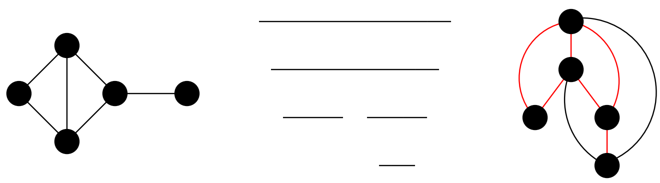
Tree-depth of Graphs: Characterisations and Obstructions



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$\mu\Pi\lambda\forall$

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Preface

> “*The mathematician’s patterns, like the painter’s or the poet’s must be beautiful; the ideas, like the colours or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics.*”
>
> G. H. Hardy (1877 - 1947)

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Typesetting

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Recycling

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Contents

Chapter 1

Introduction

Graph Theory, Logic and Complexity

Three of the most common entertaining riddles are the following:

1. (Graph Theory) Can you draw a given figure (for example, the left-most figure in Figure 1.1) without picking up your pen and overlapping lines? or Can you draw a given figure (for example, the right-most figure in Figure 1.1) without picking up your pen, overlapping lines and by beginning and ending at the same point?

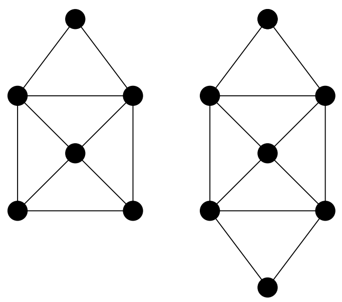


Figure 1.1: The drawing riddle

2. (Logic) A prisoner is convicted to death penalty and the judge allows him to say a last sentence in order to determine the way the penalty will be carried out. If the prisoner lies, he will be hanged, if he speaks the truth he will be beheaded. The prisoner speaks a last sentence and to everybody's surprise some minutes later he is set free because the judge cannot determine his penalty. What did the prisoner say?

3. (Complexity) Fill in the blank squares of a sudoku.

What someone finds out while trying to solve the first riddle is that you cannot draw every figure without picking up your pen and overlapping lines. More specifically, it is even more rare to find a figure that you can draw without picking up your pen or overlapping lines and by beginning and ending at the same point.

For the second riddle (while using logical reasoning) one finds out that his sentence could be the following: This sentence is a lie! (Notice that this is not the unique such sentence that the prisoner could have said.)

While solving the third riddle one realises how much harder it becomes to fill in the empty squares while the dimension of the sudoku board increases.

Behind these riddles and the hardness of finding their solution lie three of the most beautiful fields of Mathematics: Graph Theory, Logic and Complexity Theory[1] while their common component is Combinatorics.

The first result in the history of Graph Theory is a theorem by L. Eüler (1736) stating when a figure can de drawn without picking up your pen and overlapping lines (i.e. when it has an Eüler path) and when a figure can be drawn without picking up your pen or overlapping lines and by beginning and ending at the same point (i.e. when it has an Eüler cycle).

Logic has its origins in Ancient Greece where Aristotle was the first to suggest a formal system that was then used by Euclid. The riddle above is based on the paradox of Epimenides. Epimenides was a Cretan who stated the following: "All Cretans lie!", thus created a contradiction to his one sentence.

Complexity Theory is the most recent one of them but a lot of progress has been made since the first important papers in its history [7, 33] (Cook-Levin (Леонид Анатольевич Левин) theorem and other NP-completeness results of R. Karp), while one of its most important problems, $\mathrm{P} \stackrel{?}{=} \mathrm{NP}$, remains open.

About this thesis

What we study in this thesis is some problems of Graph Theory and (partially) their connection with Logic, Complexity and Algorithms. A graph G consists of two sets (V, E), where the elements of the first set are its vertices and the second consists of 2-element subsets of V (not necessarily all), called edges. We say that two vertices v, u of a graph are connected by an edge if $\{u, v\} \in E$. A graph parameter is a function mapping a graph to a non-negative integer. We explore three graph parameters, namely tree-width, path-width and tree-depth of a graph, concentrating mostly on tree-depth.

The tree-depth (also known as the vertex ranking problem [5], or the ordered colouring problem [34]) of a graph is equal to the minimum integer k such that we can colour all of its vertices (with colours $1, 2, \ldots, k$) in a way that no two

[1]Sudoku has been proven to be NP-complete.

vertices connected by an edge have the some colour and every path whose endpoints have the some colour contains some vertex of greater colour. Tree-depth has received much attention lately because of the theory of graph classes of bounded expansion, developed by J. Nešetřil and P. Ossona de Mendez in [49, 50, 51, 52, 48]. Furthermore, the tree-depth of a graph is equivalent to the minimum-height of an elimination tree of a graph [10, 12, 48] (this measure is of importance for the parallel Cholesky factorization of matrices [43]).

Moreover, we present one of the most important parts of Graph Minors Theory, due to its numerous applications in Algorithm Design, which was introduced and developed by N. Robertson and P. Seymour. Its main result, the Graph Minor Theorem (also known as the Robertson-Seymour Theorem and formerly known as Wagner's Conjecture), is that the finite graphs are well-quasi-ordered by the minor relation. A direct consequence of this result is that every minor-closed class can be characterized by a finite set of minor-minimal graphs, called its obstruction set. As the class $\mathcal{G}_k$ consisting of all graphs with tree-depth at most k is minor-closed for every $k \in \mathbb{N}$ it follows, that each one of these classes has a finite obstruction set.

We also present some important relations between tree-depth and the other parameters (tree-width and path-width) and its properties (reduction-finiteness lemmata) proven by J. Nešetřil and P. Ossona de Mendez. In this thesis we give an alternative definition of tree-depth and we examine the set of minor-minimal graphs not belonging in $\mathcal{G}_k$ for $k \geq 0$. We call these graphs *minor-obstructions for tree-depth of level* k and we denote them as $\mathbf{obs}(\mathcal{G}_k)$.

Our main result (Chapter 4) is a structural lemma that constructs new obstructions from obstructions of lower values of k. This permits us to identify all acyclic graphs in $\mathbf{obs}(\mathcal{G}_k)$ for every $k \geq 0$. So far, such a parameterized set of acyclic obstructions is known only for the classes of bounded pathwidth [68] and its variations (search number [56], proper-pathwidth [68], linear-width [69]). Moreover, using counting arguments, we prove that there are *exactly* $\frac{1}{2}2^{2^{k-1}-k}(1+2^{2^{k-1}-k})$ acyclic graphs in $\mathbf{obs}(\mathcal{G}_k)$ and this is the first time where an exact enumeration for such a class is derived.

Our next result (Chapter 5) is a general reduction lemma, on the structure of the obstructions, and the identification of the sets $\mathbf{obs}(\mathcal{G}_k)$ for $k \leq 3$. We thus derive a complete characterisation for the classes $\mathcal{G}_k$ for $k \leq 3$.

We finally (Chapter 6) proceed to present two meta-algorithmic theorems. The first one was proven by B. Courcelle [8] in 1990, and guarantees the tractability of a wide class of properties of graphs (MSOL-expressible) on graphs with bounded tree-width. The second one was proven by J. Nešetřil and P. Ossona de Mendez and guarantees the tractability of a wide class of graphs properties (FOL-expressible) on graphs of bounded expansion. We concentrate mostly on the second theorem, giving a brief introduction of the theory of classes of bounded expansion and the importance of tree-depth in this.

The results of Chapter 4 and Chapter 5 have been accepted to the *European Conference on Combinatorics, Graph Theory and Applications* (EUROCOMB '09) and an extended abstract is going to be published at the *Electronic Notes on Discrete Mathematics*. [27]

Chapter 2

Basic Notions

In this chapter we introduce some basic notions that we shall use throughout this Thesis. They include the interpretation of mathematical notation, a wider presentation of the necessary graph-theoretic notions and especially of some parameters on graphs. Finally, they include a brief introduction to logic over graphs.

2.1 Basics

$\mathbb{R}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{N}$ denote the sets of the real numbers, integers, rational numbers and natural numbers, respectively. Let A be a set, we denote by $\mathcal{P}_k(A)$ the set of all subsets of A with exactly k elements. For integers m, n the interval $\{m, m+1, \ldots, n\}$ is denoted $[m, n]$ and it is empty if $n < m$. Furthermore, let $[n] = [1, n]$. By the sign $\sim$ we mean approximately equal and we use $\alpha \doteq d$ to represent a numerical approximation of the real α by the decimal d, with the last digit of d being at most ± 1 from its actual value.

2.2 Graphs

Graph Theory begins its journey from the famous problem of The Seven Bridges of Königsberg. The city of Königsberg in Prussia (now Kaliningrad (Калининград), Russia) was set on both sides of the Pregel River (Преголя), and included two large islands which were connected to each other and the mainland by seven bridges. The problem was to find a walk through the city that would cross each bridge once and only once. The islands could not be reached by any route other than the bridges, and every bridge must have been crossed completely every time (one could not walk halfway onto the bridge and then turn around to come at it from another side). Its answer, negative, was given by L. Eüler (1736) and this is regarded as the first result in the history of graph theory.

Definition 2.2.1. A (*simple*) *graph* $G = (V, E)$ is a pair of sets where $E \subseteq \{\{v, u\} \in \mathcal{P}_2(V) \mid u \neq v\}$. The elements of V are called *vertices* (or nodes) and the elements of E are called *edges* of the graph. For every graph G we denote $V(G)$ the set of its vertices and $E(G)$ the set of its edges.

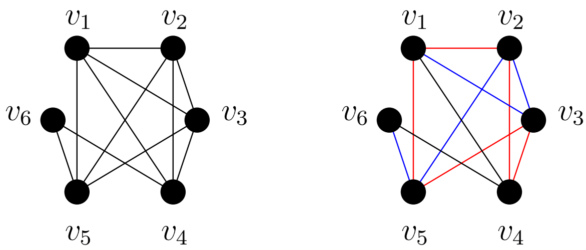


Figure 2.1: An example of a graph G. Observe that the red edges form a cycle and the blue edges form a path in the graph.

Definition 2.2.2. A *directed graph* $G = (V, E)$ is a pair of sets, a set V whose elements are called vertices and a set E of ordered pairs of vertices called directed edges, i.e., $E \subseteq \{(v, u) \in V \times V \mid v \neq u\}$. An edge $(v, u) \in E$ is directed from v to u.

We say that a graph G contains a *directed cycle* if there exist $m \geq 3$ vertices $v_1, v_2, \ldots, v_m \in V(G)$ such that $\{(v_i, v_{i+1}) \mid 1 \leq i \leq m-1\} \cup \{(v_m, v_1)\} \subseteq E(G)$ (see Figure 2.2).

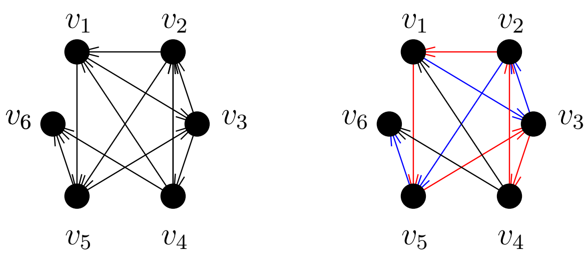
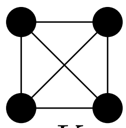


Figure 2.2: An example of a directed graph G. Observe that the red edges form a directed cycle and the blue edges form a directed path in the graph.

From now on we assume that all the graphs are simple, i.e. not directed, unless otherwise stated. The number of vertices of a graph G is its *order*, denoted as $|G|$, and its number of edges is its *size*, denoted as $\|G\|$. Given a graph G and

$v, u \in V(G)$ we say that v, u are *adjacent* if $\{v, u\} \in E(G)$. The set $N_G(v) = \{u \in V \mid u \neq v \text{ and } \{v, u\} \in E(G)\}$ is called the neighbourhood of v in G and the set $N_G[v] = N_G(v) \cup \{v\}$ is called the closed neighbourhood of v in G. For each $v \in V(G)$, $\deg_G(v) = |N_G(v)|$ and we denote $\delta(G) = \min_{v \in V(G)} |N_G(v)|$ and $\Delta(G) = \max_{v \in V(G)} |N_G(v)|$. For example, see Figure 2.1, $N_G(v_1) = \{v_2, v_3, v_4, v_5\}, \deg_G(v_i) = 4, 1 \leq i \leq 5$ and $\deg_G(v_6) = 2$. Therefore, $\Delta(G) = 4$ and $\delta(G) = 2$.

For $n \geq 1$ let K_n be the *complete graph* (or clique) with n vertices, i.e., $K_n = (V, E)$, where $V = \{i \in \mathbb{N} \mid 1 \leq i \leq n\}$ and $E = \mathcal{P}_2(V)$.



K_4

Figure 2.3: The clique with 4 vertices

The size of the largest clique in a graph G, called *clique number* and denoted $\omega(G)$, is the maximum $k \in \mathbb{N}$ such that $K_k \subseteq G$. In Figure 2.3 you can see the clique with 4 vertices and in Figure 2.1, the vertices v_1, v_2, v_3, v_4 form a clique and no set of 5 vertices forms a clique in G. Therefore, $\omega(G) = 4$.

A *path* in G of *length* $n \geq 0$ from a vertex v_0 to a vertex v_n is a sequence $v_0, v_1, \ldots, v_n \in V(G)$ of distinct vertices such that $\{v_{i-1}, v_i\} \in E(G)$ for every $i \in [n]$. A graph G is *connected* if it is non-empty and for all $v, u \in V(G)$ there exists a path from v to u. (See Figure 2.1 for an example).

The *distance* $\mathbf{dist}_G(v, u)$ between two vertices v, u of a graph G is the minimum length of a path linking v and u, or ∞ if v and u do not belong to the same connected component. The *radius* of a connected graph G is: $\rho(G) = \min_{r \in V(G)} \max_{u \in V(G)} d(r, u)$.

A *cycle* in a graph $C = (V, E)$ of *length* $n \geq 3$ is a sequence $v_1, \ldots, v_n \in V(G)$ of distinct vertices such that $\{v_i, v_{i+1}\} \in E(G)$ for all $i \in [n-1]$ and$\{v_1, v_n\} \in E(G)$. (See Figure 2.1 for an example). The minimum length of a cycle contained in a graph G is its *girth.* A graph G is called *acyclic* or a *forest* if it does not contain any cycle. Moreover, if a graph G is both acyclic and connected it is called a *tree.*

The $k \times k$ *grid* is the graph whose set of vertices is the set $V = [k] \times [k]$ and whose edge set is $\{\{(i, j), (i', j')\} \in \mathcal{P}_2(V) \mid |i - i'| + |j - j'| = 1\}$. (Figure 2.4)

A *rooted tree* is a triple $T = (V(T), E(T), r(T))$, where $(V(T), E(T))$ is a tree and $r(T) \in V(T)$ is a distinguished vertex called the *root* of the tree. A vertex v is the *parent* of a vertex u and u is a child of v if v is the predecessor of u on the unique path from the root $r(T)$ to u.

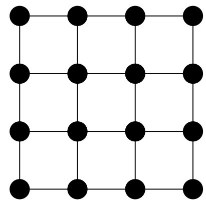


Figure 2.4: The 4×4 grid

Definition 2.2.3. A graph G is called an *interval graph* if there exists a family $\{I_v | v \in V(G)\}$ of real intervals such that $I_u \cap I_v \neq \emptyset$ if and only if $\{u, v\} \in E(G)$.

A family of real intervals $\mathcal{F} = \{I_i \mid i \in \mathcal{I}\}$ is called *nested* if for every $i, j \in \mathcal{I}$ either $I_i \subseteq I_j$ or $I_i \cap I_j = \emptyset$. A graph G is called a *nested interval graph* if there exists a family $\{I_v | v \in V(G)\}$ of nested real intervals such that $I_u \cap I_v \neq \emptyset$ if and only if $\{u, v\} \in E(G)$.

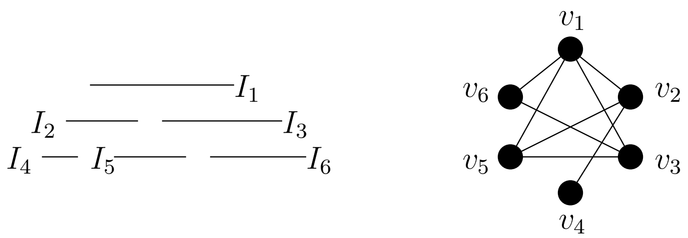


Figure 2.5: An example of an interval graph where v_i maps to $I_i, 1 \leq i \leq 6$.

Definition 2.2.4. Graphs drawn in the plane in such a way that no two edges intersect in a point other than a common end are called *plane* graphs and abstract graphs that can be drawn in such a way are called *planar* graphs. Moreover, if a graph has an embedding in the plane such that the vertices lie on a fixed circle and the edges lie inside the disk of the circle and don't intersect it is called *outerplanar*.

Given $G = (V, E)$ and $G' = (V', E')$ two graphs we set $G \cup G' = (V \cup V', E \cup E')$ and $G \cap G' = (V \cap V', E \cap E')$.

Definition 2.2.5. Let G and G' be two graphs, $v \in V(G)$ and $e = \{v_1, v_2\} \in E(G)$.

1. If $G \cap G' = \emptyset$, G and G' are *disjoint* graphs.

2. If $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$, G' is a *subgraph* of G, denoted $G' \subseteq G$. If $G' \subseteq G$ and $G \neq G'$ then G' is a *proper subgraph* of G. In any graph G, the maximal connected subgraphs of G are called (*connected*) *components*. The set of all the connected components of a graph G is denoted by $\mathcal{C}(G)$.

3. If $G' \subseteq G$ and $E(G')$ contains all the edges of $\{x, y\} \in E(G)$ with $x, y \in V(G')$, then G' is an *induced subgraph* of G. We say that $V(G')$ *induces* G' in G, and write $G' = G[V(G')]$. Thus, if $U \subseteq V(G)$ is any set of vertices, then $G[U]$ denotes the graph on U whose edges are precisely the edges of G with both ends in U.

4. The graph obtained from G by *deleting* one of its *vertices*, denoted $G \setminus v$, is the graph $G \setminus v = (V(G) \setminus \{v\}, \{e \in E(G) \mid v \notin e\}$.

5. The graph obtained from G by *deleting* one of its *edges*, denoted $G \setminus e$, is the graph $G \setminus e = (V(G), E(G) \setminus \{e\})$.

6. We say that a set $X \subseteq V(G)$ *separates* G if $G \setminus X$ is not connected, and we call X a *separator* of G. Moreover, if $X = \{v\}$ for some $v \in V(G)$ then v is an *articulation point* of G. G is called *k-connected* (for $k \in \mathbb{N}$) if $|G| > k$ and $G \setminus X$ is connected for every set $X \subseteq V(G)$ with $|X| < k$.

7. The graph obtained from G by *contracting* one of its *edges*, denoted $G \,/\, e$, is the graph $G \,/\, e = \tilde{G} = (V(\tilde{G}), E(\tilde{G}))$, where

$$\begin{aligned} V(\tilde{G}) &= (V(G) \setminus \{v_1, v_2\}) \cup \{v_{\text{new}}\} \\ E(\tilde{G}) &= (E(G) \setminus (\{e\} \cup \{e' \in E(G) \mid e \cap e' \neq \emptyset\})) \bigcup \\ &\quad \{\{v_{\text{new}}, w\} \in V(\tilde{G}) \times V(\tilde{G}) \mid \{v_1, w\} \in E(G) \text{ or } \{v_2, w\} \in E(G)\} \end{aligned}$$

8. If G' is obtained from G by applying vertex and edge deletions and edge contractions then G' is called a *minor* of G and it is denoted by $G' \leq G$.

9. G is *k-colourable* if there exists a mapping $\chi : V(G) \rightarrow [k]$ such that if $\{v, u\} \in E(G)$ then $\chi(v) \neq \chi(u)$, for every $v, u \in V(G)$. Moreover, χ is called a k-colouring and $\chi(G)$ (*the chromatic number of* G) is the minimum k such that G is k-colourable.

10. A *star colouring* of a graph G is a (proper) vertex colouring in which every path on four vertices uses at least three distinct colours. Equivalently, in a star colouring, the induced subgraphs formed by the vertices of any two colours have connected components that are star graphs. The *star chromatic number* $\chi_s(G)$ of G is the least number of colours needed to star colour G.

11. G has a *k-vertex ranking* if G is k-colourable, $\rho : V(G) \to [k]$ is the corresponding k-colouring and if P is a path from v to u where $\rho(v) = \rho(u)$ then there exists a vertex $v' \in V(P)$ such that $\rho(v') > \rho(v)$.

12. G is *homomorphic* to G' if there exists a mapping, called *homomorphism*, $\psi : V(G) \to V(G')$ which preserves adjacency: $\{\psi(x), \psi(y)\} \in E(G')$ for every $\{x, y\} \in E(G)$. G and G' are called *hom-equivalent* if there exists a homomorphism from G to G' and a homomorphism from G' to G.

13. G and G' are called *isomorphic*, denoted $G \simeq G'$, if there exists a bijection $\phi : V(G) \to V(G')$ such that $\{v, u\} \in E(G) \Leftrightarrow \{\phi(v), \phi(u)\} \in E(G')$ for every $v, u \in V(G)$. If $G = G'$, ϕ is called an *automorphism*. Furthermore, we denote by $\mathbf{Aut}(G)$ the group of the automorshisms of G. A graph G which admits only one automorphism (namely the identity map) is called *asymmetric*.

2.3 Parameters of graphs

A *graph parameter p* is a function mapping a graph to a non-negative integer k. In this section we introduce three parameters of graphs.

The first parameter is the tree-width of a graph. It became very famous due to its use in the Graph Minors Theory and the meta-algorithmic theorem of B. Courcelle [8] (see Section 6.1).

The second one is the path-width of a graph which is also of great importance as it constitutes an important part of the Graph Minors Theory and the proof of Wagner's Conjecture.

The last one is the tree-depth of a graph. This parameter has not been as much studied as the tree-width and path-width of a graph but has received a lot of attention lately as its importance became known from J. Nešetřil and P. Ossona de Mendez in their theory of classes of bounded expansion (see Section 6.2).

2.3.1 Tree-width

Definition 2.3.1. For a given set M of objects (for which intersection makes sense), the *intersection graph G_M* of these objects has $V(G_M) = M$ and $E(G_M) = \{\{m_1, m_2\} \in \mathcal{P}_2(M) \mid m_1 \neq m_2 \text{ and } m_1 \cap m_2 \neq \emptyset\}$.

Observe that an interval graph is the intersection graph of the corresponding family of intervals. (For more on Intersection Graph Theory, see [45])

Definition 2.3.2. A graph G is *chordal* (or *triangulated*) if every cycle contained in G of length $k \geq 4$ contains a chord, where a *chord* is an edge joining two nonconsecutive vertices of a cycle.

The next theorem connects the notion of subtrees of a tree to the notion of chordal graphs. Its importance on this thesis lies to the fact that this was the first notion to appear which was closely related to the tree-width of a graph.

Theorem 2.3.1 ([26])**.** *A graph is chordal if and only if it is the intersection graph of subtrees of a tree.*

The definition of a *tree decomposition* of a graph G first appeared in [60].

Definition 2.3.3. Let G be a graph, T a tree and let $\mathcal{V} = (V_t)_{t \in T}$ be a family of vertex sets $V_t \subseteq V(G)$ indexed by the vertices t of T. The pair $(T, \mathcal{V})$ is called a *tree-decomposition* of G if it satisfies the following conditions:

1. $V(G) = \bigcup_{t \in T} V_t$
2. for every edge $e \in E(G)$ there exists a vertex $t \in V(T)$ such that both ends of e lie in V_t
3. $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever t_2 is on the path between t_1 and t_3.

The *width* of a tree decomposition is $\mathbf{tw}(G, (T, \mathcal{V})) = \max\{|V_t| - 1 \mid t \in V_t\}$ and the *tree-width* of G is

$$\mathbf{tw}(G) = \min\{\mathbf{tw}(G, (T, \mathcal{V})) \mid (T, \mathcal{V}) \text{ is a tree decomposition of } G\}$$

Definition 2.3.4. A *triangulation* of a graph G is a chordal graph G' such that $V(G) = V(G')$ and $E(G) \subseteq E(G')$.

Lemma 2.3.1 ([60])**.** *Let G be a graph. Then $\mathbf{tw}(G)$ is equal to the minimum value of $\omega(G') - 1$ over all triangulations G' of G.*

From Theorem 2.3.1, Definition 2.3.3 and Definition 2.3.4 we can derive the following.

Lemma 2.3.2. *The tree-width of a graph G, is the minimum $k \in \mathbb{N}$ such that there exists a chordal graph H where $G \subseteq H$ and $\omega(H) \leq k + 1$.*

Intuitionally, the tree-width of a graph G is a way to measure how far is a graph G from looking like a tree (in the topological sense).

2.3.2 Path-width

Definition 2.3.5. The *interval thickness* of a graph G, denoted $\theta(G)$, is the minimum clique number of an interval graph H that contains G as a subgraph.

The following game, called *node searching*, on a graph G was introduced in [37] and is a slight variation of the game (known as the Edge Search Game) that was introduced in [55].

Definition 2.3.6. The *node searching* on a graph G is a one-player game with the following rules:

- Initially, all edges are contaminated.
- A move can consist of
  1. Putting a searcher on a vertex,
  2. Removing a searcher from a vertex,
  3. Moving a searcher over an edge from a vertex to an adjacent vertex.
- A contaminated vertex becomes cleared when there is a searcher on both ends of the edge.
- A clear edge becomes re-contaminated when there is a path from the edge to a contaminated edge that does not pass through a vertex with a searcher on it.

Definition 2.3.7. The *node search number* of a graph G, $\mathbf{ns}(G)$, is the minimum number of searchers needed to clear all edges of G.

Theorem 2.3.2 ([36])**.** *Let G be a graph. Then $\theta(G) = \mathbf{ns}(G)$.*

Definition 2.3.8. Let G be a graph. A *path-decomposition* $(P, \mathcal{V})$ is a tree-decomposition $(T, \mathcal{V})$ of G, where T is a path. The *width* of a path-decomposition, $\mathbf{pw}((P, \mathcal{V}))$, and the *path-width* of a graph G, $\mathbf{pw}(G)$, are defined analogously.

Definition 2.3.9. Given a graph G, a *vertex separator* in G is a subset $S \subseteq V(G)$ of vertices whose removal separates G into two components of approximately equal size. (For more on graph separation, see [42].)

Theorem 2.3.3 ([35])**.** *Let G be a graph. Then $\mathbf{pw}(G) = \mathbf{vs}(G)$.*

Theorem 2.3.4 ([20])**.** *If G is a graph then $\mathbf{ns}(G) = \mathbf{vs}(G) + 1$.*

A direct consequence of Theorems 2.3.2, 2.3.3 and 2.3.4 is the following.

Lemma 2.3.3. *The path-width of a graph G is the minimum $k \in \mathbb{N}$ such that there exists an interval graph G_I where $G \subseteq G_I$ and $\omega(G_I) \leq k+1$.*

Intuitionally, the path-width of a graph G is a way to measure how far is a graph G from looking like a path (in the topological sense).

Remark 2.3.1. Let us remark here, that analogously to path-width, there exist games defined in a graph G whose minimum solution is equal to $\mathbf{tw}(G)$. (See [11] and [65]).

2.3.3 Tree-depth

The notion of tree-depth has appeared many times in the bibliography and it is also known as the *vertex ranking* of a graph [5], the *minimum-height elimination tree* of a graph [10, 12, 48] (if the graph is connected) and the minimum number of colours in a *centered colouring* [48]. Here, however, we give the following.

Definition 2.3.10. The *tree-depth* of a graph G, denoted $\mathbf{td}(G)$, is the minimum $k \in \mathbb{N}$ such that there exists a nested interval graph G_I where $G \subseteq G_I$ and $\omega(G_I) \leq k$.

Later (see Section 3.2), we prove its equivalence to the other parameters.

2.4 Logic in graphs

A very important area of both mathematics and computer science is Logic. It has been widely explored, since Aristotle was the first to suggest a formal system that was then used by Euclid. Although Logic and its history are a very interesting subject, its exploration and presentation is out of the purpose of this brief introduction. We will, however, mention R. Dedekind (1831 - 1916), G. Peano (1858 - 1932), D. Hilbert (1862 - 1943) and K. Gödel whose contribution was determining and their results and suggestions were ahead of their time.

Here, we consider logics over graphs. (For more on Logic see [21] and [46])

2.4.1 First-order logic over graphs

Definition 2.4.1. The syntax of the *first-order logic* is the following:

- Infinite supply of *individual variables*, usually denoted by lowercase letters x, y, z.

- First-order formulas in the language of graphs are built up from *atomic formulas* $E(x, y)$ and $x = y$ by using the usual *Boolean connectives* $\neg$ (negation), $\wedge$ (conjuction), $\vee$ (disjunction), $\rightarrow$ (implication), $\leftrightarrow$ (bi-implication), existential quantification $\exists x$ and universal quantification $\forall x$ over individual variables.

Individual variables range over vertices of a graph. The atomic formula $E(x, y)$ express adjacency, and the formula $x = y$ expresses equality. From this, the free variables, the sentences and the semantics of first-order logic are defined in the obvious way.

For example, a *dominating set* in a graph $G = (V, E)$ is a set $S \subseteq V$ such that for every $v \in V$, either v belongs to S or v is adjacent to a vertex u that belongs to S. (see Figure 2.6) The following first-order sentence (parameterized by k) $\mathbf{dom}_k$ says that a graph has a dominating set of size k:

$$\mathbf{dom}_k = \exists x_1 \exists x_2 \ldots \exists x_k \left(\bigwedge_{1 \le i < j \le k} \neg(x_i = x_j) \wedge \forall y \left(\bigvee_{i=1}^{k} ((y = x_i) \vee E(y, x_i)) \right) \right)$$

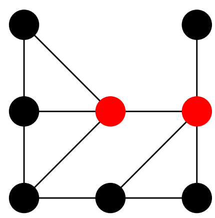


Figure 2.6: The dominating set of a graph (red vertices).

2.4.2 Monadic second-order logic over graphs

Definition 2.4.2. The syntax of the monadic-second order logic is the following:

- Infinite supply of *individual variables*, usually denoted by lowercase letters x, y, z (as above).
- Infinite supply of *set variables*, usually denoted by uppercase letters X, Y, Z.
- Monadic second-order formulas in the language of graphs are built up from *atomic formulas* $E(x, y)$, $x = y$ and $X(x)$ (for set variables X and individual variables x) by using the *Boolean connectives* $\neg$ (negation), $\wedge$ (conjuction), $\vee$ (disjunction), $\rightarrow$ (implication), $\leftrightarrow$ (bi-implication), existential quantification

$\exists x, \exists X$ and universal quantification $\forall x, \forall X$ over individual variables and set variables.

Individual variables range over vertices of a graph (as above) and set variables are interpreted by sets of vertices. The atomic formula $E(x, y)$ express adjacency (as above), the formula $x = y$ expresses equality (as above) and $X(x)$ means that the vertex x is contained in the set X. The semantics of the monadic-second order logic is defined in the obvious way.

Continuing the above example of first-order logic, the following formula says that X is a dominating set:

$$\mathbf{dom}(X) = \forall y\, (X(y) \vee \exists z(E(y, z) \wedge X(z)))$$

Chapter 3

Properties of tree-depth

In this chapter we discuss the properties of tree-depth and its relation to the other parameters.

3.1 Graph minors

Let us first discuss some important results of N. Robertson and P. Seymour in the Graph Minors Theory that we will use from now on.

Definition 3.1.1. Let $\mathcal{C}$ be a class of graphs. We say that $\mathcal{C}$ is *closed under taking of minors* (*minor-closed*) if $G \in \mathcal{C}$ and $H \leq G$ implies that $H \in \mathcal{C}$.

If $\mathcal{H}$ is any class of graphs, then the class $\mathbf{Forb}_{\leq}(\mathcal{H}) = \{G \mid H \not\leq G \text{ for all } H \in \mathcal{H}\}$ of all graphs without a minor in $\mathcal{H}$ is a *graph property*, i.e. is closed under isomorphism. Consider the following (for a proof, see [15]).

Lemma 3.1.1. *A graph property $\mathcal{P}$ can be expressed by forbidden minors if and only if it is closed under taking of minors.*

Observe that if $\mathcal{P}$ is minor-closed then $\mathcal{P} = \mathbf{Forb}_{\leq}(\overline{\mathcal{P}})$, where $\overline{\mathcal{P}}$ is the complement of $\mathcal{P}$. However, one naturally seeks to make the set of forbidden minors as small as possible. There is indeed a smallest such set:

$$\mathcal{K}_{\mathcal{P}} = \{H | H \text{ is } \leq\text{-minimal in } \overline{\mathcal{P}}\}$$

that satisfies $\mathcal{P} = \mathbf{Forb}_{\leq}(\mathcal{K}_{\mathcal{P}})$ and is contained in every other set $\mathcal{H}$ such that $\mathcal{P} = \mathbf{Forb}_{\leq}(\mathcal{H})$. $\mathcal{K}_{\mathcal{P}}$ is called the *Kuratowski set* (or *obstruction set*) for $\mathcal{P}$ and its elements are called *obstructions*. From now on the obstruction set of a minor-closed class of graphs $\mathcal{C}$ will be denoted by $\mathbf{obs}(\mathcal{C})$.

Definition 3.1.2. A reflexive and transitive relation is called a *quasi-ordering*. A quasi-ordering $\leq$ on X is a *well-quasi-ordering*, and the elements of X are well-quasi-ordered by $\leq$, if for every infinite sequence $x_0, x_1, \dots$ in X there are indices $i < j$ such that $x_i \leq x_j$.

The following theorem is one of the deepest results of modern Combinatorics.

Theorem 3.1.1 (Graph Minor Theorem)**.** *The finite graphs are well-quasi-ordered by the minor relation* $\leq$.

This theorem was first proven for trees (also known as *Vázsonyi conjecture*) by J. Kruskal [38]. In 1963 a shorter proof (again for trees) was given by Crispin St. John Alvah Nash-Williams [47]. In the general case it was proven by N. Robertson and P. Seymour [62] in their Graph Minors Series. One of its direct implications is the following.

Lemma 3.1.2. *The Kuratowski set for any minor-closed graph property is finite.*

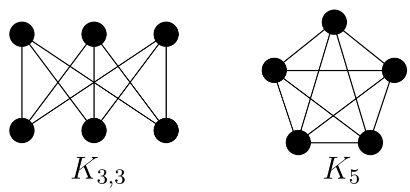


Figure 3.1: The Kuratowski set for planar graphs

The most famous example of a Kuratowski set is the set depicted in Figure 3.1. K. Kuratowski proved that a graph is planar if and only if it excludes K_5 and $K_{3,3}$ as a minor. This is the first obstruction set ever found and its proof [39] (1930) was published by K. Kuratowski[1] long before the Graph Minor Theorem was proven (2004). It is also known as the Kuratowski-Pontryagin theorem, because it is claimed that it was first proven in the unpublished notes of L. Pontryagin (Лев Семёнович Понтрягин). (Other proofs can be found in [18], [19], and [44]).

Remark 3.1.1. The algorithmic drawback of the proof of the Graph Minor Theorem and of Graph Minors Theory, in general, is that it is nonconstructive, i.e. we are assured of a finite obstruction set without being given (by the arguments that establish the theorem) a means of identifying the elements of the set, the cardinality of the set, or even the order of the largest graph in the set.

[1] K. Kuratowski proved the theorem for the relation of the *topological minors* (where instead of contracting any edge, we only contract edges whose one of the incident vertices has degree exactly 2, also called *disolving* a vertex of degree 2) and its extension to minors was observed by K. Wagner [70].

Let k be a non-negative integer. We denote by $\mathcal{G}_k$ the class of graphs with tree-depth at most k, i.e. $\mathcal{G}_k = \{G \mid \mathbf{td}(G) \leq k\}$. We also denote by $\mathcal{PW}_k$ the class of graphs with path-width at most k, i.e. $\mathcal{PW}_k = \{G \mid \mathbf{pw}(G) \leq k\}$ and by $\mathcal{TW}_k$ the class of graphs with tree-width at most k, i.e. $\mathcal{TW}_k = \{G \mid \mathbf{tw}(G) \leq k\}$.

Lemma 3.1.3 ([48, 5])**.** *If a graph H is a minor of a graph G then $\mathbf{td}(H) \leq \mathbf{td}(G)$.*

A direct consequence of Lemma 3.1.3 is the following.

Observation 3.1.1. *The class $\mathcal{G}_k$ is minor-closed for every $k \in \mathbb{N}$.*

Lemma 3.1.4 ([59])**.** *If a graph H is a minor of a graph G then $\mathbf{pw}(H) \leq \mathbf{pw}(G)$.*

Lemma 3.1.5 ([60])**.** *If a graph H is a minor of a graph G then $\mathbf{tw}(H) \leq \mathbf{tw}(G)$.*

The following is a direct implication of Lemmata 3.1.4 and 3.1.5.

Observation 3.1.2. *The classes $\mathcal{PW}_k$ and $\mathcal{TW}_k$ are minor-closed for every $k \in \mathbb{N}$.*

As it is expected by Remark 3.1.1, only a few obstruction sets have become known while a lot of graph classes are proven to be minor-closed. Therefore, the focus of research has (mostly) turned to finding lower and upper bounds on the cardinality of the obstruction set of such classes [68, 56, 69, 29, 64, 40, 17].

3.1.1 Lower bounds on the cardinality of the obstruction set of minor-closed classes

Theorem 3.1.2 ([68])**.** *The obstructions of the class $\mathcal{PW}_k$ are at least $(k!)^2$.*

Theorem 3.1.3 ([29])**.** *The obstructions of the class $\mathcal{TW}_k$ are $2^{\Omega(k \log k)}$.*

Definition 3.1.3. A *feedback vertex set*, (fvs) of a graph G is a set of vertices $X \subseteq V(G)$ such that every cycle of G passes through at least one vertex of X. For a graph G we denote by $\mathbf{fvs}(G)$ the cardinality of a minimum feedback vertex set. We denote by $\mathcal{FVS}_k$ the class of graphs $\mathcal{FVS}_k = \{G \mid \mathbf{fvs}(G) \leq k\}$. We also denote by $\mathcal{Y}_k$ the class of all outerplanar graphs in $\mathbf{obs}(\mathcal{FVS}_k)$.

Observation 3.1.3. *The class $\mathcal{FVS}_k$ is minor-closed for every $k \in \mathbb{N}$.*

Theorem 3.1.4 ([64])**.** *$|\mathcal{Y}_k| \sim \alpha \cdot k^{-\frac{5}{2}} \cdot \rho^{-k}$, where $\alpha \doteq 0.02602193$ and $\rho^{-1} \doteq 14.49381704$.*

Remark 3.1.2. It is noteworthy that the proof of Theorem 3.1.4 makes use of Analytic Combinatorics [22].

From the previous lemmata, we observe that although the cardinality of an obstruction set is finite it can be very large. In Section 4.2 we prove that the acyclic graphs in $\mathbf{obs}(\mathcal{G}_k)$ are *exactly* $\frac{1}{2}2^{2^{k-1}-k}(1+2^{2^{k-1}-k})$, thus we give a lower bound on the cardinality of $\mathbf{obs}(\mathcal{G}_k)$ that is a doubly exponential function of the parameter k.

3.1.2 Upper bounds on the cardinality of the obstruction set of minor-closed classes

We now see some results on the upper bounds on the cardinality of the obstruction set of minor-closed classes.

Theorem 3.1.5 ([40]). *Let G be a graph. If $G \in \mathbf{obs}(\mathcal{PW}_k)$, then $|E(G)|$ is at most exponential in $O(k^4)$.*

Theorem 3.1.6 ([40]). *Let G be a graph. If $G \in \mathbf{obs}(\mathcal{TW}_k)$, then $|E(G)|$ is at most doubly exponential in $O(k^5)$.*

Definition 3.1.4. Let G be a graph. A set $S \subseteq V(G)$ is a *vertex cover* of G is every edge of G is incident with a vertex in S (see Figure 3.2). We denote by $\mathbf{vc}(G)$ the minimum $k \in \mathbb{N}$ such that G has a vertex cover of cardinality k and by $\mathcal{VC}_k$ the class $\mathcal{VC}_k = \{G \mid \mathbf{vc}(G) \leq k\}$.

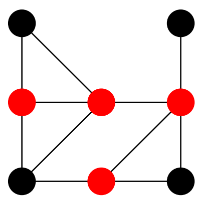


Figure 3.2: The vertex cover of a graph (red vertices).

Observation 3.1.4. *The class $\mathcal{VC}_k$ is minor-closed for every $k \in \mathbb{N}$.*

Theorem 3.1.7 ([17]). *Let G be a graph. If $G \in \mathbf{obs}(\mathcal{VC}_k)$ then $|V(G)| \leq 2(k+1)$.*

3.2 Characterisations

A *rooted forest* is a disjoint union of rooted trees. The *height* of a vertex x in a rooted forest F is the number of vertices of the path from the root (of the tree to which x belongs to) to x and is noted height(x, F). The height of F is the maximum height of the vertices of F. Let x, y be vertices of F. The vertex x is an *ancestor* of y if x belongs to the path linking y and the root of the tree to which y belongs to. The *closure* $\mathbf{clos}(F)$ of a rooted forest F is the graph with vertex set $V(F)$ and edge set $\{\{x, y\} \mid x \text{ is an ancestor of } y \text{ in } F, x \neq y\}$. A rooted forest F defines a partial order on its set of vertices: $x \leq_F y$ if x is an ancestor of y in F.

The following definition of tree-depth is attributed to J. Nešetřil and P. Ossona de Mendez and first appeared in [48].

Definition 3.2.1. Let G be a graph. The *tree-depth* of G, $\mathbf{td}(G)$, is the least $k \in \mathbb{N}$ such that there exists a rooted forest F where $G \subseteq \mathbf{clos}(F)$ and the height of F is equal to k.

The following lemma proves the equivalence of Definition 2.3.10 of tree-depth to Definition 3.2.1 .

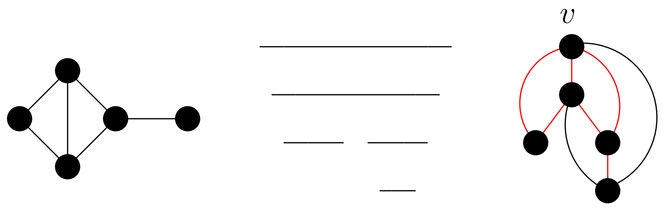


Figure 3.3: An example of the statement of Lemma 3.2.1 where v is the root of the tree and from left to right are represented the graph G, the corresponding interval graph $G_{\mathcal{I}}$ and the rooted tree $\mathcal{T}$, respectively.

Lemma 3.2.1. *Let G be a graph. Then $\mathbf{td}(G) \leq k$ if and only if there exists a rooted forest F of height lower or equal than k such that $G \subseteq \mathbf{clos}(F)$.*

Proof. Without loss of generality we may assume that G is connected. Observe that if $k = 1$ then $G = K_1$ and the lemma follows trivially. We now prove the non-trivial case where $k \geq 2$. For the straightforward let $\mathbf{td}(G) \leq k$. Then there exists a nested interval graph G_I such that $G \subseteq G_I$ and $\omega(G_I) \leq k$. Let T' be the directed graph where $V(T') = V(G_I)$ and $E(T') = \{(u, v) \in \mathcal{P}_2(V(G_I)) : ((I_v \subseteq I_u)\&(\neg\exists x \in V(G) \setminus \{u, v\})(I_v \subseteq I_x \subseteq I_u)), v \neq u\}$.

Claim. There does not exist distinct vertices $v_1, v_2, v_3 \in V(T')$ such that $(v_2, v_1), (v_3, v_1) \in E(T')$.

Proof of the Claim. Assume in contrary, that there exist distinct vertices $v_1, v_2, v_3 \in V(T')$ such that $(v_2, v_1), (v_3, v_1) \in E(T')$. Then $I_{v_1} \subseteq I_{v_i}, i = 2, 3$ and $I_{v_2} \cap I_{v_3} \neq \emptyset$. Moreover, $I_{v_1} \subseteq I_{v_2} \subseteq I_{v_3}$ or $I_{v_1} \subseteq I_{v_3} \subseteq I_{v_2}$, a contradiction.

Let T be the graph where $V(T) = V(T')$ and $E(T) = \{\{u, v\} \in \mathcal{P}_2(V(T)) \mid (u, v) \in E(T')\}$. We claim that T is acyclic. Assume in contrary, that there exists a cycle in T. Then there exists a cycle C_m in T'. From the Claim it follows that C_m is a directed cycle. Then there exist $v_1, v_2, \ldots, v_m$ distinct vertices in $V(T')$ such that $I_{v_1} \subseteq I_{v_2} \subseteq \cdots \subseteq I_{v_m} \subseteq I_{v_1}$, a contraction to the hypothesis that $k \geq 2$. Therefore T is acyclic. We claim now that T does not contain a path of length greater than k. Assume, in contrary, that T contains a path of length $k+1$ as a subgraph. Then T' contains a path P_{k+1}. From the above Claim it follows that P_{k+1} is directed. Therefore, there exist $k+1$ distinct vertices in $V(T')$ such that

$I_{v_i} \subseteq I_{v_{i+1}}, 1 \leq i \leq k$. Then $\omega(G_I) \geq k+1$, a contradiction. Finally observe that $G \subseteq \mathbf{clos}(T)$ and the straightforward direction follows.

Conversely, assume that there exists a rooted forest T of height lower or equal than k such that $G \subseteq \mathbf{clos}(T)$. Consider the following algorithm that recursively maps $V(G)$ to a family of real intervals $\mathcal{I}$. First, map the root of T to the interval $(0, 1)$. When a vertex v is already mapped to an interval I divide the interval to $\deg_G(v) - 1$ disjoint intervals and map each one of its children to one of the intervals in a way that no two vertices are mapped in the same interval. It trivially follows that $\mathcal{I}$ is a family of nested intervals, $\omega(G_{\mathcal{I}}) \leq k$ and $G \subseteq G_{\mathcal{I}}$. Therefore, $\mathbf{td}(G) \leq k$ and the lemma follows. □

Definition 3.2.2. A *centered colouring* of a graph G is a vertex colouring c, such that, for any induced connected subgraph H of G, some colour $c(H)$ appears exactly once in H. Note that a centered colouring is necessarily proper.

Definition 3.2.3. The *vertex ranking* of a graph G is the minimum $k \in \mathbb{N}$ such that G admits a k-vertex ranking.

Lemma 3.2.2 ([48])**.** *The minimum number of colours in a centered colouring of a graph G is exactly $\mathbf{td}(G)$.*

Lemma 3.2.3 ([48])**.** *Any vertex ranking is a centered colouring and conversely any centered colouring defines a vertex ranking with the same number of colours. Thus the vertex ranking of a graph G is the minimum number of colours in a centered colouring of G.*

An immediate consequence of Lemmata 3.2.2 and 3.2.3 is the following.

Corollary 3.2.1 ([48])**.** *Let G be a graph. Then $\mathbf{td}(G)$ is equal to its vertex ranking.*

Definition 3.2.4. Let G be a connected graph. An *elimination tree* for G is a rooted tree Y with vertex set $V(G)$ defined recursively as follows. If $V(G) = \{x\}$ then Y is reduced to $\{x\}$. Otherwise, choose a vertex $r \in V(G)$ as the root of Y. Let $G_1, G_2, \ldots, G_p$ be the connected components of $G \setminus r$. For each component G_i let Y_i be an elimination tree. Y is defined by making each root r_i of Y_i adjacent to r.

Lemma 3.2.4 ([48, 12])**.** *Let G be a connected graph. A rooted tree Y is an elimination tree for G if and only if $G \subseteq \mathbf{clos}(Y)$. Hence $\mathbf{td}(G)$ is the minimum height of an elimination tree for G.*

A direct implication of Lemma 3.2.4 is the following.

Lemma 3.2.5. *Let G be a graph and let $G_1, G_2, \ldots, G_p$ be its connected components. Then,*

$$\mathbf{td}(G) = \begin{cases} 1, & \text{if } |V(G)| = 1 \\ 1 + \min_{v \in V(G)} \mathbf{td}(G \setminus v), & \text{if } p = 1 \text{ and } |V(G)| > 1 \\ \max_{i=1}^{p} \mathbf{td}(G_i), & \text{otherwise} \end{cases}$$

3.3 The inequalities and the properties

In this section we discuss some interesting inequalities between tree-depth, tree-width and path-width and some reduction and finiteness lemmata for tree-depth.

3.3.1 The Inequalities

A graph and its tree-depth

Lemma 3.3.1 ([50])**.** *If G is a graph and P_k is the longest path in G then*

$$\lceil \log_2(k+1) \rceil \leq \mathbf{td}(G) \leq \binom{k+2}{2} - 1$$

Lemma 3.3.2 ([60, 6, 2, 48])**.** *Every graph G of order n with no minor isomorphic to K_h has tree-depth at most $(2+\sqrt{2})\sqrt{h^3 n}$.*

Lemma 3.3.3 ([48])**.** *Let G be a tree of size m having p leaves such that $\mathbf{td}(G) = k$. Then*

$$m \leq (2^{k-1} - 1)p$$

Tree-depth and its relation to the other parameters of a graph

Definition 3.3.1. Let G be a graph of order n. An *α-vertex separator* of G is a subset S of vertices such that every component of $G \setminus S$ contains at most αn vertices.

Lemma 3.3.4 ([48])**.** *Let G be a graph of order n and let $s_G : \{1, 2, \ldots, n\} \to \mathbb{N}$ be defined by*

$$s_G(i) = \max_{\substack{|A| \leq i \\ A \subseteq V(G)}} \min \left\{ |S| \mid S \text{ is a } \tfrac{1}{2}\text{-vertex separator of } G[A] \right\}$$

Then:

$$\mathbf{td}(G) \leq \sum_{i=1}^{\log_2 n} s_G\left(\frac{n}{2^i}\right).$$

Lemma 3.3.5 ([5, 48]). *For any connected graph G of order n,*

$$\mathbf{tw}(G) + 1 \leq \mathbf{td}(G) \leq (\mathbf{tw}(G) + 1) \cdot \log_2 n.$$

Observation 3.3.1. *If $G = P$ is a path then the upper bound is tight (Figure 3.4) and for $G = K_2$ the lower bound is tight.*

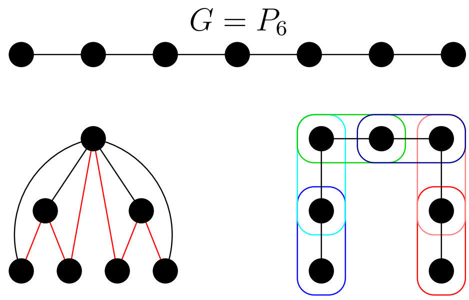


Figure 3.4: An example of the tightness of the inequality of Corollary 3.3.5

Lemma 3.3.6 ([6]). *For any connected graph G of order n,*

$$\mathbf{tw}(G) \leq \mathbf{pw}(G) \leq \mathbf{tw}(G) \cdot \log_2 n$$

Corollary 3.3.5, Lemmata 3.3.6 and 2.3.3 and Definition 2.3.10 imply the following.

Lemma 3.3.7. *For any connected graph G of order n,*

$$\mathbf{pw}(G) + 1 \leq \mathbf{td}(G) \leq (\mathbf{pw}(G) + 1) \cdot \log_2 n$$

Chordal, interval and nested interval graphs (a hierarchy)

The relation of the chordal, interval and nested interval graphs is noteworthy. That is because a hierarchy of the classes (according to refinement) follows, making it easier to comprehend most of the above inequalities.

Chordal graph
$\downarrow$
Chordal graph & AT-free = Interval graph
$\downarrow$
Interval graph & P_3-induced subgraph free = Nested interval graph

Definition 3.3.2. An *asteroidal triple* (AT) of a graph is a set of three independent vertices such that any two of them are joined by a path avoiding the closed neighbourhood of the third. A graph is called *asteroidal triple-free* (AT-free) if it does not contain an AT.

Lemma 3.3.8 ([41, 30])**.** *A graph G is an interval graph if and only if it is chordal and AT-free.*

Lemma 3.3.9 ([67])**.** *An interval graph is a nested interval graph if and only if is does not contain P_3 as an induced subgraph (P_3-induced subgraph free).*

Remark 3.3.1. The notion of the nested interval graphs was first introduced by E. Wolk ([71, 72]) in 1962 while the notion of the interval graphs is one of the oldest notions of Graph Theory.

Remark 3.3.2. It can be proven that a graph G is a nested interval graph if and only if it is (C_4, P_3)-induced subgraph free. [1]

3.3.2 The Properties

In this section we present two powerful reduction theorems (and finiteness results) related to tree-depth.

Definition 3.3.3. Let $G = (V, E)$ be a graph and $f \in \mathbf{Aut}(G)$. We say that f has the *fixed-point property* if, for every connected subgraph H of G, $f(H) \cap H$ is either empty or contains a vertex $x \in V(H)$ such that $f(x) = x$. f is also said to be *involuting* if $f \circ f$ is identical map.

Let $g : V(G) \to [N]$ be any mapping, f is said to be *g-preserving* if $f \circ g = g$.

Theorem 3.3.1 ([48])**.** *There exists a function $F : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with the following property: For any integer N, any graph G of order $n > F(N, \mathbf{td}(G))$ and any mapping $g : V(G) \to [N]$, there exists a non-trivial involuting g-preserving automorphism $\mu : G \to G$ with the fixed-point property.*

Corollary 3.3.1. *Any asymmetric graph of tree-depth at most t has order at most $F(1, t)$.*

Corollary 3.3.2. *For any graph G and any mapping g from $V(G)$ to a set of cardinality N, there exists a subset A of $V(G)$ of cardinality at most $F(N, t)$, such that G has a g-preserving homomorphism to $G[A]$.*

In particular, any graph G is hom-equivalent to one of its induced subgraphs of order at most $F(1, \mathbf{td}(G))$.

Corollary 3.3.3. *Let $k \geq 1$ be an integer. Then, the class $\mathcal{G}_k$ of all graphs G with $\mathbf{td}(G) \leq k$ includes a finite subset $\hat{\mathcal{G}}_k$ such that, for every graph $G \in \mathcal{G}_k$, there exists $\hat{G} \in \hat{\mathcal{G}}_k$ which is hom-equivalent to G and isomorphic to an induced subgraph of G.*

The previous consequences indicate that tree-depth is a good "scale" for asymmetric graphs and even cores: For each given tree-depth we get only finitely many cores.

Finally, an interesting property of tree-depth is the following.

Theorem 3.3.2 ([48])**.** *There exists a function $\mu : \mathbb{N} \to \mathbb{N}$, such that any graph G has a connected subgraph $H \subseteq G$, so that $\mathbf{td}(H) = \mathbf{td}(G)$ and $|E(H)| \leq \mu(\mathbf{td}(G))$.*

Tree-depth vs. Tree-width

Definition 3.3.4. A countable partially ordered set is said to be *universal* if it contains any countable partial order as an (induced) suborder.

Definition 3.3.5. A graph G is called *series-parallel* if it may be turned into K_2 by a sequence of the following operations:

- Replacement of a pair of parallel edges with a single edge that connects their common endpoints.
- Replacement of a pair of edges incident to a vertex of degree 2 with a single edge.

An implication of Definition 3.3.5 the following.

Lemma 3.3.10. *A graph G is series-parallel if and only if it is biconnected and $\mathbf{tw}(G) \leq 2$.*

Theorem 3.3.3 ([32])**.** *The class of all series parallel graphs of given girth is universal.*

Remark 3.3.3. By Theorem 3.3.3 and Lemma 3.3.10 follows that tree-width does not share the same nice properties of tree-depth.

3.4 The minor-hierarchy of the parameters

In this section we see how a natural hierarchy occurs to the tree-width, the path-width and tree-depth of a graph G by excluding specific graphs as minors.

Lemma 3.4.1 ([61, 63, 58, 16])**.** *For every planar graph H there exists an integer w such that for every graph G with no minor isomorphic to H, $\mathbf{tw}(G) \leq w$.*

Remark 3.4.1. What N. Robertson and P. Seymour actually proved in [61] is that a graph G has bounded tree-width if it does not contain a grid as minor and by that they derived Lemma 3.4.1.

Intuitionally, if G does not contain a grid as a minor, then it looks like a forest.

Lemma 3.4.2 ([59, 14])**.** *For every forest H there exists an integer w such that for every graph G with no minor isomorphic to H, $\mathbf{pw}(G) \leq w$.*

Intuitionally, if G does not contain a forest as a minor, then it is looks like a path.

Lemma 3.4.3 ([48])**.** *For every path H there exists an integer w such that for every graph G with no minor isomorphic to H, $\mathbf{td}(G) \leq w$.*

You can also observe that Lemma 3.4.3 also follows from Lemma 3.3.1. A hierarchy then follows trivially from Lemmata 3.4.1, 3.4.2 and 3.4.3 and is depicted in Table 3.1.

Excluded Minor of G	**Large Scale Structure of G**
Grid	Forest (bounded tree-width)
Forest	Path (bounded path-width)
Path	Point (bounded tree-depth)

Table 3.1: The minor-hierarchy of the parameters

Remark 3.4.2. Notice that the minor-hierarchy above corresponds to the hierarchy that we observed at Section 3.3.1.

Chapter 4

Lower bounds on $\mathbf{obs}(\mathcal{G}_k)$

In this chapter we prove a structural lemma that recursively constructs new obstructions from obstructions of lower levels. This permits us to identify all acyclic graphs in $\mathbf{obs}(\mathcal{G}_k)$ for every $k \geq 0$ and count them. To do so we use methods of Algebraic Graph Theory (For more information, see [3] and [28]). By this counting we derive a lower bound on the number of obstructions for the classes of tree-depth at most k, $\mathcal{G}_k$, for every $k \geq 1$.

4.1 Structural lemmata

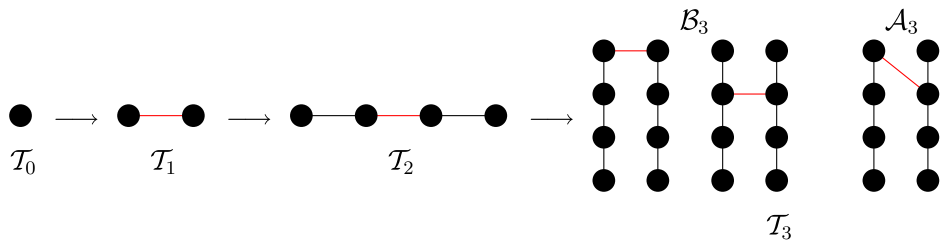


Figure 4.1: The classes $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$

Before we start proving the lemmata consider the following observations. Recall the following.

Observation 4.1.1. *For any graph G, $\mathbf{td}(G) = \max\{\mathbf{td}(C) \mid C \in \mathcal{C}(G)\}$.*

Observation 4.1.1 directly implies the following.

Observation 4.1.2. *For every $k \geq 0$, all graphs in $\mathbf{obs}(\mathcal{G}_k)$ are connected.*

Observation 4.1.3. *Let r be a positive integer. Let also G be an r-connected graph and let $\rho : V(G) \to [k]$ be a k-vertex ranking of G such that $k \geq r$. Then $|\rho^{-1}(i)| \leq 1$, where $k - r + 1 \leq i \leq k$.*

Proof. Let $r = 1$. If v_1 and v_2 are two (non-adjacent) vertices in $\rho^{-1}(k)$, then there exists a path with end-vertices v_1, v_2. Observe that all internal vertices of this path have colour smaller than k, a contradiction. Assume that the hypothesis is true for $r = m$ and let $r = m+1$. Observe that G is m-connected as G is $m+1$-connected. Therefore, by the induction hypothesis $|\rho^{-1}(i)| \leq 1$ for $k-m+1 \leq i \leq k$ and this implies that G contains $\leq m$ vertices with colour strictly greater than $k - m$. We claim that $|\rho^{-1}(k-m)| \leq 1$. Assume in contrary that v_1, v_2 are two (non-adjacent) vertices in $\rho^{-1}(k-m)$. Then by Menger's theorem there exist $m+1$ disjoint paths with end-vertices v_1, v_2. Therefore, there exist at least $m+1$ vertices with colour strictly greater than $k - m$, a contradiction and this completes the proof of the claim. □

Observation 4.1.4. *If $G \in \mathbf{obs}(\mathcal{G}_k)$ then for every $v \in V(G)$ there exists a $(k+1)$-vertex ranking ρ such that $\rho(v) = k+1$.*

Proof. As $G \in \mathbf{obs}(\mathcal{G}_k)$, $G \setminus v$ admits a k-vertex ranking ρ. Then $\rho \cup \{(v, k)\}$ is the required $(k+1)$-vertex ranking of G. □

Let G_1 an G_2 be two disjoint graphs and let $v_i \in V(G_i), i = 1, 2$. We define $\mathbf{j}(G_1, G_2, v_1, v_2) = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2) \cup \{\{v_1, v_2\}\})$.

Observation 4.1.5. *Let G_1 and G_2 be disjoint graphs where $\mathbf{td}(G_1) \leq k$ and $\mathbf{td}(G_2) \leq k$. Let also $v_i \in V(G_i), i = 1, 2$. Then the graph $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ has tree-depth at most $k+1$.*

Proof. Let ρ_i be a k-vertex ranking of $G_i, i = 1, 2$. Then $\rho = \rho_1 \cup \rho_2 \setminus \{(v_1, \rho_1(v_1))\} \cup \{(v_1, k+1)\}$ is a $(k+1)$-vertex ranking of G. □

Observation 4.1.6. *Let G_1 and G_2 be disjoint graphs where $\mathbf{td}(G_1) \geq k$ and $\mathbf{td}(G_2) \geq k$. Let also $v_i \in V(G_i), i = 1, 2$. Then the graph $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ has tree-depth at least $k+1$.*

Proof. Assume in contrary that there exists a k-vertex ranking $\rho : V(G) \to [k]$. Notice that $\rho^{-1}(k) \neq \emptyset$, otherwise $\mathbf{td}(G) < k$ contradicting the fact that $\mathbf{td}(G_1) \geq k$. Combining this fact with Observation 4.1.3, G has a unique vertex v where $\rho(v) = k$. W.l.o.g. we assume that $v \in V(G_1)$. Then ρ gives a $(k-1)$-vertex ranking of G_2, a contradiction. □

Lemma 4.1.1. *Let k be a positive integer. Let G_1, G_2 be disjoint graphs such that $G_1, G_2 \in \mathbf{obs}(\mathcal{G}_{k-1})$ and let $v_1 \in V(G_1), v_2 \in V(G_2)$. Then $\mathbf{j}(G_1, G_2, v_1, v_2) \in \mathbf{obs}(\mathcal{G}_k)$.*

Proof. Let G_1, G_2 such that $G_1, G_2 \in \mathbf{obs}(\mathcal{G}_{k-1})$ and let $v_i \in V(G_i), i = 1, 2$. We set $G = \mathbf{j}(G_1, G_2, v_1, v_2)$. We first prove that $\mathbf{td}(G) = k + 1$. Indeed, Observation 4.1.5 yields $\mathbf{td}(G) \leq k + 1$ and Observation 4.1.6 yields $\mathbf{td}(G) \geq k + 1$.

We now prove that if G' is the result of the contraction or the removal of some edge e in G, then $\mathbf{td}(G') \leq k$. We examine first the case where $e = \{v_1, v_2\}$. If $G' = G \setminus e$, then from Observation 4.1.1, $\mathbf{td}(G) = \max\{\mathbf{td}(G_1), \mathbf{td}(G_2)\} \leq k$. Suppose now that $G' = G \,/\, e$. From Observation 4.1.4, there exists a k-vertex ranking ρ_i of G_i such that $\rho_i(v_i) = k, i = 1, 2$. Then if v_{new} is the result of the contraction of e we have that $\rho : V(G') \to [k]$ where

$$\rho(x) = \begin{cases} \rho_1(x) & \text{if } x \in V(G_1) \setminus \{v_1\} \\ \rho_2(x) & \text{if } x \in V(G_2) \setminus \{v_2\} \\ k & \text{if } x = v_{\text{new}} \end{cases}$$

is a k-vertex ranking of G', therefore $\mathbf{td}(G') \leq k$.

Finally, we examine the case where e is an edge of G_1 or G_2. Without loss of generality we assume that $e_1 \in E(G_1)$. Because $G_1 \in \mathbf{obs}(\mathcal{G}_{k-1})$, there exists a $(k-1)$-vertex ranking ρ'_1 of $G_1 \,/\, e$. By Observation 4.1.4, since $G_2 \in \mathbf{obs}(\mathcal{G}_{k-1})$, there exists a k-vertex ranking ρ_2 of G_2 such that $\rho_2(v_2) = k$. It is easy to see that $\rho'_1 \cup \rho_2$ is a k-vertex ranking of G', thus $\mathbf{td}(G') \leq k$ and this completes the proof of the lemma. □

In the previous lemma we proved that if $G_1, G_2 \in \mathbf{obs}(\mathcal{G}_k)$ are disjoint graphs we can construct a graph $G \in \mathbf{obs}(\mathcal{G}_{k+1})$ by adding an edge connecting a vertex v_1 of G_1 and a vertex v_2 of G_2. In the following lemma we prove that if $G \in \mathbf{obs}(\mathcal{G}_{k+1})$ and G is a tree then there exists an edge $e \in E(G)$ such that if $\mathcal{C}(G \setminus e) = \{G_1, G_2\}$ then $G_1, G_2 \in \mathbf{obs}(\mathcal{G}_k)$.

Lemma 4.1.2. *Let G be a tree in $\mathbf{obs}(\mathcal{G}_k)$ for $k \geq 1$. Then there exists an $e \in E(G)$ such that if $\{G_1, G_2\} = \mathcal{C}(G \setminus \{e\})$ then $G_1, G_2 \in \mathbf{obs}(\mathcal{G}_{k-1})$.*

Proof. We examine the non-trivial case where $k \geq 2$. From Observation 4.1.5, we obtain that for each edge $e = \{v_1, v_2\} \in E(G)$, at least one of the connected components G_1, G_2 of $G \setminus e$ has tree depth at least k. We claim G contains at least one edge $e = \{v_1, v_2\}$ such that both connected components $G \setminus e$ have tree depth k. Suppose that this is not correct. Then we can direct each edge $e = \{v_1, v_2\}$ of $E(G)$ such that its tail belongs to the connected component of $G \setminus e$ that has tree-depth $< k$. We denote this directed tree by $\tilde{T}$. As $k \geq 2$, $\tilde{T}$ contains internal vertices. Moreover, all edges of $\tilde{T}$ that are incident to a leaf are directed away from it. It follows that $\tilde{T}$ contains an internal vertex v of out-degree 0. This means that each, say G_i, connected component of $G \setminus v$ has a $(k-1)$-vertex ranking ρ_i.

Then $\rho = \{(v,k)\} \cup \bigcup_{i=1,\dots,m} \rho_i$ is a k-vertex ranking of G, a contradiction and this completes the proof of the claim.

Let now G_i be the connected component of $G \setminus e$ that contains $v_i, i = 1,2$. If one, say G_1, is not in $\mathbf{obs}(\mathcal{G}_{k-1})$ then there is a graph $G'_1 \in \mathbf{obs}(\mathcal{G}_{k-1})$ that is a proper minor of G_1. Then, G'_1 contains a vertex v'_1 such that the graph $G' = \mathbf{j}(G'_1, G_2, v'_1, v_2)$ has tree-depth at least $k+1$ (Observation 4.1.6). This is a contradiction as G' is also a proper minor of G and lemma follows. □

For every integer $k \geq 0$, we recursively define the class $\mathcal{T}_k$ as follows. Let $\mathcal{T}_0 = \{K_1\}$ and for every $k \geq 1$ we set

$$\mathcal{T}_k = \{\mathbf{j}(G_1, G_2, v_1, v_2) \mid G_1, G_2 \in \mathcal{T}_{k-1}, v_i \in V(G_i), i = 1,2\}$$

The following is a direct consequence of Lemma 4.1.1 and Lemma 4.1.2.

Theorem 4.1.1 ([27]). *Let k be a non-negative integer. Then $\mathcal{T}_k$ is the set of all acyclic graphs in* $\mathbf{obs}(\mathcal{G}_k)$.

For an example, see Figure 4.1.

4.2 The bounds

In this section, we prove that $|\mathcal{T}_k| = \frac{1}{2}2^{2^{k-1}-k}(1 + 2^{2^{k-1}-k})$ for every integer $k \geq 1$. which gives as a lower bound on the number of obstructions for the class of graphs of tree-depth at most k for every integer $k \geq 1$.

A direct consequence from Theorem 4.1.1 is the following.

Observation 4.2.1. *Let k be a non-negative integer. If G_1, G_2 are graphs such that $G_1, G_2 \in \mathcal{T}_k$ then $|V(G_1)| = |V(G_2)| = 2^k$.*

Consider also the following.

Observation 4.2.2. *Let T^1, T^2 be two trees and $e^i = \{v^i_1, v^i_2\} \in E(T^i), i = 1,2$. Let also ϕ an isomorphism from T^1 to T^2 such that $\phi(v^1_i) = v^2_i, i = 1,2$. Let also T^j_i be the connected component of $T^j \setminus e^j$ that contains $v^j_i, i = 1,2, j = 1,2$. Then $\phi_i = \{(x,y) \in \phi \mid x \in V(T^1_i)\}$ is an isomorphism from T^1_i to $T^2_i, i = 1,2$.*

Observation 4.2.2 easily implies the following.

Observation 4.2.3. *Let T be a tree and $e = \{v_1, v_2\} \in E(T)$. Let also $\phi \in$ $\mathbf{Aut}(T)$ such that $\phi(v_i) = v_{3-i}, i = 1,2$. Let also T_i be the connected component of $T \setminus e$ that contains $v_i, i = 1,2$. Then $\phi' = \{(x,y) \in \phi \mid x \in V(T_1)\}$ is an isomorphism from T_1 to T_2.*

In the following lemma we prove that if ϕ is an isomorphism from G to G' where G, G' are graphs that have been constructed as described in Lemma 4.1.1 by the graphs $G_i, i = 1, 2$ and $e \in E(G), e' \in E(G')$ are these edges then $\phi(e) = e'$.

Lemma 4.2.1. *Let G_1, G_2 be disjoint graphs such that $G_1, G_2 \in \mathbf{obs}(\mathcal{G}_k), k \geq 1$ and let $v_i^j \in V(G_j), i = 1, 2, j = 1, 2$. Let also ϕ an isomorphism from G to G', where $G = \mathbf{j}(G_1, G_2, v_1^1, v_1^2)$ and $G' = \mathbf{j}(G_1, G_2, v_2^1, v_2^2)$. Then $\phi(\{v_1^1, v_1^2\}) = \{v_2^1, v_2^2\}$.*

Proof. Let $e_i = \{v_i^1, v_i^2\}, i = 1, 2$. Assume in contrary that at least one of the connected components of $G' \setminus e_2$, say G_1, contains an edge e' such that $\phi(e_1) = e'$. Clearly, ϕ is also an isomorphism from $G \setminus e_1$ to $G' \setminus e'$. Let G'_1, G'_2 be the connected components of $G' \setminus e'$ where $e_2 \in G'_2$. Observe that G'_1 is a proper subgraph of G_1. Therefore they cannot be isomorphic and thus $\mathbf{td}(G'_1) < \mathbf{td}(G_1) = k + 1$. Then G'_1 and G_2 are isomorphic, thus $\mathbf{td}(G_2) = \mathbf{td}(G'_1) \leq k$, a contradiction. □

A direct consequence of Lemma 4.2.1 is the following.

Observation 4.2.4. *Let G_1, G_2 be disjoint graphs such that $G_1, G_2 \in \mathbf{obs}(\mathcal{G}_k), k \geq 1$ and let $v_i \in V(G_i), i = 1, 2$. Let also $\phi \in \mathbf{Aut}(G)$, where $G = \mathbf{j}(G_1, G_2, v_1, v_2)$. Then either $\phi(v_i) = v_i, i = 1, 2$ or $\phi(v_1) = v_2$ and $\phi(v_2) = v_1$.*

In what follows we prove that if $G \in \mathcal{T}_k, k \geq 0$ and $\phi(v) = v$ for some $v \in V(G)$ and $\phi \in \mathbf{Aut}(G)$ then $\phi = \mathbf{id}$. Consider first the following.

Lemma 4.2.2. *Let $G \in \mathcal{T}_k$ for $k \geq 1$, $e = \{v_1, v_2\} \in E(G)$ the edge of Lemma 4.1.2 and $\phi \in \mathbf{Aut}(G)$. If there exists $v \in V(G)$ such that $\phi(v) = v$, then $\phi(v_i) = v_i, i = 1, 2$.*

Proof. We examine the non-trivial case where $k \geq 2$. Notice first that if $v \in e$, then the result follows directly from Observation 4.2.4, therefore we may assume that $v \notin e$. Suppose, in contrary, that $\phi(v_i) = v_{3-i}, i = 1, 2$. We denote by G_1, G_2 the connected components of $G \setminus e$ where, w.l.o.g, $v, v_1 \in V(G_1)$. By Observation 4.2.3, $\phi' = \{(x, y) \in \phi \mid x \in V(G_1)\}$ is an isomorphism of G_1 to G_2, a contradiction since $\phi'(v) = \phi(v) = v$. □

We now proceed to prove the following.

Lemma 4.2.3. *Let k be a non-negative integer. Let also $G \in \mathcal{T}_k$ and $\phi \in \mathbf{Aut}(G)$. If there exists $v \in V(G)$ such that $\phi(v) = v$, then $\phi = \mathbf{id}$.*

Proof. We use induction on k. For $k = 0$ the claim is trivial. Assume now that the claim is true for $k = n \geq 0$. Let $k = n + 1$. We denote by $e = \{v_1, v_2\} \in E(G)$ the edge of Lemma 4.1.2 and by G_1, G_2 the connected components of $G \setminus e$, where $v_i \in V(G_i), i = 1, 2$. Since $\phi \in \mathbf{Aut}(G)$, by Lemma 4.2.2, it follows that $\phi(v_i) = v_i, i = 1, 2$. Hence ϕ is an isomorphism from $G \setminus e$ to $G \setminus e$. From Observation 4.2.2, $\phi_i = \{(v, u) \in \phi \mid v \in V(G_i)\} \in \mathbf{Aut}(G_i), i = 1, 2$. Observe that $\phi_i(v_i) = \phi(v_i) = v_i, i = 1, 2$. Since $G_i \in \mathcal{T}_n, i = 1, 2$, by the induction hypothesis, $\phi_i, i = 1, 2$ is the trivial automorphism of G_i. Therefore, $\phi = \mathbf{id}$. □

Let G be a graph and $v \in V(G)$. We denote $\mathbf{tr}_G(v) = \{u \in V(G) \mid \exists \phi \in \mathbf{Aut}(G)$ such that $\phi(u) = v\}$, i.e. $\mathbf{tr}_G(v)$ is the *orbit* of the automorphism group of G that contains v.

Consider now the following two.

Lemma 4.2.4. *Let G_1, G_2 be disjoint graphs such that $G_1, G_2 \in \mathcal{T}_k$ and $v_1 \in V(G_1), v_2, v_2' \in V(G_2)$ such that $v_2 \in \mathbf{tr}_{G_2}(v_2')$. Then $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ and $G' = \mathbf{j}(G_1, G_2, v_1, v_2')$ are isomorphic.*

Proof. Notice that if $v_2 = v_2'$ the result follows directly since $G = G'$. Therefore, we may assume that $G_2 \in \mathcal{B}_k$ and $v_2 \neq v_2'$. Let $\mathbf{id} \in \mathbf{Aut}(G_1)$ and $\phi \in \mathbf{Aut}(G_2)$, such that $\phi(v_2) = v_2'$. Then $\mathbf{id} \cup \phi$ is an isomorphism from G to G'. □

Lemma 4.2.5. *Let G_1, G_2 be disjoint graphs such that $G_1, G_2 \in \mathcal{T}_k$ and $v_1 \in V(G_1), v_2, v_2' \in V(G_2)$ such that $v_2 \notin \mathbf{tr}_{G_2}(v_2')$. Then $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ and $G' = \mathbf{j}(G_1, G_2, v_1, v_2')$ are not isomorphic.*

Proof. Assume, in contrary, that ϕ is an isomorphism from G to G'. By Lemma 4.2.1 follows that either $\phi(v_1) = v_1$ and $\phi(v_2) = v_2'$ or $\phi(v_1) = v_2'$ and $\phi(v_2) = v_1$. We first exclude the case where $\phi(v_1) = v_1$ and $\phi(v_2) = v_2'$. Indeed, by Observation 4.2.2, $\phi' = \{(x, y) \in \phi \mid x \in V(G_2)\} \in \mathbf{Aut}(G_2)$ and moreover $\phi'(v_2) = \phi(v_2) = v_2'$, a contradiction since $v_2 \notin \mathbf{tr}_{G_2}(v_2')$. Thererefore, $\phi(v_1) = v_2'$ and $\phi(v_2) = v_1$. By Observation 4.2.2, $\phi_i = \{(x, y) \in \phi \mid x \in V(G_i)\}$ is an isomorphism from G_i to $G_{3-i}, i = 1, 2$. Then $\psi = \phi_1 \circ \phi_2 \in \mathbf{Aut}(G_2)$ and $\psi(v_2) = \phi_1(\phi_2(v_2)) = \phi_1(\phi(v_2)) = \phi_1(v_1) = v_2'$. It follows that $v_2 \in \mathbf{tr}_{G_2}(v_2')$, a contradiction. □

In the following we count $|\mathcal{T}_k|$. By the previous lemmata we first need to count $|\mathbf{tr}_G(v)|$ for every orbit of every graph $G \in \mathcal{T}_k$. Recall that, given a graph G we say that G is asymmetric if for every $v \in V(G)$ and every orbit of the automorphism group of G that contains v it is true that $|tr_G(v)| = 1$. We also say that a graph G is *bisymmetric* if for every $v \in V(G)$ and every orbit of the automorphism group of G that contains v it is true that $|tr_G(v)| = 2$.

Lemma 4.2.6. *Let k be a non-negative integer and let G_1, G_2 be two disjoint non-isomorphic graphs such that $G_1, G_2 \in \mathcal{T}_k$. Then the graph $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ is asymmetric.*

Proof. Suppose that $\phi \in \mathbf{Aut}(G)$ and $\phi \neq \mathbf{id}$. From Lemma 4.2.3, $\phi(v) \neq v$ for all $v \in V(G)$ and from Observation 4.2.4, $\phi(v_i) = v_{3-i}, i = 1, 2$. From Observation 4.2.3, G_1 is isomorphic to G_2, a contradiction. □

Lemma 4.2.7. *Let k be a non-negative integer, let G_1, G_2 two disjoint graphs such that $G_1, G_2 \in \mathcal{T}_k$. Let also ϕ an isomorphism from G_1 to G_2 and $v_i \in V(G_i), i = 1, 2$ such that $\phi(v_1) \notin \mathbf{tr}_{G_2}(v_2)$. Then $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ is asymmetric.*

Proof. Suppose that $\psi \in \mathbf{Aut}(G)$ and $\psi \neq \mathbf{id}$. From Lemma 4.2.3, $\psi(v) \neq v$ for all $v \in V(G)$ and from Observation 4.2.4, $\psi(v_1) = v_2$ and $\psi(v_2) = v_1$. From Observation 4.2.3, $\chi = \{(x, y) \in \psi \mid x \in V(G_1)\}$ is an isomorphism from G_1 to G_2. Moreover, $\chi(v_1) = \psi(v_1) = v_2 \in \mathbf{tr}_{G_2}(v_2)$, a contradiction to the fact that if G_1, G_2 are two disjoint graphs, ϕ an isomorphism from G_1 to G_2 and $v_i \in V(G_i), i = 1, 2$ such that $\phi(v_1) \notin \mathbf{tr}_{G_2}(v_2)$ then $\psi(v_1) \notin \mathbf{tr}_{G_2}(v_2)$ for every isomorphism ψ from G_1 to G_2. □

In the previous lemmata we proved that if G_1, G_2 are disjoint non-isomorphic graphs such that $G_1, G_2 \in \mathcal{T}_k$ then for every $v_i \in V(G_i), i = 1, 2$ the graph $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ is asymmetric. We also proved that the following is true: if G_1, G_2 are isomorphic graphs and $v_i \in V(G_i), i = 1, 2$ are vertices such that $\phi(v_1) \notin \mathbf{tr}_{G_2}(v_2)$. We now examine the remaining case where G_1, G_2 are isomorphic graphs and $v_i \in V(G_i), i = 1, 2$ are vertices such that $\phi(v_1) \in \mathbf{tr}_{G_2}(v_2)$, for an isomorphism ϕ from G_1 to G_2. Finally, we prove the following.

Lemma 4.2.8. *Let k be a non-negative integer, let G_1, G_2 two disjoint graphs such that $G_1, G_2 \in \mathcal{T}_k$, and let $\phi : V(G_1) \rightarrow V(G_2)$ be an isomorphism from G_1 to G_2. Let also $v_i \in V(G_i), i = 1, 2$ such that $\phi(v_1) \in \mathbf{tr}_{G_2}(v_2)$. Then $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ is bisymmetric.*

Proof. Let S be an orbit of the automorphism group of G that contains exactly one element. Then, from Lemma 4.2.3, $\mathbf{Aut}(G) = \{\mathbf{id}\}$. Observe also that there exists an isomorphism ψ from G_1 to G_2 such that $\psi(v_1) = v_2$ and notice that $\mathbf{id} \neq \psi \cup \psi^{-1} \in \mathbf{Aut}(G)$, a contradiction. Suppose now that S contains three distinct vertices u_1, u_2, and u_3. Then there exist $\phi_1, \phi_2 \in \mathbf{Aut}(G)$, such that $\phi_1(u_1) = u_2$ and $\phi_2(u_2) = u_3$. As u_1, u_2, and u_3 are distinct, $\phi_i \neq \mathbf{id}, i = 1, 2$. Therefore, $\phi_i(v_1) = v_2, i = 1, 2$ and $\phi_i(v_2) = v_1, i = 1, 2$. Moreover, $\phi_2 \circ \phi_1 \neq \mathbf{id}$, since $(\phi_2 \circ \phi_1)(u_1) = u_3$. However, $\phi_2(\phi_1(v_1)) = \phi_2(v_2) = v_1$, a contradiction and the lemma follows. □

A direct implication of Theorem 4.1.1 and Lemmata 4.2.6, 4.2.7 and 4.2.8 is the following.

Observation 4.2.5. *If G is a graph such that $G \in \mathcal{T}_k$ then G is either asymmetric or bisymmetric.*

For every integer $k \geq 0$, we define for following partition of $\mathcal{T}_k$:

$$\mathcal{A}_k = \{G \in \mathcal{T}_k \mid \mathbf{Aut}(G) = \{\mathbf{id}\}\} \text{ and } \mathcal{B}_k = \{G \in \mathcal{T}_k \mid \mathbf{Aut}(G) \neq \{\mathbf{id}\}\}.$$

We denote $\alpha_k = |\mathcal{A}_k|$, $\beta_k = |\mathcal{B}_k|$ and $\tau_k = |\mathcal{T}_k| = \alpha_k + \beta_k$ (see Figure 4.1). We also set $\gamma_k = 2^{k-2}$. A direct consequence of Observation 4.2.1 and Lemmata 4.2.7 and 4.2.8 is the following.

Observation 4.2.6. *Let $k \geq 2$ be an integer. Then the automorphism group of each graph in $G \in \mathcal{A}_k$ (resp. $G \in \mathcal{B}_k$) has exactly γ_{k+2} (resp. γ_{k+1}) orbits.*

Observation 4.2.7. *Clearly, $b_0 = a_1 = a_2 = 0$ and $a_0 = b_1 = b_2 = 1$.*

Theorem 4.2.1 ([27])**.** *For every integer $k \geq 1$, $\tau_k = 2^{2^k-(2k+1)} + 2^{2^{k-1}-(k+1)}$.*

Proof. First observe that for $k = 1, 2$ the claim is true. Let G be a graph. Recall that $G \in \mathcal{T}_k$ iff $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ for some $G_i \in \mathcal{T}_{k-1}$, and $v_i \in V(G_i), i = 1, 2$. Therefore, in order to count τ_k it is sufficient to count the ways to choose $G_1, G_2 \in \mathcal{T}_{k-1}$ and $v_i \in V(G_i), i = 1, 2$ and not end up to isomorphic graphs. Let G_1, G_2 be graphs such that $G_i \in \mathcal{T}_{k-1}$ and $v_i \in V(G_i), i = 1, 2$. We define

$$\begin{aligned}
\mathcal{A}_k^1 = \{G \mid G = \mathbf{j}(G_1, G_2, v_1, v_2), G_1 \not\simeq G_2, G_i \in \mathcal{A}_{k-1}, i = 1, 2, \\
\text{and } v_i \in V(G_i), i = 1, 2\} \quad (4.1)\\
\mathcal{A}_k^2 = \{G \mid G = \mathbf{j}(G_1, G_2, v_1, v_2), G_1 \not\simeq G_2, G_i \in \mathcal{B}_{k-1}, i = 1, 2, \\
\text{and } v_i \in V(G_i), i = 1, 2\} \quad (4.2)\\
\mathcal{A}_k^3 = \{G \mid G = \mathbf{j}(G_1, G_2, v_1, v_2), G_1 \not\simeq G_2, G_1 \in \mathcal{A}_{k-1}, G_2 \in \mathcal{B}_{k-1}, \\
\text{and } v_i \in V(G_i), i = 1, 2\} \quad (4.3)\\
\mathcal{A}_k^4 = \{G \mid G = \mathbf{j}(G_1, G_2, v_1, v_2), G_1 \simeq_\phi G_2, G_i \in \mathcal{A}_{k-1}, i = 1, 2, \\
\text{and } v_i \in V(G_i), i = 1, 2, \text{ such that } \phi(v_1) \notin \mathbf{tr}_{G_2}(v_2)\} \quad (4.4)\\
\mathcal{A}_k^5 = \{G \mid G = \mathbf{j}(G_1, G_2, v_1, v_2), G_1 \simeq_\phi G_2, G_i \in \mathcal{B}_{k-1}, \\
\text{and } v_i \in V(G_i), i = 1, 2, \text{ such that } \phi(v_1) \notin \mathbf{tr}_{G_2}(v_2)\} \quad (4.5)\\
\mathcal{B}_k^1 = \{G \mid G = \mathbf{j}(G_1, G_2, v_1, v_2), G_1 \simeq_\phi G_2, G_i \in \mathcal{A}_{k-1}, i = 1, 2, \\
\text{and } v_i \in V(G_i), i = 1, 2, \text{ such that } \phi(v_1) \in \mathbf{tr}_{G_2}(v_2)\} \quad (4.6)\\
\mathcal{B}_k^2 = \{G \mid G = \mathbf{j}(G_1, G_2, v_1, v_2), G_1 \simeq_\phi G_2, G_i \in \mathcal{B}_{k-1}, \\
\text{and } v_i \in V(G_i), i = 1, 2 \text{ such that } \phi(v_1) \in \mathbf{tr}_{G_2}(v_2)\}. \quad (4.7)
\end{aligned}$$

By their definitions, the above sets are a partition of $\mathcal{T}_k$. By Lemma 4.2.6 (for relations (1)–(3)) and by Lemma 4.2.7 (for relations (4) and (5)), the union of the first five is a subset of $\mathcal{A}_k$. Moreover, by Lemma 4.2.8 (applied to relations (6) and (7)) the union of the last two is a subset of $\mathcal{B}_k$. We conclude that $\mathcal{A}_k = \bigcup_{i=1,\ldots,5} \mathcal{A}_k^i$ and $\mathcal{B}_k = \mathcal{B}_k^1 \cup \mathcal{B}_k^2$.

From Observation 4.2.6, Lemmata 4.2.4 and 4.2.5, and Relations (1)–(7) we derive that

$$
\begin{aligned}
|\mathcal{A}_k^1| &= \binom{\alpha_{k-1}}{2} \cdot \gamma_{k+1}^2, \\
|\mathcal{A}_k^2| &= \binom{\beta_{k-1}}{2} \cdot \gamma_k^2, \\
|\mathcal{A}_k^3| &= \alpha_{k-1} \cdot \gamma_{k+1} \cdot \beta_{k-1} \cdot \gamma_k, \\
|\mathcal{A}_k^4| &= \alpha_{k-1} \cdot \binom{\gamma_{k+1}}{2} \\
|\mathcal{A}_k^5| &= \beta_{k-1} \cdot \binom{\gamma_k}{2} \\
|\mathcal{B}_k^1| &= \alpha_{k-1} \cdot \gamma_{k+1} \\
|\mathcal{B}_k^2| &= \beta_{k-1} \cdot \gamma_k
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\alpha_k &= \binom{\alpha_{k-1}}{2}\gamma_{k+1}^2 + \binom{\beta_{k-1}}{2}\gamma_k^2 + \alpha_{k-1}\binom{\gamma_{k+1}}{2} + \beta_{k-1}\binom{\gamma_k}{2} \\
&\quad + \alpha_{k-1}\beta_{k-1}\gamma_k\gamma_{k+1} \qquad (4.8) \\
\beta_k &= \alpha_{k-1}\gamma_{k+1} + \beta_{k-1}\gamma_k \qquad (4.9)
\end{aligned}
$$

By simplifying (4.8),

$$
\begin{aligned}
\alpha_k &= \frac{1}{2}\left[\left(\gamma_{k+1}^2\alpha_{k-1}^2 + \gamma_k^2\beta_{k-1}^2 + 2\alpha_{k-1}\beta_{k-1}\gamma_k\gamma_{k+1}\right) - \left(\alpha_{k-1}\gamma_{k+1} + \beta_{k-1}\gamma_k\right)\right] \\
&= \frac{1}{2}\left(\beta_k^2 - \beta_k\right)
\end{aligned}
$$

It follows (using Relation (4.9)) that,

$$
\tau_k = \frac{1}{2}\left(\beta_k^2 + \beta_k\right) \text{ and } \beta_k = \gamma_k\beta_{k-1}^2
$$

Let $\delta_k = 2^{k-1} - k$ and observe that $\beta_k = 2^{\delta_k} = 2^{2^{k-1}-k}$, for every integer $k \geq 2$. Then $\tau_k = 2^{2^k-(2k+1)} + 2^{2^{k-1}-(k+1)}, k \geq 3$ and the theorem follows. □

Chapter 5

Obstructions for tree-depth at most 3

Our primary goal in this chapter is to fully characterize the classes $\mathcal{G}_k$ for $k \leq 3$ by identifying their obstruction sets (who are finite by the Graph Minor Theorem 3.1.1).

It is easy to prove that $\mathbf{obs}(\mathcal{G}_0) = \{K_1\}$, $\mathbf{obs}(\mathcal{G}_1) = \{K_2\}$, $\mathbf{obs}(\mathcal{G}_2) = \{K_3, P_3\}$.

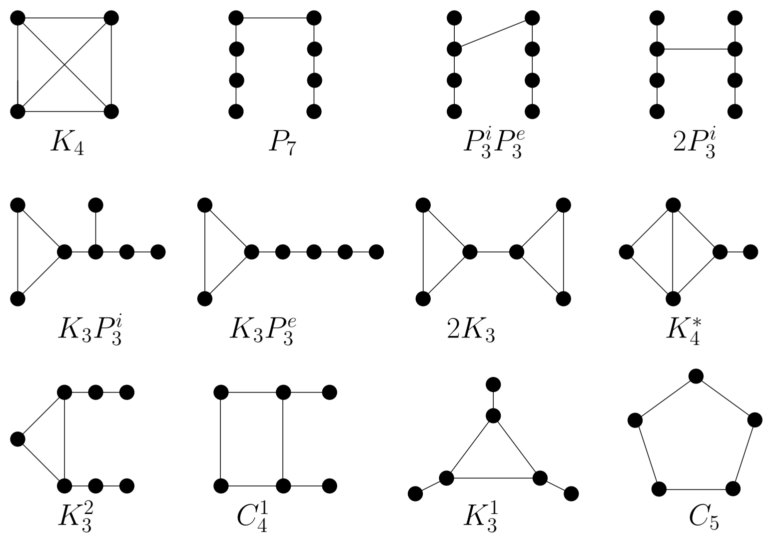


Figure 5.1: The minor obstruction set for the class of graphs with tree-depth at most 3.

Let $\mathcal{C} = \{K_4, P_7, P_3^iP_3^e, 2P_3^i, K_3P_3^i, K_3P_3^e, 2K_3, K_4^*, K_3^2, C_4^1, K_3^1, C_5\}$ be the set of the graphs in Figure 5.1. In this chapter we prove that $\mathbf{obs}(\mathcal{G}_3) = \mathcal{C}$. We call a graph $\mathcal{C}$-*minor-free* if it does not contain any of the graphs in $\mathcal{C}$ as a minor.

5.1 The reduction

Given a graph G we say that a set $S \subseteq V(G)$ is *a set of siblings* if for every $x, y \in S$, $N_G(x) = N_G(y)$. Consider the following.

Observation 5.1.1. *Let G be a graph and ρ be a k-vertex ranking of G. Let also $v_1, v_2 \in V(G)$ such that $\{v_1, v_2\} \in E(G)$ and $\rho(v_1) < \rho(v_2)$. Then $\rho(v_2) \notin \rho(N_{G \setminus \{v_2\}}(v_1))$.*

We now prove the following general reduction-lemma.

Lemma 5.1.1. *Let k be a positive integer, G be a graph and $S \subseteq V(G)$ be a set of siblings of G each of degree k. Let also $G' = G \setminus S'$ where S' is any subset of S such that $|S'| \leq |S| - k$. Then $\mathbf{td}(G) = \mathbf{td}(G')$.*

Proof. We examine the non-trivial case where $|S| \geq k + 1$. We denote $S'' = S \setminus S' = \{u_i \mid i \in |S''|\}$. As G' is a subgraph of G, it is enough to prove that $\mathbf{td}(G) \leq \mathbf{td}(G')$. Let $\rho' : V(G') \to \{1, \ldots, t\}$ be a vertex ranking of G'. Let $N = \{v_i \mid i \in [k]\}$ be the common neighbourhood of the vertices in S'' and w.l.o.g assume that $\rho'(v_i) \leq \rho'(v_{i+1}), i \in [k-1]$. Notice that $|S''| \geq k$ and w.l.o.g assume that $\rho'(u_i) \leq \rho'(u_{i+1}), i \in [|S''| - 1]$. We need the following claim.

Claim 1. Let P be a (z', z)-path in G where $z \in S''$, $z' \in (G \setminus S'') \setminus N$, and $\rho'(z) = \rho'(z')$. Let P' be the portion of P between z' and the first vertex, say x, in N (recall that N is a separator of G). Then there exists a vertex $y \in V(P') \setminus \{z'\}$ such that $\rho'(y) > \rho'(z')$.
Proof. It is enough to observe that the path $P'' = (V(P') \cup \{z\}, E(P') \cup \{\{x, z\}\})$ should contain an internal vertex y where $\rho'(y) > \rho'(z')$.

In what follows we construct a vertex ranking $\rho : V(G) \to \{1, \ldots, t\}$. Let

$$m = \begin{cases} \max\{i \mid \rho'(u_1) > \rho'(v_i)\} + 1 & \text{if } A = \{i \mid \rho'(u_1) > \rho'(v_i)\} \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

and observe that $m \leq k + 1$. We claim that

$$\rho = \{(x, \rho'(x)) \mid x \in V(G') \setminus (S'' \cup \bigcup_{i \in [m-1]} \{v_i\})\} \cup \sigma$$

where

$$\sigma = \begin{cases} \{(v_i, \rho'(u_{i+1})) \mid i \in [m-1]\} \cup \{(x, \rho'(u_1)) \mid x \in S\} & m \neq k+1 \\ \{(v_i, \rho'(u_i)) \mid i \in [m-1]\} \cup \{(x, \rho'(v_1)) \mid x \in S\} & m = k+1 \end{cases}$$

is a t-vertex ranking of G.

First we examine the case where $m = 1$. Then observe that

$$\rho'' = \{(x, \rho'(x)) \mid x \in V(G') \setminus S''\} \cup \{(x, \rho'(u_1)) \mid x \in S''\}$$

is a t-vertex ranking of G'. It is easy to observe that $\rho'' \cup \{(x, \rho'(u_1)) \mid x \in S'\}$ is a t-vertex ranking of G that is equal to ρ.

We examine now the case where $1 < m \leq k + 1$. As $A \neq \emptyset$, Observation 5.1.1 implies that

$$\rho'(u_i) < \rho'(u_{i+1}), \qquad i \in [|S''| - 1] \tag{5.1}$$

$$\rho'(v_i) < \rho'(v_{i+1}), \qquad m \leq i \leq k - 1 \tag{5.2}$$

$$\rho'(N_{G' \setminus S''}(\bigcup_{i \in [m-1]} \{v_i\})) \cap \rho'(S'') = \emptyset \tag{5.3}$$

thus, from (5.1), $|\rho'(S'')| = |S''| \geq k$. We distinguish the following cases:

Case 1. $1 < m < k + 1$. We claim that

$$\begin{aligned} \rho'' \quad = \quad & \{(x, \rho'(x)) \mid x \in V(G') \setminus (S'' \cup \bigcup_{i \in [m-1]} \{v_i\})\} \cup \\ & \{(v_i, \rho'(u_{i+1})) \mid i \in [m - 1]\} \cup \{(x, \rho'(u_1)) \mid x \in S''\} \end{aligned}$$

is a t-vertex ranking of G'. Indeed, ρ'' is a valid colouring of G' because of (5.1), (5.2), and (5.3). To prove that ρ'' is a t-vertex ranking, we consider a (z', z)-path P between two vertices $z, z' \in V(G')$ where $\rho''(z) = \rho''(z')$. We observe the following.

Claim 2. $|\rho''(N)| = k$.
Proof. It follows directly from (5.1) and (5.2).

We distinguish the following subcases.
Subcase 1.1. If one, say z, of the endpoints of P belongs to S'', then P contains at least one vertex $v_i, i \in N$. If $i \in A$ then $\rho''(v_i) \geq \rho'(u_2) > \rho'(u_1) = \rho''(z)$. If $i \in [k] \setminus A$, then $\rho''(v_i) = \rho'(v_i) > \rho'(u_1) = \rho''(z)$.
Subcase 1.2. If one, say z, of the endpoints of P belongs to $N' = \{v_i \mid i \in A\}$, then we assume that $z = v_i$ and, from Claim 2, $z' \in (V(G') \setminus S'') \setminus N$. Let P' be the portion of P between z' and the first vertex x in N. Then from Claim 1, there exists a vertex $y \in V(P') \setminus \{z'\}$ where $\rho'(y) > \rho'(z')$. Observe that $\rho'(z') = \rho''(z')$ and $\rho''(y) \geq \rho'(y)$. Therefore, $\rho''(y) > \rho''(z')$ and we are done as $y \in V(P') \subseteq V(P)$.
Subcase 1.3. If one, say z, of the endpoints of P belongs to $N \setminus N'$, then again from Claim 2, $z' \in (V(G') \setminus S'') \setminus N$. Let P' be the portion of P between z' and the first vertex x in N. If $w = z$ then $P' = P$ and we are done. If $x \neq z$, we define

$P'' = (V(P') \cup \{u_1, z\}, E(P) \cup \{\{x, u_1\}, \{u_1, z\}\})$ and observe that $\rho'(z) = \rho''(z')$ and $\rho'(z') = \rho''(z')$. Therefore, P'' contains some internal vertex y where $\rho'(y) > \rho'(z) = \rho''(z)$. Notice also that $\rho'(u_1) < \rho'(z)$, thus $y \in V(P')$. It also holds that $\rho''(y) \geq \rho'(y)$, therefore $\rho''(y) > \rho''(z)$ and we are done as $y \in V(P') \subseteq V(P)$.

Subcase 1.4. If both z, z' belong in $(V(G') \setminus S'') \setminus N$, then we examine the non-trivial case where $V(P) \cap S'' \neq \emptyset$ (recall that the new colouring, only increases the colours not in S''). Let P' (resp. P'') be the portion of P between z (resp. z') and the first vertex x (resp. x') in N. We define the path $P''' = P' \cup P'' \cup (\{u_1, x, x'\}, \{\{x, u_1\}, \{x', u_1\}\})$. Again $\rho'(z) = \rho''(z')$ and $\rho'(z') = \rho''(z')$ and let y be a vertex in P''' where $\rho'(y) > \rho'(z) = \rho''(z)$. If $y \in V(P') \cup V(P'')$ then we are done as $\rho''(y) \geq \rho'(y)$ and $V(P') \cup V(P'') \subseteq V(P)$. If $y = u_1$, then we are also done as $S'' \cap V(P) \neq \emptyset$ and the colour assigned by ρ'' to every vertex in $S'' \cap V(P) \neq \emptyset$ is equal to $\rho'(u_1)$.

We just proved that ρ'' is a t-vertex ranking of G'. It remains now to observe that $\rho'' \cup \{(x, \rho'(u_1)) \mid x \in S'\}$ is a t-vertex ranking of G that is equal to ρ.

Case 2. $m = k + 1$. We claim that

$$\begin{aligned} \rho'' &= \{(x, \rho'(x)) \mid x \in V(G') \setminus (\bigcup_{i \in [m-1]} \{v_i\} \cup S'')\} \cup \\ &\quad \{(v_i, \rho'(u_i)) \mid i \in [m-1]\} \cup \{(x, \rho'(v_1)) \mid x \in S''\} \end{aligned}$$

is a t-vertex ranking of G'.

Observe first that Claim 2 is again true from (5.1).

We distinguish the following subcases.

Subcase 2.1. If one, say z, of the endpoints of P belongs to S'', then P contains at least one vertex $v_i, i \in N$. Then $\rho''(v_i) \geq \rho'(u_1) > \rho'(v_1) = \rho''(z)$.

Subcase 2.2. If one, say z, of the endpoints of P belongs to N, then we assume that $z = v_i$ and, from Claim 2, $z' \in (V(G') \setminus S'') \setminus N$. Let P' be the portion of P between z' and the first vertex x in N. Then from the Claim 1, there exists a vertex $y \in V(P') \setminus \{z'\}$ where $\rho'(y) > \rho'(z')$. Observe that $\rho'(z') = \rho''(z')$ and $\rho''(y) \geq \rho'(y)$. Therefore, $\rho''(y) > \rho''(z')$ and we are done as $y \in V(P') \subseteq V(P)$.

Subcase 2.3. If both z, z' belong to $(V(G') \setminus S'') \setminus N$, then we examine the non-trivial case where $V(P) \cap S'' \neq \emptyset$ (recall that the new colouring, only increases the colours not in S''). Let P' (resp. P'') be the portion of P between z (resp. z') and the first vertex x (resp. x') in N. We define the path $P''' = P' \cup P'' \cup (\{u_1, x, x'\}, \{\{x, u_1\}, \{x', u_1\}\})$. Again $\rho'(z) = \rho''(z')$ and $\rho'(z') = \rho''(z')$ and let y be a vertex in P''' where $\rho'(y) > \rho'(z) = \rho''(z)$. If $y \in V(P') \cup V(P'')$ then we are done as $\rho''(y) \geq \rho'(y)$ and $V(P') \cup V(P'') \subseteq V(P)$. If $y = u_1$, then we are done as $\rho''(x) \geq \rho'(u_1)$ and $x \in V(P') \cup V(P'') \subseteq V(P)$.

We just proved that ρ'' is a t-vertex ranking of G'. It remains to observe that $\rho'' \cup \{(x, \rho'(v_1)) \mid x \in S'\}$ is a t-vertex ranking of G that is equal to ρ. □

We call a graph G *k-sibling-free* if every maximal set of siblings each of degree k has at most k elements. We say that a graph G is *reduced* if it is k-sibling-free for each $k \geq 0$.

5.2 The proof

Let G be a graph and let x be an articulation point of G. We define the *x-components* of G the graphs $G_1, \dots, G_q$ constructed as follows: Let $V_1, \dots, V_q$ be the vertex sets of the connected components of $G \setminus x$. Then $G_i = G[\{x\} \cup V_j]$. We call an articulation point x of G *critical* if it belongs to some biconnected component of G and at least two of the x-components of G are different than K_2. Given a graph H, we call an (x, x')-path P an *H-path* if $V(P) \geq 2$ and $V(G) \cap V(P) = \{x, x'\}$.

Before the proof of the result of this section consider the following auxiliary lemmata.

Lemma 5.2.1. *Let G be a connected reduced graph such that $K_4 \not\leq G$, $C_5 \not\leq G$. Then its 2-connected components are either K_3, C_4 or K_4^-.*

Proof. From [15, Proposition 3.1.3], a graph G is 2-connected if and only if it can be constructed from a cycle by successively adding H-paths to graphs H already constructed. Under the assumptions of the lemma, the construction of G should start from a graph H that is either C_3 or C_4. It is now easy to see that every addition of an H-path in H should construct K_4^- and any other application of the same rule would construct graphs they contain either K_4 or C_5 as minors or a graph that is not 2-sibling-free. □

Lemma 5.2.2. *Let G be a connected $\mathcal{C}$-minor-free reduced graph and H be a 2-connected component of G. Then H contains at most two articulation points. More specifically, if H contains two articulation points v_1, v_2 then if $H = C_4$ then $\{v_1, v_2\} \notin E(C_4)$ and if $H = K_4^-$ then $\deg_H(v_i) = 3, i = 1, 2$.*

Proof. Assume, in contrary, that H is 2-connected but contains 3 articulation points, $v_i, i = 1, 2, 3$. Then there exist three vertices $u_i, i = 1, 2, 3 \in V(G) \setminus V(H)$ such that $\{v_i, u_i\} \in E(G)$. Therefore $K_3^1 \leq G$, a contradiction. If $H = C_4$ and $\{v_1, v_2\} \in E(C_4)$ then $C_4^1 \leq G$, a contradiction. If $H = K_4^-$ and for at least one, say v_1, of it's articulation points $\deg_{K_4^-}(v_1) = 2$, then $K_4^* \leq G$, a contradiction. □

Lemma 5.2.3. *Let G be a connected $\mathcal{C}$-minor-free reduced graph and H be a 2-connected component of G. Let also $v_i, i = 1, 2$ be two distinct articulation points of H and G_1, G_2 be the two connected components of $G \setminus (V(H) \setminus \{v_1, v_2\})$. Then H contains at most one critical articulation point.*

Proof. Assume, in contrary, that there exist vertices $u_i \in V(G_i), i = 1, 2$ such that there exists a (v_i, u_2)-path in G_i of length ≥ 2. Then $K_3^2 \leq G$, a contradiction. □

Lemma 5.2.4. *Let G be a connected $\mathcal{C}$-minor-free reduced graph containing a cycle and let x be a critical articulation point of G. Then all the acyclic x-components of G are paths of length ≤ 3.*

Proof. Let H be some acyclic x-component of G. Notice that H does not contain a path of length 4, otherwise $K_3P_3^e \leq G$. Let P be a path in H of maximum length that has x as endpoint. If P has length one, then we are done. If P has length 2 then $H = P$ since G is reduced. Finally, if P has length 3, then the vertex of P adjacent to x has degree 2, otherwise $K_3P_3^i \leq G$ and the other internal vertex has also degree 2 because G is reduced and we are done. □

Lemma 5.2.5. *Let G be a connected $\mathcal{C}$-minor-free reduced graph. Then G contains at most one critical articulation point.*

Proof. Assume, towards a contradiction, that x_1 and x_2 are distinct critical articulation points in G. From Lemma 5.2.3 there are two distinct biconnected components H_1 and H_2 of G where $x_i \in V(H_i), i = 1, 2$. Then there is a path of length ≥ 1 from a vertex of H_1 to a vertex of H_2. As both H_1, H_2 can be contracted to K_3, we obtain that $2K_3 \leq G$, a contradiction. □

We now, finally, prove the following.

Theorem 5.2.1 ([27])**.** *The class $\mathcal{C}$ is the obstruction set of $\mathcal{G}_3$.*

Proof. Notice that all graphs in $\mathcal{C}$ have tree depth ≥ 4 and every minor of them has tree-depth ≤ 3. Therefore $\mathcal{C}$ is a collection of minor-minimal graphs with tree-depth ≥ 4. It remains to prove that any minor-minimal graph G where $\mathbf{td}(G) \geq 4$ is a graph in $\mathcal{C}$. Suppose in contrary, that $G \notin \mathcal{C}$. Then G is $\mathcal{C}$-minor-free. We will arrive to a contradiction by proving that $\mathbf{td}(G) \leq 3$.

First of all G is connected by Observation 4.1.2. Also from the fact that G is minor-minimal and Lemma 5.1.1 we obtain that G is reduced. Moreover, G cannot be a tree as otherwise, by Theorem 4.1.1, $G \in \mathcal{T}_3 \subseteq \mathcal{C}$, a contradiction. From Lemma 5.2.5, we distinguish two cases.
Case 1. G has no critical articulation points. Notice that G cannot have more than one biconnected components. Indeed, if H_1, H_2 are such components, then take a shortest path between vertices of H_1 and H_2 and observe that its endpoints are both critical articulation points, a contradiction to Lemma 5.2.5. Let H be the unique biconnected component of G. Then, from Lemmata 5.2.1 and 5.2.2 H is a subgraph of the graph W depicted in Figure 5.2 that has a 3-vertex ranking.
Case 2. G has a critical articulation point x. Let $G_1, \ldots, G_q$ be the x-components of G. W.l.o.g. let G_1 be the one containing a biconnected component H_1 that

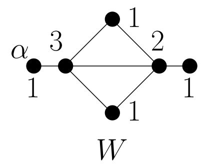


Figure 5.2: The graph W and a 3-vertex ranking of it.

contains x as a vertex. From Lemmata 5.2.1 and 5.2.2 G_1 is a subgraph of the graph $W' = W \setminus \alpha$ that has a 3-vertex ranking where x is coloured by 3 (see Figure 5.2). Let now $J = G_i, i = 2, \ldots, q$. If J is a tree then J is a subgraph of P_3 that can be coloured such that the colour of x is 3 (in Figure 5.2 P_3 can be seen as any path of length 3 starting from unique vertex coloured by 3). If J is not a tree, then each biconnected component of J should contain x as a vertex (otherwise, G would have two critical articulation points contradicting Lemma 5.2.5). Therefore J contains a unique biconnected component that contains x and from Lemmata 5.2.1 and 5.2.2, J is a subgraph of W' that has a 3-vertex ranking where x is coloured by 3 (see Figure 5.2). Let ρ_i be the 3-vertex ranking of G_i as defined above. Clearly, $\rho = \bigcup_{i=1,\ldots,q} \rho_i$ is a 3-vertex ranking of G, and we are done. $\square$

Chapter 6

Graphs, Logic and Algorithms

In 1990 B. Courcelle proved a fundamental theorem stating that graph properties definable in monadic second-order logic can be decided in linear time on graphs of bounded tree-width. This is the first in a series of meta-algorithmic theorems. But what do we mean by the term meta-algorithmic theorem?

The general form of meta-algorithmic theorems is:

All *problems* definable in a certain *logic*
on a certain class of *structures* can be solved *efficiently*.

The term meta-algorithmic refers to the fact that these results describe algorithms for whole families of problems, while usually an algorithm refers to a specific problem, whose definition typically has a logical and a structural (usually graph-theoretical) component.

Although the *problems* may be of different types (e.g., optimisation or counting problems) in this thesis we consider decision problems. Moreover, the *structures* that we consider are graphs. We will, however, consider two types of *logic*, specifically first-order and monadic second-order logic.

What remains is to explain what we mean by the term *efficient solvability*. Efficient solvability may mean, polynomial time solvability (e.g., linear or quadratic time solvability) but could also mean *fixed-parameter* solvability (i.e., given a parameter along with the structure, if we fix the parameter then the problem can be solved in polynomial time). For more on Parameterized Complexity Theory[1] see [23] and [53].

A recent meta-algorithmic theorem guarantees that all first-order definable properties of planar graphs can be decided in linear time [24] (proven by M. Frick and M. Grohe, 2001). An ever more recent meta-algorithmic theorem states that

[1]Parameterized Complexity Theory was recently developed by R. Downey and M. Fellows. For Classical Complexity, see [33], [25] and [54]

all first-order definable optimisation problems on classes of graphs with excluded minors can be approximated in polynomial time to any given approximation ratio [9] (proven by A. Dawar, M. Grohe, S. Kreutzer and N. Schweikardt, 2006).

In this chapter we will present two meta-algorithmic theorems. The first is the one that was proven by B. Courcelle and the second one was proven by J. Nešetřil and P. Ossona de Mendez and states that graph properties definable in first-order logic can be decided in linear time on graphs of bounded expansion.

6.1 A meta-algorithm for monadic second-order logic

In this section we present B. Courcelle's meta-algorithmic theorem. Before doing so, consider the following.

Theorem 6.1.1 ([4]). *For all $k \in \mathbb{N}$, there exists a linear time algorithm, that tests whether a given graph G has tree-width at most k, and if so, outputs a tree-decomposition of G with tree-width at most k.*

Remark 6.1.1. The proof of Theorem 6.1.1 is constructive in the sense that in [4] H. Bodlaender provided a way to construct an algorithm as above.

Theorem 6.1.2 ([8]). *Let $\mathcal{K}$ be a class of finite graphs $G = \langle V, E, R \rangle$ represented as τ_2 structures, that is: by two sorts of elements (vertices V and edges E) and an incidence relation R. Let also ϕ be a MSOL(τ_2) sentence. If $\mathcal{K}$ has bounded tree-width and $G \in \mathcal{K}$, then checking whether $G \models \phi$ can be done in linear time.*

Remark 6.1.2. The proof of Theorem 6.1.2 is also constructive. Whenever such an algorithm exists, it can be constructed by the proof of this Theorem combining Logic and Automata Theory (For more on Automata Theory, see [66] and [31]).

From the example of Section 2.4 it follows that if $\mathcal{K}$ is a class of finite graphs with bounded tree-width and H is a fixed graph in $\mathcal{K}$ it is decidable in linear time if it contains a dominating set of k elements.

6.2 A meta-algorithm for first-order logic

Definition 6.2.1. A *p-centered colouring* of a graph G is a vertex colouring such that, for any induced connected subgraph H, either some colour $c(H)$ appears exactly once in H, or H gets at least p colours.

The p-centered colourings were introduced in [48] as a local approximation of centered colourings.

Definition 6.2.2. Let $\mathcal{C}$ be a class of graphs. Then $\mathcal{C}$ has a *low tree-width colouring* if, for any integer $p \geq 1$, there exists an integer $N(p)$ such that any graph $G \in \mathcal{C}$ may be coloured using $N(p)$ colours so that each of the connected components of the subgraph induced by any $i \leq p$ parts has tree-width at most $(i-1)$.

Definition 6.2.3. Let $\mathcal{C}$ be a class of graphs. Then $\mathcal{C}$ has a *low tree-depth colouring* if, for any integer $p \geq 1$, there exists an integer $N(p)$ such that any graph $G \in \mathcal{C}$ may be coloured using $N(p)$ colours so that each of the connected components of the subgraph induced by any $i \leq p$ parts has tree-depth at most i.

This naturally leads to a sequence $\chi_1, \chi_2, \ldots$ of chromatic numbers χ_p, where χ_1 is the usual chromatic number, χ_2 is the star chromatic number and, more generally, χ_p is the minimum number of colours such that any $i \leq p$ parts induce a graph with tree-depth at most i.

Theorem 6.2.1 ([13]). *Any minor-closed class of graphs excluding at least one graph as a minor (proper minor-closed class) has a low tree-width colouring.*

Theorem 6.2.2 ([48]). *Any proper minor-closed class of graphs has a low tree-depth colouring.*

The following definition of the greatest reduced average density was introduced in [49].

Definition 6.2.4. Let G be a graph. A *ball* of G is a subset of vertices inducing a connected subgraph. The set of all families of balls of G is noted $\mathcal{B}(G)$.

Let $\mathcal{P} = \{V_1, V_2, \ldots, V_p\}$ be a family of balls of G.

- The *radius* $\rho(\mathcal{P})$ of $\mathcal{P}$ is $\rho(P) = \max_{X \in \mathcal{P}} \rho(G[X])$.
- The *quotient* $G/\mathcal{P}$ of G by $\mathcal{P}$ is a graph with vertex set $V(G/\mathcal{P}) = \{1, \ldots, p\}$ and edge set $E(G/\mathcal{P}) = \{\{i,j\} \mid (V_i \times V_j) \cap E(G) = \emptyset \text{ or } V_i \cap V_j = \emptyset\}$.

The *greatest reduced average density* (*grad*) of G with *rank* r is

$$\nabla_r(G) = \max_{\substack{\mathcal{P} \in \mathcal{B}(G) \\ \rho(\mathcal{P}) \leq r}} \frac{|E(G/\mathcal{P})|}{|\mathcal{P}|}$$

By relaxing the notions of the neighbourhood of a vertex in a graph and of the minor of a graph we get the following.

Definition 6.2.5. Let G and H be graphs and d any positive integer. Then

- the *d-neighbourhood* $N_G^d(u)$ of a vertex $u \in V(G)$ is the subset of vertices of G at distance at most d from u in G, i.e. $N_G^d(u) = \{v \in V(G) \mid \mathbf{dist}_G(u,v) \leq d\}$.
- the graph H is said to be a *shallow minor of* G *at depth* d, denoted $\leq_d$, if there exists a subset $\{v_1, v_2, \ldots, v_p\}$ of $V(G)$ and a collection of disjoint subsets $V_1 \subseteq N_G^d(v_1), \ldots, V_p \subseteq N_G^d(v_p)$ such that H is a subgraph of the graph obtained from G by contracting each V_i to v_i and removing loops and multiple edges. The set of all shallow minors of G at depth d is denoted by $G \triangledown i$.

Remark 6.2.1. The notion of shallow minors first appeared in [57], then called low depth minor, and is attributed to Ch. Leiserson and S. Toledo.

Remark 6.2.2. Let G, H be graphs. Notice that $H \leq_0 G$ if and only if $H \subseteq G$. Moreover, $H \leq_\infty G$ if and only if $H \leq G$. Therefore, the shallow minors define a hierarchy of relations on graphs between the relation of the subgraph and the minor of a graph.

$$\subseteq \quad = \quad \leq_0, \leq_1, \quad \ldots \quad , \leq_n, \quad \ldots \quad , \leq_\infty \quad = \quad \leq$$

It is easy to observe that the following is true.

Lemma 6.2.1. *The grad of a graph G with rank r is equal to*

$$\nabla_r(G) = \max \left\{ \frac{\|H\|}{|H|} \mid H \in G \triangledown r \right\}$$

By extension, for a class $\mathcal{C}$ of graphs, we denote by $\mathcal{C} \triangledown r$ the set of all shallows minors at depth r of graphs of $\mathcal{C}$, i.e.

$$\mathcal{C} \triangledown r = \bigcup_{G \in \mathcal{C}} (G \triangledown r)$$

Hence we have

$$\mathcal{C} \subseteq \mathcal{C} \triangledown 0 \subseteq \mathcal{C} \triangledown 1 \subseteq \cdots \subseteq \mathcal{C} \triangledown r \subseteq \cdots \subseteq \mathcal{C} \triangledown \infty$$

By $\mathcal{C} \triangledown \infty$ we denote the class of all minors of graphs of $\mathcal{C}$.

Lemma 6.2.2 ([49])**.** *For any graph G and any positive integer r:*

$$\nabla_r(G) \leq (2r+1) \binom{\chi_{2r+2}(G)}{2r+2}$$

Lemma 6.2.2 motivated the following.

Definition 6.2.6. A class of graphs $\mathcal{C}$ has *bounded expansion* if there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that for every graph $G \in \mathcal{C}$ and every $r \in \mathbb{N}$ holds

$$\nabla_r(G) \leq f(r)$$

The *expansion* of a class $\mathcal{C}$ with bounded expansion is the function f defined by

$$f(r) = \nabla_r(\mathcal{C}) = \sup_{G \in \mathcal{C}} \nabla_r(G)$$

The following theorem summarizes the relation between χ_p, p-centered colourings, low tree-width colourings, low tree-depth colourings and bounded expansion.

Theorem 6.2.3 ([49]). *Let $\mathcal{C}$ be a class of graphs. Then the following conditions are equivalent:*

1. *$\mathcal{C}$ has low tree-width colouring,*
2. *$\mathcal{C}$ has low tree-depth colouring,*
3. *for any positive integer p, $\{\chi_p(G) \mid G \in \mathcal{C}\}$ is bounded,*
4. *for any positive integer p, there exists an integer $X(p)$ such that any graph in $\mathcal{C}$ has a p-centered colouring using at most $X(p)$ colours,*
5. *$\mathcal{C}$ has bounded expansion.*

Theorem 6.2.4 ([50, 52]). *Let $\mathcal{C}$ be a class with bounded expansion and let p be a fixed integer. Let ϕ be a $FOL(\tau_2)$ sentence. Then there exists a linear time algorithm to check whether $\exists X : (|X| \leq p) \wedge (G[X] \models \phi)$.*

A direct consequence of Theorem 6.2.4 is the following.

Theorem 6.2.5. *Let $\mathcal{C}$ be a class with bounded expansion and let p be a fixed integer. Let ϕ be an existential $FOL(\tau_2)$ sentence. Then there exists a linear time algorithm to check whether an input graph $G \in \mathcal{C}$ satisfies ϕ or not.*

These theorems have great importance in the (parameterized) Complexity Theory.

Theorem 6.2.6 ([50]). *For any fixed integer k, there exists a linear time algorithm which decides whether an input graph G has tree-depth at most k or not.*

Remark 6.2.3. We should not neglect to mention that the proofs of Theorem 6.2.4 and 6.2.6 make use of the Theorem 6.1.2.

In [5] a polynomial-time algorithm is constructed that for any integer k, given a graph G with tree-width at most k, determines the vertex ranking of a graph (equivalently, the tree-depth of a graph) and finds an optimal vertex ranking of G.

From the example of Section 2.4 it follows that if $\mathcal{K}$ is a class of graphs that has bounded expansion and H is a fixed graph in $\mathcal{K}$ it is decidable in linear time if it contains a dominating set of k elements.

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