_1 - p_1) = 15(7 - 4) = 45 \tag{4.10}$$

Lets examine his utility if he underbids agent 2.

Players	*Bids*	*Prices*	*Values*	*Budgets*	*Allocation*
2	$b_2 = 4$	$p_2 = b_1$	$v_2 = 4$	$B_2 = 30$	$x_2 = \frac{30}{b_1}$
1	$b_1 < 4$	$p_1 = p_{\min} = 1$	$v_1 = 7$	$B_1 = 60$	$x_1 = \min(N - \frac{30}{b_1}, \frac{60}{1})$

Depending on b_1, the worst case for agent 1 (his minimum new utility) is when agent 2 gets the maximum amount of items, according to his budget B_2. This happens for $b_1 = p_{\min}$. Even if agent 1 bids $b_1 = p_{\min} = 1$, his new utility $u_1 = x_1(v_1 - p_1) = 60(7 - 1) = 360$ is higher than his previous utility (4.10). Thus, agent 1 desires to change his position.

- If $b_2 > b_1$

In this case, agent's 2 bid should be over his value because of $b_1 > v_2$. But this is not legitimate due to conservative bids of the theorem's hypothesis.

In both cases, there is an agent who wants to change his position, therefore we have no PNE.

◀

Chapter 5

Budgeted Second-Price Ad Auctions with Envy-Free equilibria

"In competitive behavior,
someone always loses."

-John Nash
A beautiful mind

In this chapter, we present another Second-Price Budgeted Auction model of an on-going paper of J. Diaz, Y. Giotis, L. M. Kirousis, E. Markakis and M. Serna. This auction model concerns the selling of advertising space on web pages. It is focused on the existence of envy-free assignments and their relation with pure Nash equilibria under budget constraints.

5.1 The auction model

The ad space is assumed to be divided into slots. There exist k slots and n players ($n > k$) who compete with each other for those slots through a second price auction. Every slot has different visibility and therefore, every ad in each slot has different probability of being clicked. These probabilities are called click-through-rates (CTR) and we symbolize them with θ_i for each slot $i \in \{1, \dots, k\}$. Depending on slots' position on the web page, we have $\theta_1 > \theta_2 > \cdots > \theta_k > 0$.

Every player i has a private value v_i and a public budget constraint B_i which expresses the total payment they are willing to pay. Each player is assumed to be assigned to at most one slot. The auctioneer ranks the players according to their bids and renames them such that $b_1 > b_2 > \cdots > b_n$. The bids need not to be conservative, therefore a player could overbid his value,

$b_i > v_i$. In case player i gets slot s_i, he will pay $\theta_{s_i} \cdot p_i$, where $p_i = b_{i+1}$ (second price auction). In case he gets no slot, he pays $p_i = 0$.

After players are ranked by decreasing bid, each player is assigned to the slot with the highest CTR that is currently available and within his budget. For instance, assume that player i is assigned to slot $j = s_i$ which is the slot with highest CTR that is currently available, it has not been taken so far by another player $i' < i$ in the ranking. Then, player i must also be able to afford this slot, that is $p_i \theta_{s_i} \leq B_i \Rightarrow b_{i+1}\theta_{s_i} \leq B_i$. In case there is no slot left that player i can aford in his turn, he gets nothing.

Categories of players

Definition 5.1.1 (winner). *A player is called* winner *if he is assigned to a slot.*

The utility of a winner player i being assigned to slot s_i is $u_i = \theta_{s_i}(v_i - p_i) = \theta_{s_i}(v_i - b_{i+1})$

Definition 5.1.2 (loser). *A player is called* loser *if he gets no slot.*

The utlity of a loser player is $u_i = 0$.

5.2 Types of Second-Price auctions

Three types of budgeted second price auctions are defined below.

- In the *non-excluding-budget* auction, the mechanism ranks first the players according to their bids in a decreasing order. Each player pays per click the bid of the player who is ranked exactly below him. Then, the slots are also assigned to the players in a decreasing order. However, the mechanism ignores the budget constraints. This means that it is possible for some budget constraints to be violated. The utility of those players whose budget constraints are violated is $u = -\infty$.

- In the *excluding-budget* auction, the mechanism ranks first the players according to their bids in a decreasing order. Then, it assigns to each player a price per click, equal to the bid of the player who is ranked exactly below him. Respecting players' decreasing order and budget constraints, the mechanism assigns to each player the highest available slot. In this model, it is possible for a player i to be assigned to a higher[1] slot than the slot of a player i' for $i' < i$. In other words, a

[1] The slot of player i has higher click-through-rate than the slot of player i'.

player assigned to a higher slot (player i) may pay less than another player who is assigned to a lower slot (player i'). Notice that some slots may remain unassigned in case that no player can afford them. Also, in case that the lowest bidding player is awarded a slot, he pays zero.

- In the *decreasing* auction, a price point is lowered, starting from infinity. A player requests a slot if the price is equal or lower than his bid and his budget in case he would be asked to pay for that slot. The highest slot is assigned to the first player who requests it. The price of a player will be the price point where a second player (assigned or unassigned) requests that slot. Then, the same procedure follows for the next slot, resetting the price point to infinity.

In all mechanisms, in the case of bid ties, the players are ordered according to their original index, ranking first the player with the lower original index.

5.3 Envy and envy-free assignment

In compare with the definition of envy (3.1.2) on page 29, the envy definition in this budgeted model differs. due to the additional paremeter, the budget constraint to each player. More precisely, in order for a player i to envy the place of another player j, he must also be able to afford j's place.

Rationality

Definition 5.3.1. *A price assignment is called* rational *if for every player i his price is at most his value, $p_i \leq vi$.*

Envy

Definition 5.3.2. *Given a rational price and slot assignment, we say that a player j* envies *a winner i with assigned slot s_i if j can afford s_i and j would be strictly better off in i's place than his present situation, i.e. $\theta_{s_i} p_i \leq B_j$ and $\theta_{s_j}(v_j - p_j) < \theta_{s_i}(v_j - p_i)$.*

We should notice that the definition of j envying i does not involve the value v_i or the budget B_i of player i.

Envy-free assignment

Definition 5.3.3. *We say that the assignment is* envy-free (or is in envy-free equilibrium) *if it is rational and no player envies another player. That is, for every player j*

$$\theta_{s_j}(v_j - p_{s_j}) \geq \begin{cases} \theta_{s_i}(v_j - p_{s_i}), & \text{if } \theta_{s_i} \cdot p_{s_i} \leq B_j, \\ 0, & \text{otherwise,} \end{cases}$$

where s_j and p_{s_j} is the slot assigned to player j and the price[2] *per click he would be asked to pay, respectively.*

Notice that in an envy-free equilibrium, a player j may not envy another player i because he cannot afford i's place, $\theta_{s_i} p_{s_i} > B_j$.

5.4 Envy-free equil. vs PNE under budgets

By property (3.1.5) on page 30, we have that the class of envy-free equilibria is a subset of pure Nash equilibria. In that model, there were no budget constraints. This implies that this property is also satisfied under the *non-excluding-budget* second-price auction model.

However, this property under the *excluding-budget* second-price auction model. Since the slot allocation respects the budget constraints, a player could alter the allocation in his benefit forcing other player to get out of their budget, increasing his utility. We present here an example as it follows. First, we have an envy-free assignment without budget constraints. This is also a pure Nash equilibrium. Then, we add a budget restriction to each player. Even though the assignment is still envy-free (according to definition (5.3.2) on page 61), we show that it is not a PNE.

▶*Example.*

Envy-free assignment without budget contraints

The following table depicts an envy-free assignment of slots, s_1, s_2, to four players, A, B, C, D, associated with particular prices, p_A, p_B, p_C, p_D (proof of envy-freeness in Appendix B, page 71).

Players	Bids	Prices	Values	CTR
A	$b_A = 20$	$p_A = 15$	$v_A = 30$	$\theta_{s_1} = 0.3$
B	$b_B = 15$	$p_B = 12$	$v_B = 20$	$\theta_{s_2} = 0.2$
C	$b_C = 12$	$p_C = 0$	$v_C = 12$	no slot
D	$b_D = 9$	$p_D = 0$	$v_D = 10$	no slot

By property (3.1.5) on page 30, this is also a Pure Nash equilibrium.

Envy-free assignment under excluding-budget setting

[2] p_{s_j} equals to zero should player j awarded no slot.

Lets now add a budget constraint to each player. The allocation is still an envy-free assignment according to definition (5.3.2) (proof in Appendix B.2, page 73). However, we will show that this is not a PNE.

Notice that player A cannot afford slot s_1 since $\theta_{s_1} \cdot p_A = \frac{3}{10} \cdot 15 = \frac{9}{2} > \frac{7}{2} = B_A$. Thus, he gets the next currently available slot within his budget. Slot s_2 is within player's A budget, since $\theta_{s_2} \cdot p_A = \frac{2}{10} \cdot 15 = 3 < \frac{7}{2} = B_A$.

Players	Bids	Prices	Values	Budgets	CTR
A	$b_A = 20$	$p_A = 15$	$\upsilon_A = 30$	$B_A = \frac{7}{2} = 3.5$	$\theta_{s_2} = 0.2$
B	$b_B = 15$	$p_B = 12$	$\upsilon_B = 20$	$B_B = 3.6$	$\theta_{s_1} = 0.3$
C	$b_C = 12$	$p_C = 0$	$\upsilon_C = 12$	$B_C = 1$	no slot
D	$b_D = 9$	$p_D = 0$	$\upsilon_D = 10$	$B_D = \frac{1}{2} = 0.5$	no slot

Not a pure Nash equilibrium

Proof.
The utility of player A is $u_A = \theta_{s_2}(\upsilon_A - p_A) = \frac{2}{10}(30 - 15) = \frac{2}{10} \cdot 15 = 3$.

If player A underbids player C by bidding $b_A = 10$, then we have the following. Player B becomes first and can still afford slot s_1 since $\theta_{s_1} \cdot p_B = \frac{3}{10} \cdot 12 = 3.6 = B_B$, so s_2 is assigned to B. Player C cannot afford slot s_2, since $\theta_{s_2} \cdot p_C = \frac{2}{10} \cdot 10 > 1 = B_C$ (if he could afford slot s_2, he would pay $p_C = b_A = 10$). Thus, he gets no slot and $p_C = 0$. However, player A can afford slot s_2, since $\theta_{s_2} \cdot p_A = \frac{2}{10} \cdot 9 = 1.8 < 3.5 = B_A$. Therefore, slot s_2 is assigned to player A.

Players	Bids	Prices	Values	Budgets	CTR
B	$b_B = 15$	$p_B = 12$	$\upsilon_B = 20$	$B_B = 3.6$	$\theta_{s_1} = 0.3$
C	$b_C = 12$	$p_C = 0$	$\upsilon_C = 12$	$B_C = 1$	no slot
A	$b_A = 10$	$p_A = 9$	$\upsilon_A = 30$	$B_A = \frac{7}{2} = 3.5$	$\theta_{s_2} = 0.2$
D	$b_D = 9$	$p_D = 0$	$\upsilon_D = 10$	$B_D = \frac{1}{2} = 0.5$	no slot

The new utility of player A is $u_A = \theta_{s_2}(\upsilon_A - p_A) = \frac{2}{10}(30 - 9) = \frac{42}{10} = 4.2$ which is higher than before, so he wants to change his position. Therefore, this is not a PNE. □

◀

5.5 Main Results

One of the main relults of this working paper is the existence of envy-free equilibria under the *excluding-budget* auction.

Theorem 5.5.1. *There exists bids such that the excluding-budget second-price auction produces an envy-free assignment of prices (per click) to the players and of players to the slots.*

Proof Idea.
! This is only the main idea of the theorem's proof. It will be presented and described in details in [4].

The assignment process goes as follows. Initially, to every slot is tagged the positive price ∞ such that no player can afford any slot at that price. Each time the price of an arbitrary slot is continuously lowered until a player can afford it or envies it (in case he is already assigned to another slot). If he envies it, he leaves his first slot and is assigned to the new one. His previous slot is now free and its price starts being lowered until, similarly, a player can afford it or envies it. The procedure is repeated for all slots. At the end, is proved that all slots will be assigned to the players. Moreover, the allocation of prices and slots becomes with a particular method such that the following properties are satisfied. The final assignment respects the budget constraints, is envy-free and rational, which completes the proof. □

Theorem 5.5.2. *There are settings under excluding-budget auction where no Nash equilibrium exists.*

The following table depicts one of those settings.

Players	Values	Budgets
1	$v_1 = 50$	$B_1 = 50$
2	$v_2 = 16$	$B_2 = 5$
3	$v_3 = 8$	$B_3 = 2$

Theorem 5.5.3. *There are settings under non-excluding-budget auction where no Nash equilibrium exists.*

Since the class of envy-free equilibria is a subset under the *non-excluding-budget* auction, we have the following.

Corollary 5.5.4. *There are settings under non-excluding-budget auction where no envy-free equilibrium exists.*

The proofs of the all the previous theorems will be analytically presented in [4].

Appendix A

Counterexamples

A.1 Property 3.1.7

Positions	Values	Bids	Prices	CTRs
1	$\upsilon_1 = 10$	$b_1 = 8$	$p_1 = 7$	$\theta_{s_1} = 0.4$
2	$\upsilon_2 = 11$	$b_2 = 7$	$p_2 = 4$	$\theta_{s_2} = 0.2$
3	$\upsilon_3 = 5$	$b_3 = 4$	$p_3 = 2$	$\theta_{s_3} = 0.15$
4	$\upsilon_4 = 4$	$b_4 = 2$	$p_4 = 0$	no slot

Player 1's utility

- In his position: $u_1 = \theta_{s_1}(\upsilon_1 - p_1) = 0.4(10 - 7) = 1.2$.
- In 2's position: $u_{1\rightsquigarrow 2} = \theta_{s_2}(\upsilon_1 - p_2) = 0.2(10 - 4) = 1.2$.
- In 3's position: $u_{1\rightsquigarrow 3} = \theta_{s_3}(\upsilon_1 - p_3) = 0.15(10 - 2) = 1.2$.
- In 4's position: he gets no slot, so $u_{1\rightsquigarrow 4} = 0$.

Player 2's utility

- In his position: $u_2 = \theta_{s_2}(\upsilon_2 - p_2) = 0.2(11 - 4) = 1.4$.
- In 1's position: $u_{2\rightsquigarrow 1} = \theta_{s_1}(\upsilon_2 - b_1) = 0.4(11 - 8) = 1.2$.
- In 3's position: $u_{2\rightsquigarrow 3} = \theta_{s_3}(\upsilon_2 - p_3) = 0.15(11 - 2) = 1.35$.
- In 4's position: he gets no slot, so $u_{2\rightsquigarrow 4} = 0$.

Player 3's utility

- In his position: $u_3 = \theta_{s_3}(\upsilon_3 - p_3) = 0.15(5 - 2) = 0.45$.
- In 1's position: $u_{3\rightsquigarrow 1} = \theta_{s_1}(\upsilon_3 - b_1) = 0.4(5 - 8) = -1.2$.

- In 2's position: $u_{3 \rightsquigarrow 2} = \theta_{s_2}(v_3 - b_2) = 0.4(5 - 7) = -0.4$.
- In 4's position: he gets no slot, so $u_{3 \rightsquigarrow 4} = 0$.

Player 4's utility

- In his position: he gets no slot, so $u_4 = 0$.
- In 1's position: $u_{4 \rightsquigarrow 1} = \theta_{s_1}(v_4 - b_1) = 0.4(4 - 8) = -1.6$.
- In 2's position: $u_{4 \rightsquigarrow 2} = \theta_{s_2}(v_4 - b_2) = 0.2(4 - 7) = -0.6$.
- In 3's position: $u_{4 \rightsquigarrow 3} = \theta_{s_3}(v_4 - b_3) = 0.15(4 - 4) = 0$.

A.2 Property 3.1.8

Positions	Values	Bids	Prices	CTRs
1	$v_1 = 10$	$b_1 = 8$	$p_1 = 7$	$\theta_{s_1} = 0.4$
2	$v_2 = 11$	$b_2 = 7$	$p_2 = 4$	$\theta_{s_2} = 0.2$
3	$v_3 = 5$	$b_3 = 4$	$p_3 = 1.9$	$\theta_{s_3} = 0.15$
4	$v_4 = 4$	$b_4 = 1.9$	$p_4 = 0$	no slot

Player 1's utility

- In his position: $u_1 = \theta_{s_1}(v_1 - p_1) = 0.4(10 - 7) = 1.2$.
- In 2's position: $u_{1\rightsquigarrow 2} = \theta_{s_2}(v_1 - p_2) = 0.2(10 - 4) = 1.2$.
- In 3's position: $u_{1\rightsquigarrow 3} = \theta_{s_3}(v_1 - p_3) = 0.15(10 - 1.9) = 1.215$.
- In 4's position: he gets no slot, so $u_{1\rightsquigarrow 4} = 0$.

Player 2's utility

- In his position: $u_2 = \theta_{s_2}(v_2 - p_2) = 0.2(11 - 4) = 1.4$.
- In 1's position: $u_{2\rightsquigarrow 1} = \theta_{s_1}(v_2 - b_1) = 0.4(11 - 8) = 1.2$.
- In 3's position: $u_{2\rightsquigarrow 3} = \theta_{s_3}(v_2 - p_3) = 0.15(11 - 1.9) = 1.365$.
- In 4's position: he gets no slot, so $u_{2\rightsquigarrow 4} = 0$.

Player 3's utility

- In his position: $u_3 = \theta_{s_3}(v_3 - p_3) = 0.15(5 - 2) = 0.45$.
- In 1's position: $u_{3\rightsquigarrow 1} = \theta_{s_1}(v_3 - b_1) = 0.4(5 - 8) = -1.2$.
- In 2's position: $u_{3\rightsquigarrow 2} = \theta_{s_2}(v_3 - b_2) = 0.4(5 - 7) = -0.4$.
- In 4's position: he gets no slot, so $u_{3\rightsquigarrow 4} = 0$.

Player 4's utility

- In his position:he gets no slot, so $u_4 = 0$.
- In 1's position: $u_{4\rightsquigarrow 1} = \theta_{s_1}(v_4 - b_1) = 0.4(4 - 8) = -1.6$.
- In 2's position: $u_{4\rightsquigarrow 2} = \theta_{s_2}(v_4 - b_2) = 0.2(4 - 7) = -0.6$.
- In 3's position: $u_{4\rightsquigarrow 3} = \theta_{s_3}(v_4 - b_3) = 0.15(4 - 4) = 0$.

A.3 Property 3.1.9

Positions	Values	Bids	Prices	CTRs
1	$v_1 = 10$	$b_1 = 8$	$p_1 = 7$	$\theta_{s_1} = 0.4$
2	$v_2 = 11$	$b_2 = 7$	$p_2 = 4$	$\theta_{s_2} = 0.2$
3	$v_3 = 5$	$b_3 = 4$	$p_3 = 4$	$\theta_{s_3} = 0.15$
4	$v_4 = 4$	$b_4 = 4$	$p_4 = 0$	no slot

Player 1's utility

- In his position: $u_1 = \theta_{s_1}(v_1 - p_1) = 0.4(10 - 7) = 1.2$.
- In 2's position: $u_{1\rightsquigarrow 2} = \theta_{s_2}(v_1 - p_2) = 0.2(10 - 4) = 1.2$.
- In 3's position: $u_{1\rightsquigarrow 3} = \theta_{s_3}(v_1 - p_3) = 0.15(10 - 4) = 0.9$.
- In 4's position: he gets no slot, so $u_{1\rightsquigarrow 4} = 0$.

Player 2's utility

- In his position: $u_2 = \theta_{s_2}(v_2 - p_2) = 0.2(11 - 4) = 1.4$.
- In 1's position: $u_{2\rightsquigarrow 1} = \theta_{s_1}(v_2 - b_1) = 0.4(11 - 8) = 1.2$.
- In 3's position: $u_{2\rightsquigarrow 3} = \theta_{s_3}(v_2 - p_3) = 0.15(11 - 4) = 1.05$.
- In 4's position: he gets no slot, so $u_{2\rightsquigarrow 4} = 0$.

Player 3's utility

- In his position: $u_3 = \theta_{s_3}(v_3 - p_3) = 0.15(5 - 4) = 0.15$.
- In 1's position: $u_{3\rightsquigarrow 1} = \theta_{s_1}(v_3 - b_1) = 0.4(5 - 8) = -1.2$.
- In 2's position: $u_{3\rightsquigarrow 2} = \theta_{s_2}(v_3 - b_2) = 0.4(5 - 7) = -0.4$.
- In 4's position: he gets no slot, so $u_{3\rightsquigarrow 4} = 0$.

Player 4's utility

- In his position:he gets no slot, so $u_4 = 0$.
- In 1's position: $u_{4\rightsquigarrow 1} = \theta_{s_1}(v_4 - b_1) = 0.4(4 - 8) = -1.6$.
- In 2's position: $u_{4\rightsquigarrow 2} = \theta_{s_2}(v_4 - b_2) = 0.2(4 - 7) = -0.6$.
- In 3's position: $u_{4\rightsquigarrow 3} = \theta_{s_3}(v_4 - b_3) = 0.15(4 - 4) = 0$.

A.4 Theorem 4.8.2 on page 53, (bidding non-conservatively).

Agents	Values	Budgets
1	$v_1 = 4$	$B_1 = 9$
2	$v_2 = 3$	$B_2 = 12$

and $N = 3$, $p_{\min} = 3$

The conditions are satisfied:

- $\frac{B_1}{v_2} \geq N \Leftrightarrow \frac{9}{3} \geq 3$
- $c_2 = 3 \leq 4 = c_1$ (The proof of critical bids is at the end.)

Assume that agent 1 bids $b_1 = v_1 = 4$ and agent 2 bids $b_2 = v_2 = 3$.

Agents	Bids	Prices	Values	Budgets	Allocation
1	$b_1 = v_1 = 4$	$p_1 = b_2 = 3$	$v_1 = 4$	$B_1 = 9$	$x_1 = \min\{\frac{B_1}{p_1}, N\} = N$
2	$b_2 = v_2 = 3$	$p_2 = p_{\min} = 3$	$v_2 = 3$	$B_2 = 12$	$x_2 = N - x_1 = 0$

The bids b_1, b_2 are conservative and $b_1 = 4 > 3 = c_2$, so by Theorem 4.8.2, this is a PNE.

But what would happen if agent 2 could change his bid non-conservatively remaining at bottom? We show that in this case the previous bids are not in a PNE because agent 2 has a higher utility bidding non-conservatively.

Proof.
Assume that agent 2 changes his bid by bidding $b_2 = \frac{7}{2} > 3 = v_2$.

Agents	Bids	Prices	Values	Budgets	Allocation
1	$b_1 = v_1 = 4$	$p_1 = b_2 = \frac{7}{2}$	$v_1 = 4$	$B_1 = 9$	$x_1 = \min\{\frac{9}{7/2}, N\} = \frac{18}{7}$
2	$b_2 = \frac{7}{2} > v_2$	$p_2 = p_{\min} = 3$	$v_2 = 3$	$B_2 = 12$	$x_2 = \min\{N - x_1, \frac{B_2}{p_2}\} = \frac{3}{7}$

We observe that agent 2 forces agent 1 to pay more. As a result agent 2 gets more items and his utility becomes positive which is higher than his original zero utility. Thus, he prefers to deviate $\Rightarrow$ this is not a PNE.

□

Checking the critical bids

Firstly, we assume that both agents bid $b_1 = b_2 = 3$ and we show that agent 2 is indifferent between top and bottom.
$u_{2-top} = x_2(v_2 - p_2) \overset{v_2 = p_2}{=} 0.$

Agents	Bids	Prices	Values	Budgets	Allocation
2	$b_2 = 3$	$p_2 = b_1 = 3$	$v_2 = p_{min} = 3$	$B_2 = 12$	$x_2 = \min\{\frac{12}{3}, N\} = N$
1	$b_1 = 3$	$p_1 = p_{min} = 3$	$v_1 = 4$	$B_1 = 9$	$x_1 = 0$

$u_{2-bottom} = x_2(v_2 - p_2) \overset{x_2=0}{=} 0.$

Agents	Bids	Prices	Values	Budgets	Allocation
1	$b_1 = c_2 = 3$	$p_1 = b_2 = 3$	$v_1 = 4$	$B_1 = 9$	$x_1 = \min\{\frac{9}{3}, N\} = N$
2	$b_2 = c_2 = 3$	$p_2 = p_{min} = 3$	$v_2 = p_{min} = 3$	$B_2 = 12$	$x_2 = 0$

Indeed, $c_2 = 3$.

Secondly, we do the same for agent 1. Assume that both agents bid $b_1 = b_2 = c_1 = 4$ and then we show that $u_{1-top} = u_{1-bottom}$.
$u_{1-top} = x_1(v_1 - p_1) \overset{v_1=p_1}{=} 0.$

Agents	Bids	Prices	Values	Budgets	Allocation
1	$b_1 = c_1 = 4$	$p_1 = b_2 = 4$	$v_1 = 4$	$B_1 = 9$	$x_1 = \min\{\frac{9}{4}, N\} = \frac{9}{4}$
2	$b_2 = c_1 = 4$	$p_2 = p_{min} = 3$	$v_2 = p_{min} = 3$	$B_2 = 12$	$x_2 = \frac{3}{4}$

$u_{1-bottom} = x_1(v_1 - p_1) \overset{x_1=0}{=} 0.$

Agents	Bids	Prices	Values	Budgets	Allocation
2	$b_2 = c_1 = 4$	$p_2 = b_1 = 4$	$v_2 = p_{min} = 3$	$B_2 = 12$	$x_2 = \min\{\frac{12}{4}, N\} = N$
1	$b_1 = c_1 = 4$	$p_1 = p_{min} = 3$	$v_1 = 4$	$B_1 = 9$	$x_1 = 0$

Indeed, $c_1 = 4$.

◀

Appendix B

Envy-free assignments

B.1 No budget constraints

Players	Bids	Prices	Values	CTR
A	$b_A = 20$	$p_A = 15$	$\upsilon_A = 30$	$\theta_{s_1} = 0.3$
B	$b_B = 15$	$p_B = 12$	$\upsilon_B = 20$	$\theta_{s_2} = 0.2$
C	$b_C = 12$	$p_C = 0$	$\upsilon_C = 12$	no slot
D	$b_D = 9$	$p_D = 0$	$\upsilon_D = 10$	no slot

Proof. Firstly, it is obvious that the assignment is rational since $\upsilon_i \geq p_i$ for $i = A, B, C, D$. Now we will show that no player envies another one, according to the definition (3.1.2) on page 29.

Player A's utility

- In his position: $\theta_{s_1}(\upsilon_A - p_A) = \frac{3}{10}(30 - 15) = 4.5$.
- In B's position: $\theta_{s_2}(\upsilon_A - p_B) = \frac{2}{10}(30 - 12) = \frac{36}{10} = 3.6$.
- In C's position: He gets no slot, so $u_A = 0$.
- In D's position: He gets no slot, so $u_A = 0$.

Player B's utility

- In his position: $\theta_{s_2}(\upsilon_B - p_B) = \frac{2}{10}(20 - 12) = \frac{16}{10} = 1.6$.
- In A's position: $\theta_{s_1}(\upsilon_B - p_A) = \frac{3}{10}(20 - 15) = 3 \cdot \frac{15}{10} = 1.5$.
- In C's position: He gets no slot, so $u_B = 0$.
- In D's position: He gets no slot, so $u_B = 0$.

Player C's utility

- In his position: $u_C = 0$.
- In A's position: $\theta_{s_1}(\upsilon_C - p_A) = \frac{3}{10}(12 - 15) = \frac{3}{10} \cdot (-3) = -\frac{9}{10}$.
- In B's position: $\theta_{s_2}(\upsilon_C - p_B) = \frac{2}{10}(12 - 12) = 0$.
- In D's position: He gets no slot, so $u_C = 0$.

Player D's utility

- In his position: $u_D = 0$.
- In A's position: $\theta_{s_1}(\upsilon_D - p_A) = \frac{3}{10}(10 - 15) = \frac{3}{10} \cdot (-5) = -\frac{15}{10}$.
- In B's position: $\theta_{s_2}(\upsilon_D - p_B) = \frac{2}{10}(10 - 12) = \frac{2}{10} \cdot (-2) = -\frac{4}{10}$.
- In C's position: He gets no slot, so $u_D = 0$.

For any player, his utility at his position is at least equal with his utility in any other player's position. So this is an envy-free assignment. □

B.2 Under budget constraints

Players	Bids	Prices	Values	Budgets	CTR
A	$b_A = 20$	$p_A = 15$	$\upsilon_A = 30$	$B_A = \frac{7}{2} = 3.5$	$\theta_{s_2} = 0.2$
B	$b_B = 15$	$p_B = 12$	$\upsilon_B = 20$	$B_B = \frac{18}{5} = 3.6$	$\theta_{s_1} = 0.3$
C	$b_C = 12$	$p_C = 0$	$\upsilon_C = 12$	$B_C = 1$	no slot
D	$b_D = 9$	$p_D = 0$	$\upsilon_D = 10$	$B_D = \frac{1}{2} = 0.5$	no slot

Proof. - Firstly, players A, B can rationally afford slots s_1, s_2 respectively.

$$\upsilon_A = 30 \geq 15 = p_A \quad \text{and} \quad p_A\theta_{s_2} = 15 \cdot \frac{2}{10} = 3 \leq \frac{7}{2} = B_A,$$
$$\upsilon_B = 20 \geq 12 = p_B \quad \text{and} \quad p_B\theta_{s_1} = 12 \cdot \frac{3}{10} \leq 3.6 = B_B.$$

- Secondly, no player envies any other player, according to the definition (5.3.2) on page 61.

Player A's utility

- In his position: $\theta_{s_2}(\upsilon_A - p_A) = \frac{2}{10}(30 - 15) = 3$.
- In B's position: $\theta_{s_1}(\upsilon_A - p_B) = \frac{3}{10}(30 - 12) = \frac{3}{10} \cdot 18 = \frac{54}{10}$, but it is out of his budget ($\theta_{s_1} \cdot p_B = \frac{3}{10} \cdot 12 = \frac{36}{10} > \frac{35}{10} = \frac{7}{2} = B_A$).
- In C's position: He gets no slot, so $u_A = 0$.
- In D's position: He gets no slot, so $u_A = 0$.

Player B's utility

- In his position: $\theta_{s_1}(\upsilon_B - p_B) = \frac{3}{10}(20 - 12) = \frac{24}{10}$.
- In A's position: $\theta_{s_2}(\upsilon_B - p_A) = \frac{2}{10}(20 - 15) = 1$.
- In C's position: He gets no slot, so $u_B = 0$.
- In D's position: He gets no slot, so $u_B = 0$.

Player C's utility

- In his position: $u_C = 0$.
- In A's position: $\theta_{s_2}(\upsilon_C - p_A) = \frac{2}{10}(12 - 15) = \frac{2}{10} \cdot (-3) = -\frac{6}{10}$.
- In B's position: $\theta_{s_1}(\upsilon_C - p_B) = \frac{3}{10}(12 - 12) = 0$.
- In D 's position: He gets no slot, so $u_C = 0$.

Player D's utility

- In his position: $u_D = 0$.
- In A's position: $\theta_{s_2}(\upsilon_D - p_A) = \frac{2}{10}(10 - 15) = -1$.
- In B's position: $\theta_{s_1}(\upsilon_D - p_B) = \frac{3}{10}(10 - 12) = -\frac{6}{10}$.
- In C 's position: He gets no slot, so $u_D = 0$.

No player envies another player, so this is an envy-free assignment.

□

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