NATIONAL AND KAPODISTRIAN UNIVERSITY OF ATHENS
Mathematics Department

Graduate Program in Logic, Algorithms and Computation

M.Sc Thesis

On The Relation Between Treewidth and Toughness

by
Konstantinos Gerakios

Supervised by
Lefteris Kirousis

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Πρόλογος

Το πρόβλημα εύρεσης του δενδροπλάτους (treewidth) και της σκληρότητας (toughness) ενός γραφήματος, είναι γνωστά NP-πλήρη προβλήματα. Το δενδροπλάτος αποτελεί, σε γενικές γραμμές, ένα μέτρο συνεκτικότητας του γραφήματος καθώς και ένα μέτρο αποδοτικού υπολογισμού για γνωστά NP-hard προβλήματα. Συγκεκριμένα, είναι γνωστό ότι πολλά NP-hard προβλήματα σε γραφήματα, μπορούν να επιλυθούν με Δυναμικό Προγραμματισμό σε πολυωνυμικό χρόνο, αν το δενδροπλάτος τους είναι φραγμένο από σταθερά.Η σκληρότητα ενός γραφήματος από την άλλη είναι ένα μέτρο κυκλικότητας ενός γραφήματος, και αποτελεί μία παράμετρο ένδειξης κυκλικών δομών σε αυτό. Ένα παράδειγμα αποτελεί η ύπαρξη χαμιλτονιανών κύκλων σε ένα γράφημα, η οποία σχετίζεται στενά με την έννοια της σκληρότητας. Στην παρούσα διπλωματική εργασία συσχετίζουμε το δενδροπλάτος ενός γραφήματος με την σκληρότητα, και δίνουμε άνω φράγματα για την σκληρότητα ενός γραφήματος.

Ευχαριστίες

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Contents

1 Treewidth

1.1 Introduction

In this section we will describe the notion of treewidth, a graph parameter that expresses a measure of connectivity in graphs. This graph parameter has many interesting theoretical and practical applications. Many NP-Hard problems on general graphs, can be solved in polynomial time if the input graph has bounded treewidth. Intuitively, small treewidth means that the graph can be decomposed recursively into small subgraphs with small overlap between them. Therefore we can view a graph having small treewidth, as graph that has a tree-like structure. When the input graph has bounded treewidth, we can take andvantage of the underlying tree-structure of the graph and apply dynamic programming techniques to achieve polynomial time complexity.

The notion of treewidth was introduced by Robertson and Seymour in their work on graph minors [1], as a measure of the topological resemblance of a graph to the structure of a tree. The actual term 'treewidth' was defined in terms of tree decompositions. There is another, equivalent definition of treewidth in terms of partial k-trees. Finding the treewidth of an arbitrary graph is NP-Hard, as it is reduced to another NP-Hard problem, the pathwidth.

1.2 Tree Decompositions

As stated earlier, treewidth is defined via tree decompositions. Tree decompositions are defined as follows :

Definition 1. Let $G = (V, E)$ be a graph, $T = (I, F)$ be a tree and let $\mathcal{V} = (\{X_i, i \in I\})$ be a collection of subsets of V (called bags). The pair $(\mathcal{V}, T)$ is called a *tree decomposition* of G, if it satisfies the following three conditions :

1. For all $v \in V(G),$ there exists an $i \in I$ such that $v \in X_i$
2. For all $\{v, w\} \in E(G),$ there exists an $i \in I$ such that $v, w \in X_i$.
3. For all $v \in V(G), T_v = \{i \in I \mid v \in X_i\}$ forms a connected subtree of T.

The last condition of definition 1 can be replaced by the following equivalent condition :

$3'$. If $i, j, k \in I$ and k is on the path from i to j in T then $X_i \cap X_j \subseteq X_k$

We can verify the previous statement as follows. Let $i, j, k \in I$ and k be on the path from i to j in T. Let $P = i, v_1, ..., v_t, j$ be that path. Assume also that $v \in X_i \cap X_j$. Then $i, j \in T_v = \{i \in I \mid v \in X_i\}$, and from condition

3. T_v is connected. Thus, there exists a path $P' = i, v'_1, ..., v'_s, j$ in T_v. If $k \in P'$ then $k \in T_v$ and therefore $v \in X_k$ from the definition of T_v. If $k \notin P'$, then $P \cup P'$ forms a cycle in T which is a contradiction. So we conclude that $v \in X_i \cap X_j \Rightarrow v \in X_k$.

Conversely, assume that we have replaced condition 3 with 3' in definition 1. Let $T_v = \{i \in I \mid v \in X_i\}$ and let $i, j \in T_v$ (thus $v \in X_i \cap X_j$), and let k be any vertex on the path from i to j in T. Then we have that $X_i \cap X_j \subseteq X_k$ and consequently $v \in X_k$. So $k \in T_v$, and that holds for every other vertex on that path. Thus T_v is connected.

We give the following examples of tree decompositions :

Example 1. Let G be the graph shown in Figure 1 (a). The tree shown in (b) is a tree decomposition of G.

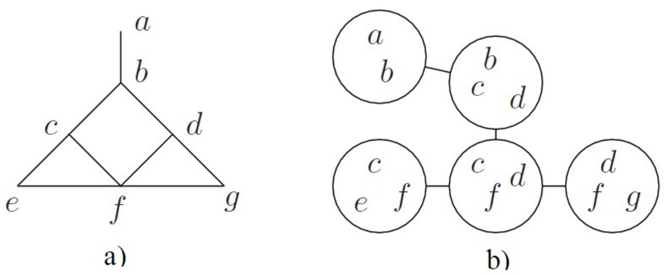

Figure 1: A graph G and a tree decomposition of G

An interesting example of a tree decomposition of a graph is that of a $(k \times k)$-grid. The $(k \times k)$-grid is the graph on $\{1, ..., k\}^2$ with edge set

$$\{(i, j), (i', j') \ : \ |i - i'| + |j - j'| = 1\}$$

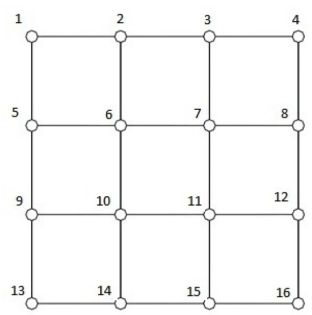

Figure 2: A (4×4)-grid

Example 2. Let G be the (4×4)-grid in figure 1.2. A tree decomposition of G is the following path : $\{2,1,5,9,13\}$,$\{2,6,5,9,13\}$, $\{2,6,10,9,13\}$,$\{2,6,10,14,13\}$, $\{3,2,6,10,14\}$,$\{3,7,6,10,14\}$,$\{3,7,11,10,14\}$, $\{3,7,11,15,14\}$,$\{4,3,7,11,15\}$,$\{4,8,7,11,15\}$, $\{4,8,12,11,15\}$,$\{4,8,12,16,15\}$. It is easy to verify that every vertex and edge of G is contained in some vertex set in the path, thus satisfying the first two conditions of a tree decomposition. It is also easy to verify that for every $v \in [1,k]$, $T_v = \{i \in I \mid v \in X_i\}$ forms a subpath of the original path and hence it forms a connected subtree of the tree decomposition. Thus, the path is a valid tree decomposition of G, with a size of largest bag of 5. The same line of thought can be applied to a $(k \times k)$-grid, for an arbitrary k, giving a tree decomposition with size of largest bag of $k+1$. We will later see that this tree decomposition is optimal, with respect to the size of the largest bag.

In the next section, we describe some important properties of tree decompositions. However, we haven't yet answered a critical question. Which graphs have tree decompositions? The answer is that every graph has at least one tree decomposition. The most trivial tree decomposition is just one node with a corresponding bag the entire vertex set of the original graph. It is easy to verify that it satisfies all three conditions of definition 1. Tree decompositions for the same graph can differ in the number of nodes, the size of the bags and of course in the context of the bags. We are mostly interested in tree decompositions with the minimum possible largest bag and with the least possible number of nodes. In the context of dynamic programming, we are interested in tree decompositions that do not have too many nodes. One way to achieve this, is by running the following procedure :

Algorithm 1

procedure TRANSFORMTONONREDUNANT($\{X_i \mid i \in I\}, T = (I,F)$)
$\qquad\qquad\qquad\qquad$ $\triangleright$ The input is a tree decomposition
$\quad$ **while** there exists an edge $(i,j) \in F$ with $X_i \subseteq X_j$ **do**
$\qquad$ Contract edge (i,j) and let the new node be node j and remove
$\qquad$ X_i from the set of bags
$\quad$ **end while**
$\quad$ **return** the new tree decomposition
end procedure

The output of algorithm 1 is a non-redundant tree decomposition.

1.3 Properties of Tree Decompositions

Tree decompositions have many useful properties, especially the tree-like separation properties that we will use in section 3. A useful lemma on tree decompositions is the following. While this lemma is well known, we include a proof of it that is rather different from what we usually come accross in the literature.

Lemma 1. *Let* $(\{X_i \mid i \in I\}), T = (I, F))$ *be a tree decomposition of graph* $G = (V, E)$*. Let* W *be a clique in* G*. Then there exists an* $i \in I$ *with* $W \subseteq X_i$*.*

Proof. We will prove it by induction on $n = |W|$.

For $n = 1$ and $n = 2$, it is immediate from the conditions 1,2 of the definition of a tree decomposition (Definition 1), since $\forall v \in V(G)\ \exists i \in I\ [v \in X_i]$ and $\forall \{v, u\} \in E(G)\ \exists i \in I[\{v, u\} \subseteq X_i]$.

Suppose now that $n > 2$, $v \in W$ and also let $W' = W \setminus \{v\}$. Then $|W'| < n$ and we can apply the induction hypothesis on W'. So it holds that $\exists k \in I\ [W' \subseteq X_k]$. If $v \in X_k$ there is nothing to prove. So assume that $v \notin X_k$. From condition 2 of definition 1 and the fact that $\{v, u\} \in E(G),\ \forall u \in W'$, we have that :

$$\forall u \in W'\ \exists j \in I\ [\{v, u\} \subseteq X_j] \quad (*)$$

Let $X_{j_1}, .., X_{j_s}$ be all the sets that satisfy the condition $\exists u \in W'\ \{v, u\} \subseteq X_{j_t}$ $(1 \leq t \leq s)$. If $s = 1$ then j_s is the desired i because of (*). So assume further that $s > 1$. Let now $T_u = \{j \in I \mid u \in X_j\}$. Then the following hold as well :

1. For all $1 \leq t \leq s\ \exists u \in W'\ [j_t \in T_u]$, by their definition
2. For all $1 \leq t \leq s\ [v \in X_{j_t}]$, by their definition
3. $\forall u \in W'\ [k \in T_u]$, since $u \in X_k$

Now let $T_{W'} = \cup_{u \in W'} T_u$. Then $T_{W'}$ is connected because $\forall u \in W'$, T_u is connected and $\forall u \in W'\ k \in T_u$ (from 2). We also have that $j_1, ..., j_s \in T_{W'}$. In more detail, each T_u is connected and k is a common node.

Now, it cannot be the case that node k is on the path from j_l to $j_{l'}$ $(1 \leq l < l' \leq s)$ in $T_{W'}$ because from condition 3' of tree decompositions we would have that $v \in X_{j_l} \cap X_{j_{l'}} \subseteq X_k$, which yields a contradiction because of the assumption that $v \notin X_k$. Then $T_{W'}$ must have the following form : set node k as the root of $T_{W'}$. Then k is connected to only one node r having the property $r \in \{j_1, ..., j_s\}$ or r has descendants in $\{j_1, ..., j_s\}$.

Suppose first that there exists a node $t \in \{j_1, ...j_s\}$ such that nodes in $\{j_1, ..., j_s\}$ are in the subtree of $T_{W'}$ below t. Then node t is on the path between node k and node $r' \in \{j_1, ..., j_s\} \setminus \{r\}$. Hence, $X_k \cap X_{r'} \subseteq X_t$, for any $r' \in \{j_1, ..., j_s\} \setminus \{t\}$ and consequently $W' \subseteq X_t$ (because of (*)). Furthermore, $v \in X_t$, thus $W \subseteq X_t$.

Suppose now that there is no such node and let r be the more distant from k ancestor of all nodes in $\{j_1, ...j_s\}$ in $T_{W'}$. Then, there exist nodes

$r_1, r_2 \in \{j_1, ...j_s\}$, such that r_1 and r_2 are on different branches of r. Then node r is on the path between node r_1 and r_2 in $T_{W'}$, which implies that $v \in X_{r_1} \cap X_{r_2} \subseteq X_r$. Now, r is also in the path between k and nodes $j_1, ...j_s$, which means that $W' \subseteq X_r$ (again using (*)). Hence, $W \subseteq X_r$.

□

A tree has two useful properties. The first is that deleting an edge causes the tree to split into two connected components. The second is that deleting a node causes the tree to split into a number of components equal to the degree of that node. In fact, condition 3' in the definition of tree decomposition (Definition 1) leads to similar tree-like separation properties of the initial graph, which means that separations of the tree decomposition of the graph translate to separation of the graph. So let $G = (V, E)$ be a graph and $(\{X_i \mid i \in I\}), T = (I, F))$ be a tree decomposition of G. For every subtree T_s of T, let $G_{T_s} = G(\cup_{i \in T_s} X_i)$ denote the subgraph of G induced by the vertices in the bags indexed by nodes of T_s. Naturally, $G_T = G$. The following lemmata describe those tree-like separation properties of G (their proofs can be found in [4]). The first one describes the analog of the vertex separation property of a tree :

Lemma 2. *Let $i \in I$ and assume that $T \setminus \{i\}$ has components $T_1, ..., T_k$. Then the subgraphs $G_{T_1} \setminus V_i, ..., G_{T_k} \setminus V_i$ have no vertices in common, and there are no edges between them.*

Assume that the precondition of Lemma 2 holds. From condition 1 of Definition 1, we have that for every $v \in V(G)$, there exists an $i \in I$ such that $v \in X_i$. That means that for every $v \in V \setminus V_i$ there exits a $j \in I \setminus \{i\}$ such that $v \in X_j$. That node j must belong to some component T_s in $T \setminus \{i\}$. Thus for every $v \in V \setminus V_i$ there exists an $s \in [1, k]$ such that v belongs to subgraph $G_{T_s} \setminus V_i$. From Lemma 2, v does not belong to any other subgraph G_{T_l} for $l \neq s$, and is not connected to a vertex belonging to any other subgraph either. So the set V_i actually separates G, into connected components $G_{T_1} \setminus V_i, ..., G_{T_k} \setminus V_i$, where k is the degree of node i in T. The following is a corollary of Lemma 2 :

Corollary 1. *Let $i \in I$ and assume that $T \setminus \{i\}$ has components $T_1, ..., T_k$. Then $G \setminus V_i$ has components $G_{T_1} \setminus V_i, ..., G_{T_k} \setminus V_i$.*

The following lemma describes the analog of the edge separation property of a tree :

Lemma 3. *Let $i, j \in I$ be two adjacent nodes in T and let X, Y denote the two components of T after the deletion of the edge (i, j). Then removing $V_i \cap V_j$ from G, disconnects G into the two subgraphs, $G_X \setminus (V_i \cap V_j)$ and*

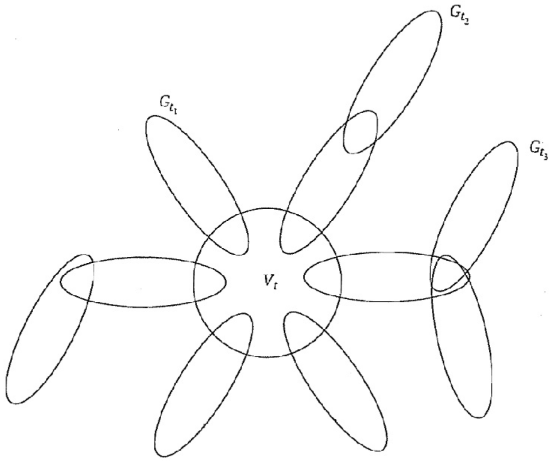

Figure 3: A tree decomposition T of a graph G and t a node of T. $G \setminus V_t$ has components $G_{T_1} \setminus V_t, ..., G_{T_k} \setminus V_t$.

$G_Y \setminus (V_i \cap V_j)$ that do not share any vertices and there is no edge connecting them.

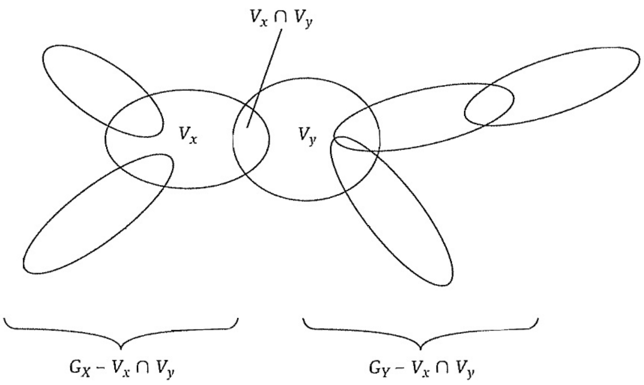

Figure 4: A tree decomposition T of a graph G. Vertex set $V_X \cap V_Y$ disconnects $G \setminus (V_i \cap V_j)$ into two components.

An interesting property of tree decompositions is that they are passed on to subgraphs :

Lemma 4. *For every $H \subseteq G$, the pair $(\{X_i \cap V(H) \mid i \in I\}, T = (I, F))$ is a tree decomposition of H.*

In the previous section we described a procedure (algorithm 1), for deleting nodes that are redundant in a tree decomposition, resulting in a tree decomposition of the same graph with fewer nodes that is nondedundant. The following proposition states that at the end of this procudure the number of nodes of the tree decomposition produced is at most n :

Proposition 1. *Let G be a graph on n vertices. Any nonredundant tree decomposition of G has at most n nodes (bags).*

1.4 Treewidth

The definition of treewidth is based on the definition of the width of tree decompositions :

Definition 2. The *width* of a tree decomposition $(\{X_i \mid i \in I\}, T = (I, F))$ is $\max_{i \in I}|X_i| - 1$.

So we have the following definition of treewidth in terms of tree decompositions :

Definition 3. The *treewidth*, $tw(G)$, of a graph G is the minimum *width* over all tree decompositions of G.

An *optimal* tree decomposition of a graph G is one with the smallest maximum bag.

The following lemma asserts the correctness of algorithm 1 :

Lemma 5. *[37] Let $G = (V, E)$ be a graph with $tw(G) \leq k$. Then there exists a tree decomposition of G, $(\{X_i \mid i \in I\}, T = (I, F))$ of width at most k, such that for all $\{i, j\} \in F$: $X_i \not\subseteq X_j$ and $X_j \not\subseteq X_i$.*

Let $\omega(G)$ be the size of the largest clique in graph G. Then we have the following corollaries of lemma 1 :

Corollary 2. *For every graph G, $tw(G) \geq \omega(G) - 1$.*

Corollary 3. *The complete graph on n vertices has treewidth $tw(K_n) = n - 1$.*

Let $\alpha(G)$ be the size of the maximum independent set in graph G. The following lemma bounds treewidth using the size of the maximum independent set.

Lemma 6. *Let G be a graph on n vertices. Then $tw(G) \leq n - \alpha(G)$.*

Proof. If $\alpha(G) = 0$ then G is the complete graph on n vertices, thus $tw(G) = n-1 \leq n = n-\alpha(G)$. Suppose now that $\alpha(G) = k > 0$, and let $v_1, ..., v_k$ be k vertices forming a maximum independent set. Now set $X_i = \{v_i\} \cup N_G(v_i)$ for $1 \leq i \leq k$, $X_{k+1} = V(G) - \{v_1, ..., v_k\}$, $I = \{1, .., k+1\}$ and $F = \{\{i, k+1\} \mid i \in I\}$. Then the pair $(\{X_i \mid i \in I\}, T = (I, F))$ is a valid tree decomposition of G. Now let $n_i = |\{v_i\} \cup N_G(v_i)|$ for $1 \leq i \leq k$ and $n_{k+1} = n-k$ be the sizes of the bags $X_1, ..., X_k, X_{k+1}$ respectively. Observe that $n_i - 1 \leq n_{k+1} = n-k$ for $1 \leq i \leq k$, since the neighborhood of each v_i can not include vertices in the independent set. Hence, $\max_{i \in I} |X_i| \leq n - k + 1 = n - \alpha(G) + 1$. Consequently, $tw(G) \leq \max_{i \in I} |X_i| - 1 \leq n - \alpha(G)$ (the tree decomposition is not necessarily optimal). □

Let $\delta(G)$ denote the minimum degree of a graph G. The next lemma presents a lower bound on treewidth, using the minimum degree as parameter.

Lemma 7. *Let G be a graph. Then $tw(G) \geq \delta(G)$.*

Proof. Let $(\{X_i \mid i \in I\}, T = (I, F))$ be an optimal nonredundant tree decomposition of G. If $|I| = 1$ then $tw(G) = n - 1$ and G must be complete (see lemma 8), hence $\delta(G) = n - 1 = tw(G)$. Now suppose that $|I| > 1$ and let $i \in I$ be any node of the tree decomposition with $deg_T(i) = 1$, and let j be its neighboor. Since the tree decomposition is nonredundant, there must exist a $v \in X_i$ such that $v \notin X_j$. Now, from the condition 2 of the definition of tree decomposition, the neighborhood of v must also belong in X_i, that is $N_G(v) \subset X_i$. Therefore, $|X_i| \geq |\{v\} \cup N_G(v)| \geq \delta(G) + 1$. Finally $tw(G) = \max_{i \in I} |X_i| - 1 \geq \delta(G)$.

□

Lemma 4 asserts that if G is a graph and $H \subseteq G$, then given a tree decomposition for G we can construct a tree decomposition T for H, simply by constraining the bags of T to vertices in $V(H)$. So we have the following corollary of lemma 4 :

Corollary 4. *Let G be a graph. For every $H \subseteq G$, $tw(H) \leq tw(G)$.*

Combining corollaries 3 and 4, we easily get the following lemma.

Lemma 8. *A graph G on n vertices has treewidth $n - 1$ if and only if it is complete.*

Proof. By corollary 3, if $G = K_n$ then $tw(G) = n - 1$. Now suppose that G is a noncomplete graph. Then it is easy to see that G is a subgraph of $H \equiv K_n - \{(v, u)\}$ for some $v, u \in V(G)$. Now, set $X_1 = V(G) - v, X_2 = V(G) - u$. It is easy to verify that the tree decomposition $(\{X_1, X_2\}, T = (\{1, 2\}, \{(1, 2)\}))$ is a valid tree decomposition of H. Thus, we get $tw(H) \leq$

$n-2$, since the maximum bag contains $n-1$ elements. Since G is a subgraph of H, by corollary 4 we obtain $tw(G) \leq tw(H) \leq n-2$. □

A well known result is the following :

Lemma 9. *Let $G = (V, E)$ be a graph, and $H = (W, F)$ a minor of G. Then $tw(H) \leq tw(G)$.*

The next lemma, shows that we can restrict the family of graphs we examine, to connected graphs only :

Lemma 10. *[35] The treewidth of a graph equals the maximum treewidth over its connected components*

Treewidth can be defined via several equivalent notions :

Definition 4. A graph $G = (V, E)$ is *chordal*, if and only if every cycle in G of length greater than three has a chord (i.e, an edge between non-succesive vertices in the cycle).

Definition 5. A *perfect elimination ordering* of a graph $G = (V, E)$ is an ordering of $V(G)$, $v_1, ..., v_n$, such that for all $v_i \in V(G)$, its higher numbered neighbours form a clique, i.e, for every $j, k > i$, if $\{v_i, v_j\}, \{v_i, v_k\} \in E(G)$, then $\{v_j, v_k\} \in E(G)$.

Definition 6. A graph $G = (V, E)$, is an *intersection graph of a family of subtrees of a tree*, if and only if there is a tree decomposition $(\{X_i \mid i \in I\}, T = (I, F))$ of G, and for every vertex $v \in V(G)$ a subtree $T_v = \{i \in I \mid v \in X_i\}$ of T, such that for all $v, w \in V(G)$ with $v \neq w$, $\{v, w\} \in E(G)$ if and only if $T_v \cap T_w \neq \emptyset$.

Definition 7. A *clique tree* for a graph $G = (V, E)$, is a tree $T = (V_T, E_T)$ where V_T is the set of maximal cliques of G.

Definition 8. A *junction tree* for a graph $G = (V, E)$, is a clique tree $T = (V_T, E_T)$ for G that satisfies the following property : For any cliques C_1, C_2, C_3 in the clique tree, if C_3 is on the path connecting C_1 and C_2 in T then $C_1 \cap C_2 \subseteq C_3$.

We can view a junction tree as a tree decomposition $(\{X_i \mid i \in I\}, T = (I, F))$, where for every $i \in I$, X_i is a clique. The maximality criterion in definition 7 is just a consequence of lemma 1 and our restriction on non-redundant tree decompositions. It turns out that these definitions describe the same family of graphs, as we can see in the following theorem :

Theorem 1. *[82, 88] Let $G = (V, E)$ be a graph. The following statements are equivalent :*
1. G is a chordal graph.

2. G has a perfect elimination ordering.
3. G is the intersection graph of subtrees of a tree.
4. G has a junction tree.

Definition 9. A *triangulation* of a graph $G = (V, E)$ is a chordal graph $H = (V, F)$ with $E \subseteq F$. A triangulation $H = (V, F)$ of a graph $G = (V, E)$ is a *minimal triangulation*, if there does not exist a triangulation $H' = (V, F')$ of G with $E \subseteq F' \subset F$. We denote the set of all triangulations of a graph G by :

$$\mathcal{T}(G) = \{H \mid H \text{ is a triangulation of } G\}$$

We are especially interested in the following measure :

Definition 10. For any graph G, let $mmc(G)$ denote the maximum clique size of the minimum triangulation of a graph G :

$$mmc(G) = \min_{H \in \mathcal{T}(G)} \omega(H)$$

Suppose that we are given a tree decomposition $(\{X_i \mid i \in I\}, T = (I, F))$ of a graph G. Then we can produce a triangulation $H = (V, F)$ of G with the fill-in procedure in algorithm 2 :

Algorithm 2

procedure TRIANGULATEGIVENTREEDECOMPOSITION(Graph G,Tree Decomposition $(\{X_i \mid i \in I\}, T = (I, F))$)
    $H \leftarrow G$
    for all $i \in I$ **do**
        for all $w, u \in X_i : w \neq u$ **do**
            if $(w, u) \notin E(H)$ **then**
                $E(H) \leftarrow E(H) \cup \{(w, u)\}$
            end if
        end for
    end for
    return H
end procedure

Algorithm 2 correctly produces a triangulation for a graph G, given a tree decomposition for G, because in each X_i it adds edges between every non-adjacent pair of vertices, thus turning each bag of T into a clique. Consequently, the tree decomposition T becomes a clique tree satisfying condition 3' of the definition of tree decompositions, hence T becomes a junction tree for H, which means that H is a chordal graph that has G as a subgraph (theorem 1). Therefore H is a triangulation of graph G. The width of the tree decomposition is exactly one less than the maximum clique size of H. So we have the following lemmata :

Lemma 11. *Given a graph $G = (V, E)$ and a tree decomposition $T = (\{X_i \mid i \in I\}, T = (I, F))$ for G, let H be the graph obtained by adding edges to G so that each X_i becomes a clique. Then H is chordal.*

Lemma 12. *For any graph G :*

$$tw(G) = \min_{H \in \mathcal{T}(G)} \omega(H) - 1$$

Definition 11. Let $G = (V, E)$ be a graph and $v \in V(G)$ be a vertex. We call *elimination* of v, the operation that adds an edge between every pair of non-adjacent neighbours of v, and then removes v.

Given an ordering $\pi = v_1, ..., v_n$ of the vertices of a graph G, we can produce a triangulation $H_\pi(G)$ with the following fill-in procedure :

Algorithm 3

procedure FILLIN(Graph G,Ordering π)
    $H \leftarrow G$
    for $i = 1$ to n **do**
        $v \leftarrow \pi^{-1}(i)$ ▷ The ith vertex in the ordering π
        for all $w, u \in N_H(v) : w \neq u, \pi(w) > \pi(v)$ and $\pi(u) > \pi(v)$ **do**
            if $(w, u) \notin E(H)$ **then**
                $E(H) \leftarrow E(H) \cup \{(w, u)\}$
            end if
        end for
    end for
    return H
end procedure

Here we use the denotation $N_G(v)$ to stand for the neighborhood of vertex v in graph G.

We can view Algorithm 3 as an "elimination game" in which we eliminate sequentialy the vertices appearing in the ordering. The graph H produced contains all edges of the initial graph plus the edges that were added during the elimination process. For i, $1 \leq i \leq n$, we connect every pair (v_k, v_l) of yet not adjacent higher numbered neighbours of v_i $(i < k < l)$. This procedure defines a sequence of graphs $(G_i)_{i=0..n}$, such that $G_0 \equiv G$, and $G_n \equiv H_\pi(G)$, where $H_\pi(G)$ is a triangulation of G. The reason that,

$H_\pi(G) = FillIn(G, \pi)$ is a triangulation of G is that, the ordering π is a perfect elimination ordering for $H_\pi(G)$ and thus $H_\pi(G)$ is chordal (theorem 1).

Let $H_\pi(G) = FillIn(G, \pi)$. We see that π is a perfect elimination ordering of $H_\pi(G)$, so $H_\pi(H) = H$. It is also clear that if G_1, G_2 are two graphs with the same vertex set and such that $E(G_1) \subseteq E(G_2)$, then if π is an ordering of their vertices, $FillIn(G_1, \pi) \subseteq FillIn(G_2, \pi)$. Suppose now that we are given a graph G and a triangulation H of G. Since H is a triangulation of a graph, it is chordal and thus from theorem 1 we conclude that it has a perfect elimination ordering π. Consequently, $H_\pi(G) = FillIn(G, \pi) \subseteq FillIn(H, \pi) = H_\pi(H) = H$. So from corollary 4, we have that $tw(H_\pi(G)) \leq tw(H)$. That means that if we want to find a triangulation of a graph G that achieves the minimum treewidth, we can constrain our search into the set of triangulations derived by process $FillIn$ and the set $\Pi_{V(G)}$ of orderings of the vertices of the graph. We sum up these observations with the following lemmata :

Lemma 13. *Let G be a graph and H any triangulation of G. Then there exists an ordering π of $V(G)$ such that $H_\pi(G) \subseteq H$.*

Lemma 14. *Let G be a graph and let Π be the set of all the orderings of $V(G)$. Then the following relation holds :*

$$\min_{H \in \mathcal{T}(G)} \omega(H) = \min_{\pi \in \Pi} \omega(H_\pi(G))$$

Combining lemmata 12,14, we get the following lemma :

Lemma 15. *Let G be a graph and let Π be the set of all the orderings of $V(G)$, then :*

$$tw(G) = \min_{\pi \in \Pi} \omega(H_\pi(G)) - 1$$

The next theorem is immediate from lemmata described in this section :

Theorem 2. *[36] Let $G = (V, E)$ be a graph, and let $k \leq |V(G)|$ be a non-negative integer . The following are equivalent :*
1. G has treewidth k.
2. There exists a triangulation H of G with $\omega(H) \leq k + 1$.
3. There exists an ordering π of $V(G)$, such that $\omega(H_\pi(G)) \leq k + 1$.
4. There exists an ordering $\pi = v_1, ..., v_n$ of $V(G)$, such that there does not exist an $i < n$ such that $|N_{H_\pi(G)}(v_i) \cap \{v_{i+1}, ..., v_n\}| > k$ in $H_\pi(G)$.

Given an ordering of the vertices of a graph, we can build a tree decomposition of it, using the following procedure :

Algorithm 4

procedure ORDERINGTOTREEDECOMPOSITION(Graph G,Ordering $\pi = v_1, ..., v_n$)
    if n = 1 **then**
        $X_1 = \{v_1\}$
        return $(\{X_1\}, (\{1\}, \emptyset))$
    end if
    Let $G' = (V', E')$ be the graph, obtained from G by eliminating v_1
    Let $(\{X_i \mid i \in I\}, T = (I, F))$ be the result of
      $OrderingToTreeDecomposition(G', (v_2, ..., v_n))$
    $j \leftarrow \min\{i \in I \mid \{v_1, v_i\} \in E(G)\}$
    $X_{v_1} \leftarrow N_G(v_1)$
    $I' \leftarrow I \cup \{v_1\}$
    $F' \leftarrow F \cup \{\{v_1, v_j\}\}$
    return $(\{X_i \mid i \in I\} \cup \{v_1\}, T = (I', F'))$
end procedure

The correctness of algorithm 4 is shown by the following result :

Lemma 16. *[36] Let $G = (V, E)$ be a graph, and $\pi = (v_1, ..., v_n)$ be an elimination ordering of G. Let $H = (V, E(H)) = H_\pi(G)$ be the filled graph of G with respect to G. The output of Algorithm 4, when given as input the graph G and vertex ordering π, is a tree decomposition $(\{X_v \mid v \in V\}, T = (V, F))$, such that :*
1. For all $v_i \in V$, $X_{v_i} = \{v_i\} \cup \{v_j \mid j > i \ \wedge\ \{v_i, v_j\} \in E(H)\}$.
2. The width of the tree decomposition is $\omega(H) - 1$.

The following corollary is immediate from lemmata 15,16 and theorem 2 :

Corollary 5. *Let $G = (V, E)$ be a graph, and $\Pi_{V(G)}$ the set of orderings of $V(G)$. Then*

$$tw(G) = \min_{\pi=(v_1,...,v_n)\in\Pi_{V(G)}} \max_{1\le i\le n} |N_{H_\pi(G)}(v_i) \cap \{v_{i+1}, ..., v_n\}|$$

We have already mentioned that treewidth can be defined alternatively via partial k-trees. We need the following definitions :

Definition 12. Let $G = (V, E)$ be a graph. We call a vertex $v \in V(G)$ *simplicial* if its adjacency set induces a clique (i.e $N_G(v)$ forms a clique in G)

Definition 13. The class of k-trees is defined recursively as follows :
1. The complete graph on k vertices is a k tree.
2. Given a k-tree G with $|V(G)| = n \geq k$, we can construct a k-tree H with $|V(H)| = n + 1$, by adding a vertex and connecting it to exactly k vertices which form a clique in G.

We can test if a given graph $G = (V, E)$ is a k-tree by recursively removing simplicial vertices of degree k. If there are no more such vertices and what remains is the complete graph on k vertices, then the graph is a k-tree. The sequence of vertices removed defines a perfect elimination ordering for G, in which each vertex has exactly k higher numbered adjacent vertices. The existence of a perfect elimination scheme for G means that G is chordal (theorem 1). Also by theorem 2, G has treewidth k. So we have the following fact :

Corollary 6. *Every k-tree is chordal.*

The following theorem provides us with several alternative characterizations of k-trees :

Theorem 3. *Let $G = (V, E)$ be a graph. The following statements are equivalent :*
1. G is a k-tree.
2. G is connected, it contains a k-clique but no $(k + 2)$-clique, and every minimal separator of G is a k-clique.
3. G is connected, with $|E(G)| = k|V(G)| - \frac{k(k+1)}{2}$ and every minimal separator of G is a k-clique.
4. G has a k-clique, but not a $(k + 2)$-clique, and every minimal separator of G is a clique, and for all $v, u \in V(G)$ with $v \neq u$ and $\{v, u\} \notin E(G)$, there exist exactly k vertex disjoint paths from v to u.

Definition 14. A *partial k-tree* is spanning subgraph of a k-tree (i.e it contains the same vertex set and a subset of the edge set).

The following theorem gives us the relationship between partial k-trees and treewidth :

Theorem 4. *[108] Let $G = (V, E)$ be a graph. Then G is a partial k-tree if and only if $tw(G) \leq k$.*

Combining theorems 3,4 we get the following :

Corollary 7. *Let $G = (V, E)$ be a graph with $tw(G) \leq k$. Then*

$$|E(G)| \leq k|V(G)| - \frac{k(k+1)}{2}$$

Definition 15. Let $G = (V, E)$ be a graph, and let us say that two subsets of $V(G)$ *touch* if they have a vertex in common or $E(G)$ contains an edge between them. A set $\mathcal{B}$ of mutually touching connected vertex sets is a *bramble.* A subset of $V(G)$ is said to *cover* $\mathcal{B}$ if it meets every element of $\mathcal{B}$. The least number of vertices covering a bramble is the *order* of that bramble.

An easy example of a bramble is the set of crosses in a grid (see definition 2). Suppose we have a $(k \times k)$-grid. Then the *crosses* of this grid are the k^2 sets :

$$C_{i,j} = \{\{i, l\} \mid l = 1, ..., k\} \cup \{\{l, j\} \mid l = 1, ..., k\}$$

The set of these k^2 crosses form a bramble of order k. The following result, shown by Seymour and Thomas, gives us a useful relationship to help us classify graphs of specified treewidth :

Theorem 5. *[129] Let k be a non-negative integer, and $G = (V, E)$ be a graph. Then $tw(G) \geq k$ if and only if G contains a bramble of order strictly greater than k.*

A $k \times k$-grid is actually a bramble of order k, so by the backward direction of theorem 5 it has treewidth at least $k - 1$. In fact, the $(k \times k)$- grid has treewidth k. In example 2, we saw a tree decomposition for a (4×4)-grid with treewidth 4 (note that the largest bag in the tree decomposition has 5 elements). We also noted that the same construction can be applied to a $(k \times k)$-grid for arbitrary k. So large grids have large treewidth.

The next theorem was proved by Robertson and Seymour in 1986 :

Theorem 6. *[126] For every integer r there exists an integer k such that every graph of treewidth at least k has an $(r \times r)$-grid minor.*

We have already mentioned that given a tree decomposition $(\{X_i \mid i \in I\}, T = (I, F))$ of a graph G, and a node $i \in I$ with $deg_T(i) > 1$, the removal of vertex set V_i from G disconnects G into connected components (the number of which equals the degree of i in T). The smaller the width of the tree decomposition, the smaller bags that can disconnect G we have, and the bigger the number of the bags becomes. Thus, having a tree decomposition of small width in general implies that the graph has many separators of small size. So, if we have a graph of small treewidth we can find many separators of small size, and that property enables many dynamic programming algorithms on graphs of bounded treewidth.

We have already argued that as treewidth gets smaller, the graph having this treewidth tends to bear resemblance to a tree. Actually the following theorem establishes the fact that only trees have treewidth of 1 (see [106]) :

Lemma 17. *A connected graph has treewidth 1, if and only if it is a tree.*

Building a tree decomposition for a tree is easy. For each vertex v of the tree we construct a bag X_v, for each edge $\{v, u\}$ of the tree we construct a bag X_{vu}. Then for every edge $\{v, u\}$ we connect X_{vu} with X_u and X_v. This construction is a valid tree decomposition, because we clearly have covered every vertex and every edge is contained in some bag, and $T_v = T[\{i \in I \mid v \in X_i\}]$ is a connected subtree of T (all the edge bags containing v are connected to node X_v and there are not any other nodes in T_v).

Several important graph classes have been proven to have bounded treewidth. Trees have treewidth 1, series parallel networks and outerplanar graphs (graphs with an embedding in the plane such that all vertices can be placed on the outward face) have treewidth 2 and Halin graphs have treewidth 3. The class of planar graphs does not in general have graphs of small treewidth, as we have already noted that the $(k \times k)$-grid has treewidth k. Therefore, nor does the class of bipartite graphs have bounded treewith in general (grid graphs are bipartite). A trivial example of a graph with unbounded treewidth is the complete graph on n vertices, K_n, because from lemma 1 this graph has a tree decomposition consisting of a single bag, containing all vertices, thus achieving treewidth $n - 1$. In the case of planar graphs we have the following theorem :

Theorem 7. *[125] Let $k > 0$ be an integer. Every planar graph G with $tw(G) \geq 6k - 5$ has a $(k \times k)$-grid minor.*

That means that every planar graph of treewidth k has a $\Omega(k) \times \Omega(k)$-grid minor.

Finally we have the following complexity result, with respect to treewidth :

Theorem 8. *[3] Given a graph $G = (V, E)$ and an integer $k < |V|$, it is NP-Complete to decide if $tw(G) \leq k$.*

2 Toughness

2.1 Introduction

In this section we will describe the notion of graph toughness. Graph toughness is a graph parameter, introduced by Chvátal in 1973 ([57]), in an attempt to capture relevant structural properties related to cycle structures in graphs. We could view graph toughness as a measure of acyclicity or alternatively as an indicator parameter of the existence of cyclic structures in graphs. It is closely related to the connectivity of a graph, which refers to the minimum number of vertices that must be removed in order to disconnect the graph, taking also into consideration the number of components that arise from the removal of these vertices. Chvátal's interpretation of graph toughness was that it measures in a simple way how tightly various pieces of a graph hold together.

Research on toughness generally involves relating toughness conditions to the existence of cycle structures, the most important of which (mainly from a computational complexity point of view) is hamiltonian cycles. Most of the research has been based on a number of conjectures introduced in [57], with the most intriguing of which being the following :

Conjecture 1. *There exists a finite constant t_0 such that every t_0-tough graph is hamiltonian.*

This conjecture is still open, but it was falsified in 2000 for $t_0 = 2$. The importance of this conjecture for $t_0 = 2$ was significant, because if it held, it would imply a number of related results and conjectures. We now know that the t_0-tough conjecture holds for some graph classes, including planar graphs, claw-free graphs and chordal graphs. Toughness has also been researched in the context of computational complexity. It is now well known that it is NP-hard to compute the toughness of a graph [15].

2.2 Preliminaries

Let $G = (V, E)$ be a graph and let $c(G)$ denote the number of connected components of G.

Definition 16. Let $G = (V, E)$ be a graph. A *cutset* of G is any set $S \subseteq V(G)$ such that $c(G - S) > 1$.

Definition 17. A graph $G = (V, E)$ is t-tough if for every cutset S of G , we have $|S| \geq tc(G - S)$.

More intuitively, a graph $G = (V, E)$ is called t-tough if, for every integer $k > 1$, G cannot be split into k different connected components by the removal of fewer than tk vertices. It is also clear that if a graph is $(t + c)$-tough, where $c > 0$, then it is also t-tough.

Definition 18. A graph G is called *minimally t-tough* if $\tau(G) = t$ and there does not exists a proper spanning subgraph H of G with $\tau(H) = t$.

Definition 19. The *toughness* of a graph $G = (V, E)$, denoted $\tau(G)$, is :

$$\tau(G) = \max\{t \mid \text{G is } t \text{ tough}\}$$

Naturally, the complete graph on n vertices K_n does not have a value t for which it is t-tough, so we set $\tau(K_n) = \infty$ for all $n \geq 1$. If G is not complete, then

$$\tau(G) = \min\left\{ \frac{|S|}{c(G-S)} \;\middle|\; S \text{ is a cutset of } G \right\}$$

Definition 20. Let $G = (V, E)$ be a graph. A cutset $S \subseteq V(G)$ is called a *tough set* if $\tau(G) = \frac{|S|}{c(G-S)}$.

Definition 21. A *tough component* of a graph G is any component of $G - S$, where S is a tough set.

The terminology presented below will be needed in the rest of the chapter. Let $\alpha(G)$ denote the size of a maximum *independent set* of G and $circum(G)$ denote the *circumference* of G (i.e the length of the longest cycle of G). The girth of a graph G, denoted $girth(G)$, is the length of the shortest cycle in G. The *connectivity* of a non-complete graph G, denoted $\kappa(G)$, is the size of the minimum cutset of G. We say that a graph G is k-connected if $\kappa(G) \geq k$. The *genus* of a graph G, denoted $\gamma(G)$, is the minimal integer n such that the graph can be drawn without crossing itself on a sphere with n handles (i.e. an oriented surface of genus n). A *Hamilton cycle* in a graph G, is a simple cycle containing every vertex of G. A graph is *hamiltonian* if it contains a Hamilton cycle. A *Hamilton path* in a graph, is a simple path containing all vertices of the graph. A graph is called *traceable* if it contains a Hamilton path. A graph $G = (V, E)$ is called *pancyclic* if it contains cycles of every length between 3 and $|V(G)|$. A cycle C in a graph G is called a *dominating cycle* of G, if every edge of G has at least one endpoint in C (clearly $G - V(C)$ is an independent set). A graph in which every vertex has the same degree is called *regular*. A graph is *k-regular* if every vertex has degree k. A *k-factor* of a graph G is a k-regular spanning subgraph of G (i.e a subgraph of G defined on the same vertex set, in which all vertices have degree k). A cycle itself is a 2-factor, and a hamiltonian cycle in a graph G is a 2-factor of G. A graph G is called k-chordal if every chordless cycle of G has length at most k. Let $N_G(v) = \{u \mid \{v, u\} \in E(G)\}$ be the neighborhood of vertex v in graph G, $deg_G(v) = |N_G(v)|$ be the degree of vertex v in G, $\delta(G) = \min_{v \in V(G)} deg_G(v)$ be the minimum degree in G, and $\Delta(G) = \max_{v \in V(G)} deg_G(v)$ be the maximum degree in G. Let $\Delta^*(G)$ denote the minimum over all spanning trees of G of their maximum degree.

The distance between two vertices v, u of a connected graph G, denoted $dist_G(v, u)$, is the length of a shortest path connecting them.

Let $\mathcal{I}_G^k$ denote the set of independent sets of k vertices of graph G. For $k \leq \alpha(G)$, let

$$\sigma_k(G) = \min_{S \in \mathcal{I}_G^k} \sum_{v \in S} deg_G(v)$$

and let

$$NC_k(G) = \min_{S \in \mathcal{I}_G^k} |\cup_{v \in S} N_G(v)|$$

For $k > \alpha(G)$, we set $\sigma_k(G) = k(|V(G)| - \alpha(G))$ and $NC_k(G) = |V(G)| - \alpha(G)$. If graph $G = (V, E)$ has a noncomplete component, let

$$NC2(G) = \min_{v,u \in V(G)} \{|N_G(v) \cup N_G(u)| \ : \ dist_G(v, u) = 2\}$$

otherwise set $NC2(G) = |V(G)| - 1$.

Before we proceed to the next subsections, it would be useful to view a few examples on toughness.

Example 3. A path P_n on $n \geq 3$ vertices has connectivity $\kappa(P_n) = 1$, since removing any vertex of degree 2 yields a disconnected graph with 2 components. Thus, the toughness of P_n for $n \geq 3$ is $\tau(P_n) = \frac{1}{2}$.

Example 4. The circle C_n on $n \geq 4$ vertices has connectivity $\kappa(C_n) = 2$, since we have to remove two nonadjacent vertices to disconnect the circle into two components. Removing those vertices, the ratio $\frac{|S|}{c(C_n - S)}$ becomes 1. If we further remove a vertex of degree 2, it breaks a component in half giving a ratio $\frac{|S|}{c(C_n - S)}$ of $\frac{3}{3} = 1$. Removing a vertex of degree 1 does not raise the number of components, so the minimum ratio $\frac{|S|}{c(G-S)}$ is 1, that is $\tau(C_n) = 1$ (where each tough set S is constructed by repeatedly removing vertices of degree 2). This means that every cycle is 1-tough,conluding that hamiltonian graphs are 1-tough.

Example 5. Let $T = (V, E)$ be a tree on $n \geq 3$ vertices. A tree has connectivity $\kappa(T) = 1$ because like in the case of a path, we have to remove only one vertex of degree > 1 to disconnect it. It is easy to verify that the toughness of a tree is $\tau(T) = \max_{v \in V(T)} \frac{1}{deg_T(v)} = \frac{1}{\Delta(T)}$. The star graph $K_{1,n}$ is a tree, so it is $\frac{1}{n}$-tough.

As a last example consider the case of a complete bipartite graph :

Example 6. Let $m \leq n$, $n \geq 2$ and $K_{m,n}$ be a complete bipartite graph. Then, removing the appropriate m vertices (that induce an independent set), yields a disconnected graph with n components. This removal gives us the best possible ratio $\frac{|S|}{c(K_{m,n}-S)}$ of $\frac{m}{n}$. Consequently, $\tau(K_{m,n}) = \frac{m}{n}$.

2.3 Bounds on toughness

In this section we present some upper and lower bounds on toughness.

Theorem 9. *[57] Let G be a graph. Then*
1. $\tau(G) = 0$ if and only if G is not connected.
2. $\tau(G) = \infty$ if and only if G is complete.

Theorem 10. *[57, 87] For any graph G*

$$\frac{\kappa(G)}{\Delta(G)} \leq \tau(G) \leq \frac{\kappa(G)}{2}$$

It is easy to see from example 5, that a tree T has $\kappa(T) = 1$, so $\tau(T) \geq \frac{1}{\Delta(T)}$ by theorem 10. In example 5 we established that $\tau(T) \leq \frac{1}{\Delta(T)}$, so we verify that a tree T has toughness $\frac{1}{\Delta(T)}$. The equality with the upper bound can be achieved in the case of noncomplete $K_{1,3}$-free graphs (claw-free graphs).

Proposition 2. *[114] If G is a noncomplete $K_{1,3}$-free graph, then $\tau(G) = \frac{\kappa(G)}{2}$*

The next conjecture is relevant :

Conjecture 2. *[86] Let G be an r-regular graph. Then G is $\frac{r}{2}$-tough if and only if G is r-connected and $K_{1,3}$-free.*

Clearly, the 'if' direction of conjecture 2 is implied by proposition 2. The following theorem uses the graph parameters $\alpha(G)$ and $\kappa(G)$ to narrow down the possible values for $\tau(G)$. Notice first that $\max_{S\subset V(G)} c(G-S) \geq \alpha(G)$, because we can remove all vertices not in the maximum independent set, and obtain as components the vertices of the maximum independent set.

Theorem 11. *[57] For any graph G on n vertices,*

$$\frac{\kappa(G)}{\alpha(G)} \leq \tau(G) \leq \frac{n-\alpha(G)}{\alpha(G)}$$

Let $\chi(G)$ denote the chromatic number of a graph G. Then the following bounds also hold :

Lemma 18. *[5] For any graph G on n vertices,*

$$\tau(G) + 1 \leq \frac{n}{\alpha(G)} \leq \chi(G)$$

It is easy to verify that $\tau(G)+1 \leq \frac{n}{\alpha(G)}$ if we consider that in any graph if we remove all those vertices not in a maximum independent set ($n-\alpha(G)$ of them), then we obtain a graph with $\alpha(G)$ components. Thus $\tau(G) \leq \frac{n-\alpha(G)}{\alpha(G)} = \frac{n}{\alpha(G)} - 1$.

Theorem 12. *[81] For any graph G, $\Delta^*(G) - 3 \leq \frac{1}{\tau(G)} \leq \Delta^*(G)$.*

Proposition 3. *[122] If G is any noncomplete graph, $\tau(G-v) \geq \tau(G) - \frac{1}{2}$.*

Proposition 4. *[87] If G is a nonempty graph and m is the largest integer such that $K_{1,m}$ is an induced subgraph of G, then $\tau(G) \geq \frac{\kappa(G)}{m}$.*

Clearly the lower bound $\tau(G) \geq \frac{\kappa(G)}{\Delta(G)}$ is implied by proposition 4. A tree has connectivity 1, and any nontrivial tree contains $K_{1,m}$ for $m \geq 2$ as an induced subgraph. We have already seen in example 5 that a tree has $\tau(G) \leq \frac{1}{\Delta(G)}$. Also, a k-regular and k-conneted graph has a $K_{1,k}$ induced subgraph as well. Thus, we have the following :

Corollary 8.
1. If G is a nontrivial tree then $\tau(G) = \frac{1}{\Delta(G)}$.
2. If G is k-connected and k-regular then $\tau(G) \geq 1$.

The next result establishes lower bounds on the toughness of a graph in terms of its connectivity and genus.

Theorem 13. *[85] If G is a connected graph of genus $\gamma(G) = \gamma$ and connectivity $\kappa(G) = \kappa$ then*

1. $\tau(G) > \frac{\kappa}{2} - 1$, if $\gamma = 0$, and

2. $\tau(G) \geq \frac{\kappa(\kappa-2)}{2(\kappa-2+2\gamma)}$, if $\gamma \geq 1$.

The lower bound given in theorem 13.2 cannot be achieved if $\gamma(G) = 1, \kappa(G) = 3$ and $girth(G) = 6$ as is shown next :

Lemma 19. *[85] If G is a graph with $\gamma(G) = 1$, $\kappa(G) = 3$ and $girth(G) = 6$, then $\tau(G) \geq 1$.*

2.4 Toughness of subgraphs and related graphs

In the case of spanning subgraphs, we have the next proposition :

Proposition 5. *[57] Let G be a graph and H be a spanning subgraph of G, then $\tau(H) \leq \tau(G)$.*

The next result relates the toughness of a component of $G-S$, where S is a tough set.

Proposition 6. *[124] Let G be a noncomplete graph and C be a tough component of G. Then $\tau(C) \geq \frac{\lceil \tau(G) \rceil}{2}$.*

Theorem 14. *[48] Let G be a graph different from $K_1, K_2, ..., K_{\lfloor \frac{4i+7}{3} \rfloor}$, with $\tau(G) > i$ for some positive integer i. Then there exists a spanning subgraph H of G with $\frac{2i+1}{3} \leq \tau(H) \leq i$.*

Setting $i = 1$ in theorem 14, we get the following result as a special case :

Corollary 9. *[48] Let G be a graph, different from K_1, K_2, K_3, with $\tau(G) > 1$. Then there exists a spanning subgraph H of G with $\tau(H) = 1$.*

We will need the following definition :

Definition 22. Let G, H be two disjoint graphs. We denote the *join* of G, H by :

$$G * H = (V(G) \cup V(H), E(G) \cup E(H) \cup \{\{v, u\} \mid v \in V(G) \ \wedge \ u \in V(H)\})$$

$G * H$ is the graph obtained from $G \cup H$ by joining every vertex of G to every vertex of H.

The following two results relate the toughness of a graph G with the toughness of the join of a complete graph and a component of $G - S$, where S is a tough set of G.

Lemma 20. *[48] Let k be a positive integer, and let G be a graph with $\tau(G) \geq k$, S an arbitrary tough set of G and H_i an arbitrary component of $G - S$. Then $\tau(K_k * H_i) \geq k$.*

This result can be further improved if we consider maximum tough sets instead of arbitrary.

Lemma 21. *[48] Let k be a positive integer, and let G be a graph G with $\tau(G) \geq k$, S a maximum tough set of G, and H_i an arbitrary component of $G - S$. Then $\tau(K_{k-1} * H_i) \geq k$.*

For the rest of the subsection we need the following definitions.

Definition 23. The *square* graph G^2 of a graph G is the graph obtained from G by joining all vertices at distance 2 in G.

Definition 24. Let G and H be graphs, and $s \geq 0$ an integer. We call G an *s-subdivision of* H, if G can be obtained from H by replacing every edge $\{u, v\}$ of H by a path between u and v with at least s internal vertices. We call G an *s-subdivision*, if it is an s-subdivision of some graph H.

Lemma 22. *[48] Let H be a 3-subdivision of a 2-connected graph and let $G = H^2$. Then $\tau(G) = 2$.*

The next theorem states that the square of 4-subdivision of a 2-connected 3-regular graph is minimally 2-tough.

Theorem 15. *[48] Let H be a 4-subdivision of a 2-connected 3-regular graph and let $G = H^2$. Then G is minimally 2-tough.*

The following theorem states that if two neighbors u and v with $d(v) + d(u) \geq 6$ exist in a 2-connected graph H, then H^2 is minimally 2-tough.

Theorem 16. *[48] Let H be a 2-connected graph with $deg_H(v)+deg_H(u) \geq 6$ for some $u, v \in V(H)$ with $\{v, u\} \in E(H)$, and let $G = H^2$. Then G is minimally 2-tough.*

2.5 Sufficient conditions for t-toughness

In this section we present some sufficient conditions for a graph to be t-tough. Chvátal and Erdös in 1971 proved the following :

Theorem 17. *[59] Let G be a graph with at least three vertices. If $\alpha(G) \leq \kappa(G)$, then G is hamiltonian.*

Since every hamiltonian graph is also 1-tough (every cycle is 1-tough) we have the following :

Corollary 10. *Let G be a graph with at least three vertices. If $\alpha(G) \leq \kappa(G)$, then G is 1-tough.*

This result easily extends to t-tougness :

Lemma 23. *[48] Let G be a graph. If $t\alpha(G) \leq \kappa(G)$, then G is t-tough.*

The condition of lemma 23 cannot be relaxed, as can be easily seen from the graph $K_p * \bar{K}_q$ with $p > q$.

The reason that $G \equiv K_p * \bar{K}_q$ with $p < tq$ is not t-tough, is that $\alpha(G) = q$, $\kappa(G) = p$ and $\tau(G) = \frac{p}{q} = \frac{\kappa(G)}{\alpha(G)} < t$.

We can apply the same line of thought, to extend degree conditions for hamiltonicity, and hence 1-toughness, to conditions for t-toughness. We have the following result from Ore :

Theorem 18. *[119] Let $G = (V, E)$ be a graph with at least 3 vertices, such that for every pair v, u of nonadjacent vertices of G, $deg_G(v) + deg_G(u) \geq n$. Then G is hamiltonian.*

Consequently we have the following :

Corollary 11. *Let $G = (V, E)$ be a graph with at least 3 vertices, such that for every pair v, u of nonadjacent vertices of G, $deg_G(v) + deg_G(u) \geq n$. Then G is 1-tough.*

The next theorem is the genaralization of theorem 18 :

Theorem 19. *[48] Let $G = (V, E)$ be a kt-connected graph, with k, t integers and $kt \geq 1$, on n vertices with $\sigma_{k+1} \geq (k+1)\frac{nt}{t+1}$. Then G is t-tough.*

The following result makes use of the minimum degree of a graph to show t-toughness :

Theorem 20. *[18] Let G be a graph on n vertices with $\delta(G) \geq \frac{nt}{t+1}$. Then G is t-tough.*

For $t = 1$, theorem 20 is a consequence of a well known theorem from Dirac :

Theorem 21. *[65] Let G be a graph on n vertices with $\delta(G) \geq \frac{n}{2}$. Then G is hamiltonian.*

There is a generalization of thorem 18 for t-toughness :

Theorem 22. *[48] Let G be a graph on n vertices, such that for every pair v, u of nonadjacent vertices of G, $deg_G(v) + deg_G(u) \geq \frac{2nt}{t+1}$, with $1 \leq t \leq n-1$. Then G is t-tough.*

2.6 Toughness of special graph classes

Proposition 7. *[122]*

$$1.\ \tau(P_n) = \begin{cases} \infty \text{ , } \textit{if } n = 1, 2 \\ \frac{1}{2} \text{ , } \textit{if } n \geq 3 \end{cases}$$

$$2.\ \tau(C_n) = \begin{cases} \infty \text{ , } \textit{if } n = 3 \\ 1 \text{ , } \textit{if } n \geq 4 \end{cases}$$

Notice that for $n = 1, 2$, P_n, and for $n = 3$, C_n, are isomorphic to K_n . In previous subsection we gave as an example of the toughness of a graph, the complete bipartite graph $K_{m,n}$ with $m \leq n$ (example 6). The following proposition, gives us the toughness of that graph :

Proposition 8. *[57] If $m \leq n$, then $\tau(K_{m,n}) = \frac{m}{n}$.*

For the cartesian product of two complete graphs, we have the next result from Chvátal.

Theorem 23. *[57] For all $m, n \geq 2$, $\tau(K_m \times K_n) = \frac{m+n}{2} - 1$.*

The following theorem, due to Chvátal, states that the square of a k-connected graph is k-tough.

Theorem 24. *[57] For any k-connected graph G,* $\tau(G^2) \geq \kappa(G)$.

The toughness of the complete multipartite graph is given in the following result by Goddard and Swart.

Proposition 9. *[86] Let* $m \geq 2$, *for* $1 \leq i \leq m$ *let* a_i *be integers with* $\max_{1\leq i\leq m} a_i \geq 2$ *and* $n = \sum_{i=1}^{m} a_i$. *Then*

$$\tau(K_{a_1,\ldots,a_m}) = \min_{1\leq i\leq m} \left\{ \frac{n-a_i}{a_i} \right\}$$

The reason that we require that there exists an $i \leq m$ where a_i is greater or equal than 2, is that if all a_i's where equal to one and we viewed K_n as $K_{1,1,\ldots,1}$ (with n ones), proposition 9 would then lead us to $\tau(K_n) = n-1$, which contradicts the fact that $\tau(K_n) = \infty$.

For the rest of the subsection we provide some results on the toughness of the cartesian product of specific graphs. Note that, if G, H are connected non-trivial graphs, then the cartesian product of G and H, $G \times H$, contains $K_{1,3}$ as an induced subgraph unless both G and H are complete.

Theorem 25. *[86] The following graphs have toughness 1*
1. $P_m \times P_n$ *for* mn *even and* $m, n \geq 2$.
2. $P_m \times C_n$ *for* n *even, and*
3. $C_m \times C_n$ *for* m *and* n *even.*

Theorem 26. *[86] For* m *and* n *odd and* $m, n \geq 3$,

$$\tau(P_m \times P_n) = \frac{mn-1}{mn+1}$$

An $m \times n$-grid is the cartesian product of two paths, P_m and P_n. So we have the following :

Corollary 12. *Let G be the* $m\times n$*-grid graph with* $m, n \geq 2$. *Then* $\tau(G) \leq 1$.

This is easy to verify, because removing the two neighbors of a vertex with degree 2 in a grid, disconnects the graph into two components.

Theorem 27. *[86] Let* $m, n \geq 3$. *Then*

1. $\tau(P_m \times K_n) = \frac{n+1}{3}$,

2. $\tau(C_m \times K_n) = \begin{cases} \frac{n}{2} & \text{, if } m \text{ is even} \\ \frac{n}{2} + \frac{1}{m-1} & \text{, if } m \text{ is odd} \end{cases}$

and finally for cartesian products we have the following bounds :

Theorem 28. *[86] Let n be odd, $n \geq 5$. Then*

$$1.\tau(P_m \times C_n) \leq \begin{cases} \frac{n}{n-1} & \text{, if } m \text{ is even,} \\ \frac{mn-1}{mn-m}, & \text{if } m \text{ is odd} \end{cases}$$

$$2.\tau(C_m \times C_n) \leq \tfrac{n}{n-1} \text{ for } m \text{ even.}$$

We continue with a bound on the toughness of cubic graphs. *Cubic graphs* are graphs in which all vertices have degree 3 (i.e cubic graphs are 3-regular graphs). In section 2.10, we will see that it is NP-hard to determine if a cubic graph is 1-tough. However, we can obtain upper bounds on the toughness of cubic graphs in terms of its independence number.

Theorem 29. *[84] Let G be a noncomplete cubic graph on n vertices. Then*

$$\tau(G) \leq \min\left\{\frac{2n - 3\alpha(G)}{n - \alpha(G)}, \frac{2\alpha(G)}{4\alpha(G) - n}\right\}$$

A special class of cubic graphs is that of cycle permutation graphs. In particular, a *cycle permutation graph* is a cubic graph on $2m$ vertices, obtained by taking two vertex disjoint cycles on m vertices and adding a matching between the vertices of the two cycles (see definition 40 in section 2.11). In [121], it was conjectured that the toughness of such graphs is at most $\frac{4}{3}$. Goddard then obtained a bound very close to $\frac{4}{3}$.

Theorem 30. *[84] Let G be a cycle permutation graph on $2m$ vertices. Then*

$$\tau(G) \begin{cases} \leq \frac{4}{3}, & \text{if } m \equiv 0, 1 \bmod 4, \\ < \frac{4}{3}, & \text{if } m \equiv 2 \bmod 4, \\ \leq \frac{4}{3} + \frac{4}{9m-3}, & \text{if } m \equiv 3 \bmod 4, \end{cases}$$

One special graph class that has received much attention is that of triangle-free graphs, as they have a number of interesting properties. A number of results on triangle free graphs have been made in the direction of finding the best possible minimum degree conditions for the extistence of 2-factors and Hamilton cycles. We begin with a result, concerning the existence of a 2-factor.

Theorem 31. *[27] Let G be a 1-tough triangle-free graph on $n \geq 3$ vertices. If $\delta(G) \geq \frac{n+2}{4}$, then G has a 2-factor.*

The bound on the minimum degree in theorem 31 is the best possible. Let $C(G)$ denote the set of cycle lengths of a graph G. Brandt in [42] has showed the following :

Theorem 32. *[42] Let $G \neq C_5$ be a triangle-free, nonbipartite graph on n vertices. If $\delta(G) > \frac{n}{3}$, then $C(G) = \{4, 5, ..., r\}$, where $r = \min\{n, 2(n - \alpha(G))\}$.*

In the case of balanced bipartite graphs we have the following result by Moon and Moser in [115]:

Theorem 33. *[115] Let G be a balanced bipartite graph on n vertices. If $\delta(G) > \frac{n}{4}$, then G is hamiltonian.*

By lemma 18, we have that $\frac{n}{\alpha(G)} \geq \tau(G)+1$. Therefore, if $\tau(G) \geq 1$, then $\frac{n}{\alpha(G)} \geq 2$ or equivalently $\alpha(G) \leq \frac{n}{2}$. In [5] Bauer et al. noticed that since $\alpha(G) \leq \frac{n}{2}$ in any 1-tough graph, and 1-tough bipartite graphs are balanced, combining theorems 32 and 33 we obtain the following result.

Theorem 34. *[5] Let G be a 1-tough triangle-free graph on $n \geq 3$ vertices. If $\delta(G) > \frac{n}{3}$, then G is hamiltonian.*

Every hamiltonian graph contains a 2-factor and since the bound on the minimum degree in 31 is the best possible, to guarantee that a 1-tough triangle free graph G is hamiltonian it must hold that $\delta(G) \geq \frac{n+2}{4}$. So, combining theorems 31, 34, the best possible minimum degree guaranteeing that a 1-tough triangle free graph is hamiltonian lies somewhere between $\frac{n+2}{4}$ and $\frac{n}{3}$.

Chvátal made the following conjecture in [57] :

Conjecture 3. *There exists a positive constant t_1 such that every t_1-tough graph is pancyclic.*

Later, Jackson and Katerinis in [96] conjectured that :

Conjecture 4. *There exists a positive constant t_2 such that every t_2-tough graph contains a triangle.*

Notice that conjecture 4 is implied by conjecture 3, since a pancyclic graph contains at least one cycle of length 3. One year later, Bauer et al. in [26] falsified both conjectures :

Theorem 35. *[26] There exist arbitrarily tough, triangle-free graphs.*

The proof of theorem 35 involved the construction of a sequence of "layered graphs". This sequence is built, starting with a triangle-free graph. As the sequence is constructed, the toughness of the graphs approaches infinity without losing the triangle-free property. Later, Alon established the following result :

Theorem 36. *[2] For every t and g there exists a t-tough graph with $girth(G) \geq g$.*

Brandt, Faundree and Goddard in [44] also showed that conjecture 3 is false.

Definition 25. A graph is called *weakly pancyclic* if it contains cycles of every length between $girth(G)$ and $circum(G)$.

Clearly, if a graph is not weakly pancyclic, then it is not pancyclic. They showed that there is no sufficiently large value of toughness that will ensure that a graph is weakly pancyclic.

In, [5], Bauer et al. note that if some vertex in a t-tough graph G on n vertices has degree greater than $\frac{n}{t+1}$, then G must contain a triangle. This is easy to verify if we consider a vertex v in G with $deg_G(v) > \frac{n}{t+1}$. Then $t > \frac{n-deg_G(v)}{deg_G(v)}$, thus if we delete all vertices except from those in $N_G(v)$, the neighborhood of v does not constitute an independent set, since $\tau(G) \geq t > \frac{n-deg_G(v)}{deg_G(v)}$. Hence, there must be an edge between neighbors of v and therefore there exists a triangle in G. It is reasonable to ask if there exists an $\frac{n}{t+1}$ regular t-tough triangle-free graph for arbitrary large t. In [26], the sequence of layered graphs appeared to have this property, so it was concjectured that such graphs exist. In [26] it was proven for $1 \leq t \leq 3$. For arbitrary t it was proven independently by Brandt in [43] and Brouwer in [50]. Brandt's result was the following :

Theorem 37. *[43] For every $\epsilon > 0$ there exists a real number t_0 such that for every $t > t_0$ there is a triangle-free graph G on n vertices with toughness $\tau(G) = \frac{n}{\delta(G)} - 1$ and $t - \epsilon \leq \tau(G) \leq t + \epsilon$.*

In theorem 35 we stated that there exist arbitrarily tough triangle-free graphs. It has also been shown [62, 105, 116, 138] that there exist triangle-free graphs with arbitrary large chromatic number. Erdös in [73] showed that there exists graphs with arbitrarily high girth and arbitrarily high chromatic number, having also arbitrarily high $\frac{n}{\alpha(G)}$ ratio. We remind the reader, the bound on $\tau(G)$ relative to $\alpha(G)$ and $\chi(G)$ in lemma 18 :

$$\tau(G) + 1 \leq \frac{n}{\alpha(G)} \leq \chi(G)$$

Therefore, theorem 35 generalizes those results, since large toughness implies high $\frac{n}{\alpha(G)}$ ratio and in turn that implies high chromatic number. Brandt in [43], showed that there exists an appropriate sequence of layered graphs, such that the inequalties in the relation above can also be satisfied by equality.

Theorem 38. *[43] For every positive integer k there exists a triangle-free graph G with $\chi(G) = k = \frac{n}{\alpha(G)} = \tau(G) + 1$.*

Brandt made also the following conjecture :

Conjecture 5. *[43] Let G be a t-tough graph on n vertices, with $\delta(G) > \frac{n}{t+1}$. Then G is pancyclic.*

We have already mentioned that a t-tough graph with $\delta(G) > \frac{n}{t+1}$ contains a triangle. We will also see in section 2.7 that every t-tough graph on at least three vertices with $\delta(G) > \frac{n}{t+1}$ is hamiltonian (theorem 68). Thus a t-tough graph of order n contains cycles of length 3 and n. The following result shows that conjecture 5 is true for a t-tough graph if $t < 3 - \frac{4000}{n}$.

Theorem 39. *[44] Let G be a graph on n vertices with $\delta(G) \geq \frac{n}{4} + 250$ that contains a triangle and a Hamilton cycle. Then G is pancyclic.*

We now turn to some resutls concerning minimum toughness conditions that ensure hamiltonicity in chordal graphs. In section 2.8 we will see that an infinite class of $(\frac{7}{4} - \epsilon)$-tough nontraceable chordal graphs can be constructed (see theorem 104). That means that not even 1-tough chordal graphs need to be hamiltonian. In particular it is proven in [39] that not even 1-tough planar chordal graphs need to be hamiltonian. However, the following was proven :

Theorem 40. *Let G be a chordal, planar graph with $\tau(G) > 1$. Then G is hamiltonian.*

Gerlach in [83] then showed that we can replace the chordality assumption in theorem 40 with the assumption that separating cycles of length greater than 3 have chords. The fact that the assumption $\tau(G) > 1$ can't be lowered, was shown using the notion of shortness exponent.

Definition 26. Let Σ be a class of graphs. The *shortness exponent* of class Σ is given by the following expression :

$$\sigma(\Sigma) = \liminf_{H_n \in \Sigma} \frac{\log circum(H_n)}{\log|V(H_n)|}$$

where H_n is a sequnce of graphs from Σ with $|V(H_n)| \to \infty$ as $n \to \infty$.

In [39], it was shown that when Σ is the class of 1-tough chordal planar graphs we have $\sigma(\Sigma) \leq \frac{\log 8}{\log 9}$. Thus, there exists a sequence $G_1, G_2, ...$ of 1-tough chordal planar graphs with $\frac{circum(G_i)}{|V(G_i)|} \to 0$ as $i \to \infty$. From proposition 2, we know that if a claw-free graph G is 2-connected then it is 1-tough ($\tau(G) = \frac{\kappa(G)}{2}$). Combining that with a result of Balakrishnan and Paulraja in [4] stating that a 2-connected claw-free chordal graph is hamiltonian, we obtain the following corollary :

Corollary 13. *Every 1-tough $K_{1,3}$-free chordal graph is hamiltonian.*

Actually, in the case of claw-free graphs we can drop the chordality assumption and obtain a $\frac{7}{2}$-toughness upper bound for hamiltonicity. First, we have the following result from Ryjacek :

Theorem 41. *[127] Every 7-connected $K_{1,3}$-free graph is hamiltonian.*

Using the fact that claw-free graphs have toughness equal to half their connectivity (proposition 2), we get the following corollary :

Corollary 14. *Every $\frac{7}{2}$-tough $K_{1,3}$-free graph is hamiltonian.*

That means that the t_0-conjecture is true for claw-free graphs.

We have already mentioned that $\frac{3}{2}$-tough graphs need not be hamiltonian (since from theorem theorem 104 there are $(\frac{7}{4} - \epsilon)$- tough nontraceable chordal graphs). Consider however the case of spit graphs :

Definition 27. A graph G is called *split graph* if $V(G)$ can be partitioned into an independent set and a clique.

Clearly, split graphs is a subclass of chordal graphs (the neighborhood of a vertex in the independent set must induce a clique, so if we place in an ordering, first the vertices of the independent set and then the rest of the vertices then we obtain a perfect elimination ordering. This means that the graph is chordal by theorem 1). The following two theorems are resutls on split graphs :

Theorem 42. *[107] Every $\frac{3}{2}$-tough split graph is hamiltonian.*

Theorem 43. *[107] There is a sequence $\{G_n\}_{n=1}^{\infty}$ of split graphs with no 2-factor and $\tau(G_n) \to \frac{3}{2}$.*

Altough $\frac{3}{2}$-tough chordal graphs are not necessarily hamiltonian, it was proven in [17] that they necessarily contain a 2-factor.

Theorem 44. *[17] Let G be a $\frac{3}{2}$-tough 5-chordal graph. Then G has a 2-factor.*

Notice that if a 5-chordal graph G has a 2-factor, then every chordal graph having G as a spanning subgraph will have a 2 factor, and that every $\frac{3}{2}$-tough chordal graph has a 5-chordal graph as a spanning subgraph. So, we have the following corollary of theorem 44.

Corollary 15. *Let G be a $\frac{3}{2}$-tough chordal graph. Then G has a 2-factor.*

There are examples in [21] that show that there exists 6-chordal graphs without a 2-factor. In section 2.8 we shall see a result from Cvátal (theorem 101) stating that for every $\epsilon > 0$ there exists a $(\frac{3}{2} - \epsilon)$-tough graph without a 2-factor. The examples used to prove this statement were all chordal.

Theorem 45. *[57] For every $\epsilon > 0$, there exist $(\frac{3}{2} - \epsilon)$-tough chordal graphs without a 2-factor.*

Hence, corollary 15 is indeed best possible with respect to the toughness of the graph. Also, theorem 44 is best possible with respect to the maximum value k for which a $\frac{3}{2}$-tough k-chordal graph can have a 2-factor. It is reasonable to ask if there exists a $t_1 > 0$ such that every t_1-tough chordal graph is hamiltonian. Chen et al. in [54] answered this questing in the affirmative.

Theorem 46. *[54] Every 18-tough chordal graph is hamiltonian.*

As we shall see in section 2.8 the 2-tough conjecture has been falsified. In light of this result, it is natural to ask if the same holds for chordal graphs as well. In particular, are all 2-tough chordal graphs hamiltonian? The same question can be asked with regard to triangle-free graphs. The following conjecture was made in [27] :

Conjecture 6. *For all $\epsilon > 0$, there exists a $(2-\epsilon)$-tough triangle-free graph that does not even contain a 2-factor.*

Obviously, if the conjecture were true, then there would exist $(2-\epsilon)$-tough triangle-free graphs without a Hamilton path. Ferland in [78] has found an infinite class of nonhamiltonian triangle-free graphs whose toughness is at least $\frac{5}{4}$.

Theorem 47. *[78] There exist $\frac{5}{4}$-tough nonhamiltonian triangle-free graphs.*

Altough the question of whether all 2-tough chordal graphs are hamiltonian is still open, it has been settled for a subclass of chordal graphs, namely the 2-trees (notice that every k-tree has a perfect elimination ordering by its definition, hence it is chordal).

Theorem 48. *[46] Let $G \neq K_2$ be a k-tree. Then G is hamiltonian if and only if G contains a 1-tough spanning 2-tree.*

This result is best possible if we take into consideration that 1-toughness is a necessary condition for hamiltonicity. Theorem 48 can be generalized to a result on k-trees, for $k \geq 2$.

Theorem 49. *[46] If $G \neq K_2$ is a $(\frac{k+1}{3})$-tough k-tree, with $k \geq 2$, then G is hamiltonian.*

In [46], Broersma et al. also present infinite classes of nonhamiltonian 1-tough k-trees for each $k \geq 3$.

In [136], Win considered the relationship between the toughness of a graph and the existence of spanning trees of connected graphs, with maximum degree at most k.

Theorem 50. *Let G be a connected graph. Suppose $k \geq 2$, and that for any subset $S \subseteq V(G)$, $c(G-S) \leq 2 + (k-2)|S|$. Then G has a spanning tree with maximum degree at most k.*

Consider now the case when $k = 2$. Then, for every $S \subseteq V(G)$, $c(G - S) \leq 2$. We remind the reader that $\max_{S \subseteq V(G)} c(G - S) \geq \alpha(G)$, since we can pick the cutset S to be the vertices not in the maximum independent set and obtain as components the vertices in the maximum independent set. Hence, for $k = 2$, theorem 50 simply says that a connected graph G with $\alpha(G) \leq 2$ has a Hamilton path (since a spanning tree with maximum degree at most k is just a Hamilton path). For $k \geq 3$, we have the following corollary of theorem 50 :

Corollary 16. *Let $k \geq 3$. If $\tau(G) \geq \frac{1}{k-2}$, then G has a spanning tree of maximum degree at most k.*

2.7 Toughness and circumference

In this section we present some results relating toughness with circumference. We have already seen in theorem 21 by Dirac, that if $\delta(G) \geq \frac{n}{2}$, then $circum(G) = n$ (the graph is hamiltonian). The next result is also due to Dirac.

Theorem 51. *[65] Let G be a 2-connected graph on n vertices. Then $circum(G) \geq \min\{n, 2\delta(G)\}$.*

In theorem 18 we saw a generalization of theorem 51 from Ore in 1960. More specifically the theorem states that if for every pair of nonadjacent vertices (i.e that belong in independent sets of size 2) their degree sum is at least n, then the graph is hamiltonian. Puting it in terms of circumference and minimum degree sum of independent sets, we can rephrase theorem 18 as :

Corollary 17. *Let G be a graph on $n \geq 3$ vertices with $\sigma_2(G) \geq n$. Then $circum(G) = n$ (i.e G is hamiltonian).*

Theorem 18 was further improved independently by Bermond, Bondy and Linial (clearly theorem 18 implies 2-connectivity) :

Theorem 52. *[40, 30, 111] Let G be a 2-connected graph on $n \geq 3$ vertices. Then $circum(G) \geq \min\{n, \sigma_2(G)\}$.*

The next result shows that if we require that the graph in the precondition of corollary 17 be 1-tough, then the lower bound on $\sigma_2(G) \geq n$ in Ore's theorem can be lowered by 4, as was shown by Jung in 1978.

Theorem 53. *[98] Let G be a 1-tough graph on $n \geq 11$ vertices with $\sigma_2(G) \geq n - 4$. Then $circum(G) = n$ (i.e G is hamiltonian).*

If $\delta(G) \geq \frac{n-4}{2}$, then $deg_G(v) + deg_G(u) \geq n - 4$ for every $v, u \in V(G)$ and consequently $\sigma_2(G) \geq n-4$. Thus, Jung's theorem implies the following weaker theorem with a minimum degree condition :

Theorem 54. *Let G be a graph on $n \geq 11$ vertices with $\delta(G) \geq \frac{n-4}{2}$. Then $circum(G) = n$.*

It is reasonable to ask, how much the lower bound $cirucm(G) = n$ can be improved. Ainouche and Christofides took the first step towards this direction.

Theorem 55. *[1] Let G be a 1-tough graph on $n \geq 3$ vertices. Then $circum(G) \geq \min\{n, \sigma_2(G) + 1\}$.*

Ainouche and Christofides also conjectured that the lower bound can be further improved, replacing $\sigma_2(G) + 1$ with $\sigma_2(G) + 2$ in theorem 55. Their conjecture was then proved correct by Bauer and Schmeichel.

Theorem 56. *[19] Let G be 1-tough graph on $n \geq 3$ vertices. Then $circum(G) \geq \min\{n, \sigma_2(G) + 2\}$.*

The lower bound $\sigma_2(G) \geq n - 4$ in Jung's theorem can be slightly improved if $\tau(G) > 1$, as is shown in the following result.

Theorem 57. *[13] Let G be a graph on $n \geq 30$ vertices with $\tau(G) > 1$. If $\sigma_2(G) \geq n - 7$, then $circum(G) = n$.*

Theorem 57 gives us the best possible guarantee for hamiltonicity in graphs, with respect to $\sigma_2(G)$. We shall see later that there exist 1-tough graphs, with $\sigma_3 \geq (3n - 24)/2$ that are not hamiltonian.

Then next theorem, due to Nash-Williams, is a useful intermediate result, concering the existence of a longest dominating cycle.

Theorem 58. *[117] Let G be a 2-connected graph on n vertices with $\delta(G) \geq \frac{n+2}{3}$. Then every longest cycle in G is a dominating cycle.*

In the same paper, Nash-Williams gave the following characterization for a graph to be hamiltonian :

Theorem 59. *[117] Let G be a 2-connected graph with $\delta(G) \geq \max\{\frac{n+2}{3}, \alpha(G)\}$. Then $circum(G) = n$.*

A generalization of theorem 58 was made by Bondy in 1980 :

Theorem 60. *[41] Let G be a 2-connected graph on n vertices with $\sigma_3(G) \geq n + 2$. Then every longest cycle in G is a dominating cycle.*

In the same paper, Bondy also gave a generalization of theorem 59.

Theorem 61. *[41] Let G be a 2-connected graph on n vertices with $\sigma_3(G) \geq \max\{n + 2, 3\alpha(G)\}$. Then $circum(G) = n$.*

Theorem 61 can be generalized, as is shown in the result.

Theorem 62. *[28] Let G be a 2-connected graph on n vertices with $\sigma_3(G) \geq n+2$. Then $circum(G) \geq \min\{n, n + \frac{\sigma_3(G)}{3} - \alpha(G)\}$.*

If we drop the 2-connectivity of the graph,and assume 1-toughness instead ,then the bounds in theorems 58-60 can be improved. The following two theorems are due to Bigalke and Jung.

Theorem 63. *[34] Let G be a 1-tough graph on n vertices with $\delta(G) \geq \frac{n}{3}$. Then every longest cycle in G is a dominating cycle.*

Theorem 64. *[34] Let G be a 1-tough graph on $n \geq 3$ vertices with $\delta(G) \geq \max\{\frac{n}{3}, \alpha(G) - 1\}$. Then $circum(G) = n$.*

The next theorem is a generalization of theorem 63.

Theorem 65. *[28] Let G be a 1-tough graph on n vertices with $\sigma_3(G) \geq n$. Then every longest cycle in G is a dominating cycle.*

With 1-toughness replacing 2-connectivity, we can improve a bit the lower bound $\sigma_3(G) \geq n+2$ in theorem 62.

Theorem 66. *[28] Let G be a 1-tough graph on $n \geq 3$ vertices with $\sigma_3(G) \geq n$. Then $circum(G) \geq \min\{n, n + \frac{\sigma_3(G)}{3} - \alpha(G)\}$.*

From theorem 11 we have the following bound for a graph G on n vertices, $\tau(G) \leq \frac{n-\alpha(G)}{\alpha(G)}$, or equivalently $\alpha(G) \leq \frac{n}{\tau(G)+1}$. If G is 1-tough, then $\tau(G) \geq 1$ and the previous upper bound becomes $\alpha(G) \leq \frac{n}{2}$. Similarly, if G is 2-tough, then $\alpha(G) \leq \frac{n}{3}$. Now, if G is 1-tough graph on n vertices with $\delta(G) \geq \frac{n}{3}$, then $\sigma_3(G) \geq n$, which according to theorem 66 means that $circum(G) \geq \frac{5n}{6}$. Note also that $\sigma_3(G) \geq n$ implies that $\sigma_2(G) \geq \frac{2n}{3}$ (we can always remove from an independent set of three vertices the one contributing most to the sum of their degrees, getting a minimum degree sum for independent sets of size two), which using theorem 57 implies that $circum(G) \geq \frac{2n}{3} + 2$. So, with a few elementary observations we used theorem 66 to improve the lower bound in $circum(G)$ in theorem 56 by $\frac{n}{6} - 2$. Using the fact that a 2-tough graph is also a 1-tough graph and that $\alpha(G) \leq \frac{n}{3}$ for any 2-tough graph G, we get the following corollary of theorem 66 :

Corollary 18. *[28] Let G be a 2-tough graph on $n \geq 3$ vertices. If $\sigma_3(G) \geq n$, then $circum(G) = n$.*

An improvement of theorem 66 came from Hoa :

Theorem 67. *[94] Let G be a 1-tough graph on $n \geq 3$ vertices, with $\sigma_3(G) \geq n$. Then $circum(G) \geq \min\{n, n + \frac{\sigma_3(G)}{3} - \alpha(G) + 1\}$.*

As we mentioned earlier, if G is 1-tough, then $\alpha(G) \leq \frac{n}{2}$. Thus we have the following corollary of theorem 67 :

Corollary 19. *Let G be a 1-tough graph on $n \geq 3$ vertices, with $\sigma_3(G) \geq n$. Then circum$(G) \geq \frac{5n}{6} + 1$.*

Li improved this result, as is shown in the next theorem :

Theorem 68. *[109] If G is a 1-tough graph on $n \geq 3$ vertices with $\delta(G) \geq \frac{n}{3}$, then*

$$circum(G) \geq \min\left\{n, \frac{2n+1+2\delta(G)}{3}, \frac{3n+2\delta(G)-2}{4}\right\} \geq \min\left\{\frac{8n+3}{9}, \frac{11n-6}{12}\right\}.$$

In [5] the following conjecture was made by Bauer, Broesma and Schmeichel.

Conjecture 7. *Let G be a 1-tough graph on $n \geq 3$ vertices with $\sigma_3(G) \geq n$. Then circum$(G) \geq \min\{n, \frac{3n+1}{4} + \frac{\sigma_3(G)}{6}\}$.*

Using it's hypothesis and if it were true, then we would conclude that $circum(G) \geq \frac{11n+3}{12}$, shrinking the gap between $\frac{8n+3}{9}$ and $\frac{11n-6}{12}$ of theorem 68 to $\frac{8n+3}{9}$ and $\frac{11n+3}{12}$. The conjecture 1, if true, would also imply the following generalization of Jung's theorem (theorem 53), which was proved by Faßbender :

Theorem 69. *[75] Let G be a 1-tough graph on $n \geq 13$ vertices with $\sigma_3(G) \geq \frac{3n-14}{2}$. Then circum$(G) = n$.*

The following theorem has had a number of applications, including theorem 69, and due to its importance, it is worthwhile mentioning it.

Theorem 70. *[28] Let G be a 1-tough graph on $n \geq 3$ vertices with $\sigma_3(G) \geq n$. Then every longest cycle in G is a dominating cycle. Moreover, if G is not hamiltonian, then G contains a longest cycle C such that* $\max\{deg_G(v) \mid v \in V(G) \setminus V(C)\} \geq \frac{\sigma_3}{3}$.

The next result of Li is related to theorem 70. First, we need to give some definitions. Let G be a graph on n vertices and let $X \subseteq V(G)$. Let $G[X] = (X, \{\{v,u\} \in E(G) \mid v,u \in X\})$ be the subgraph of G induced by X. Let $\alpha(X) = \alpha(G[X])$ and $\sigma_k(X) = \sigma_k(G[X])$, be the size of the maximum independent set and the minimum degree sum over independent sets of size k in $G[X]$ respectively. A cycle C of G is called *X-longest*, if for any other cycle C' of G, $|C' \cap X| \leq |C \cap X|$, and C is called *C-dominating* if for each vertex $v \in X - V(C)$, $N_G(v) \subseteq V(C)$.

Theorem 71. *[110] Let G be a 1-tough graph on n vertices and $X \subseteq V(G)$. If $\sigma_3(X) \geq n$, then G has an X-longest cycle C such that C is an X-dominating cycle and $|V(C) \cap X| \geq \min\{|X|, |X| + \frac{\sigma_3(G)}{3} - \alpha(X)\}$.*

So far, we have seen results that assumed some lower bounds on either minimum degree, or minimum degree sum of vertices that belong to an independent set of a specified size (combined with a level of toughness or connectivity). If we consider neighborhood unions, NC_k, we can give more tight bounds, since $NC_k(G) \leq \sigma_k(G)$ for any k. More specifically, the next theorem is an improvement of theorem 66. First we need the following definition, let

$$\varepsilon(i) = \begin{cases} 0, & \text{if } i \equiv 0 \bmod 3 \\ 2, & \text{if } i \equiv 1 \bmod 3 \\ 1, & \text{if } i \equiv 2 \bmod 3 \end{cases}$$

Theorem 72. *[47] Let G be a 1-tough graph on $n \geq 3$ vertices with $\sigma_3(G) \geq n + r$, where $r \geq 0$. Then $circum(G) \geq \min\{n, n + NC_{r+5+\varepsilon(n+r)} - \alpha(G)\}$.*

In [47] it is shown that the lower bound on $circum(G)$ and the subscript $r + 5 + \varepsilon$ of NC cannot be increased in general. $NC_k(G)$ is a nondecreasing function of k because if we had $NC_{k+1}(G) < NC_k(G)$ for some k, then we could take those $k + 1$ vertices that form an independent set and achieve a cardinality of their neighborhoods union of $NC_{k+1}(G)$, and remove any of them, yielding for the rest of the vertices a cardinality of their neighborhoods union at most $NC_{k+1}(G)$, which leads to a contradiction. It also holds that $NC_3(G) \geq \frac{\sigma_3(G)}{3}$, since every vertex in the union of the neighborhoods of the 3 vertices in the independent set achieving the $\sigma_3(G)$ value, can be added at most 3 times in the sum of the degrees in $\sigma_3(G)$, and therefore in the worst case every such vertex was taken into account exactly 3 times, in which case the neighborhoods of the three vertices coincide, giving a cardinality of the union of their neighborhoods one third of the sum of their degrees. Thus, theorem 68 implies theorem 66. It also holds that $NC_k \leq n - \alpha(G)$, because from the $\alpha(G)$ independent vertices (the union of their neighborhoods can have at most $n - \alpha(G)$ vertices), we could just pick k of them. Thus theorem 72 also implies the following corollary :

Corollary 20. *Let G be a 1-tough graph on $n \geq 3$ vertices with $\sigma_3(G) \geq n + r$, where $r \geq 0$. Then $circum(G) \geq \min\{n, 2NC_{r+5+\varepsilon(n+r)}\}$.*

Corollary 20 also implies theorem 69. In [47] it is shown that the subscript of NC in the conclusion of corollary 20, can be replaced by $\lfloor \frac{n+6r+17}{8} \rfloor$ which is an improvement if $r \leq \frac{n}{2} - 19$. The following theorem is relative to corollary 20, but the lower bound on $circum(G)$ in the conclusion is expressed in terms of NC_2 instead of NC_k.

Theorem 73. *[14] Let G be a 1-tough graph on $n \geq 3$ vertices with $\sigma_3(G) \geq n$. Then $circum(G) \geq \min\{n, 2NC_2(G)\}$.*

In [14] it was also conjectured that the lower bound on $circum(G)$ in the conclusion of theorem 73 can be replaced by $circum(G) \geq \min\{n, 2NC_2(G) +$

4}. Theorem 73 is the last result in this section involving neighborhood unions. The next result is an other application of theorem 66, that uses the fact that $\sigma_3(G) \geq 3\delta(G)$ and that $\alpha(G) \leq \frac{n}{\tau(G)+1}$.

Theorem 74. *Let G be a t-tough graph on $n \geq 3$ vertices, where $1 \leq t \leq 2$. If $\delta(G) \geq \frac{n}{t+1}$, then circum$(G) = n$.*

To apply theorem 66 it was essential that $\tau(G) \leq 2$ (because of the upper bound on $\alpha(G)$). However the bound on t in the precondition of theorem 74 is not necessary, as we can see in the following result.

Theorem 75. *[8] Let G be a t-tough graph on $n \geq 3$ vertices with $\delta(G) \geq \frac{n}{t+1}$. Then circum$(G) = n$.*

We conclude from theorem 75 that Chvátal's conjecture, that there exists a finite constant t_0 such that every t_0-tough graph is hamiltonian, is true within the class of graphs G having $\delta(G) \geq \epsilon n$, for any fixed $\epsilon > 0$.

A genaralization of theorem 51 and theorem 75, due to Jung and Wittmann is the next result.

Theorem 76. *[100] Let G be a 2-connected t-tough graph on n vertices. Then circum$(G) \geq \min\{n, (t+1)\delta(G)+1\}$.*

Another result related to theorem 75 is the following :

Theorem 77. *[9] Let G be a t-tough graph on n vertices, with $t \geq 1$ and $\delta(G) > \frac{n}{t+2}$. Then G contains a dominating cycle.*

The next theorem provides us with a sufficient condition for a 1-tough graph to be hamiltonian, using the vertex connectivity $\kappa(G)$ of G. If $\kappa(G)$ is small then this result greatly improves Dirac's theorem in the case of 2-connected graphs. The theorem appears in [5] but the background has it's origins in a theorem of Häggkvist and Nicoghossian in [90].

Theorem 78. *Let G be a 2-connected graph on n vertices with $\delta(G) \geq \frac{n+\kappa(G)}{3}$. Then circum$(G) = n$.*

A generalization of theorem 78 is the following :

Theorem 79. *[12] Let G be a 2-connected graph on n vertices with $\sigma_3(G) \geq n + \kappa(G)$. Then circum$(G) = n$.*

Theorems 78 and 79 are best possible. In the case of 1-tough graphs, Bauer and Schmeichel improved theorem 78, with the following result :

Theorem 80. *[20] Let G be a 1-tough graph on $n \geq 3$ vertices with $\delta(G) \geq \frac{n+\kappa(G)-2}{3}$. Then circum$(G) = n$.*

There are examples where theorem 80 is the best possible when $\kappa(G) = 2$ or $\kappa(G) = \frac{n-5}{2} \geq 11$.

Later, Wei generalized theorem 80 :

Theorem 81. *[135] Let G be a 1-tough graph on $n \geq 3$ vertices with $\sigma_3(G) \geq n + \kappa(G) - 2$. Then $circum(G) = n$.*

Hoa improved the lower bound on $\sigma_3(G)$ in theorem 81.

Theorem 82. *[93] Let G be a 1-tough graph on $n \geq 3$ vertices with $\sigma_3(G) \geq n + \kappa(G) - \alpha(G)$. Then $circum(G) = n$.*

The next result is due to Brandt and Veldman :

Theorem 83. *[45] Let G be a 1-tough graph on $n \geq 2$ vertices with $\min\{deg_G(v) + deg_G(u) \mid \{v, u\} \in E(G)\} \geq n$. Then G is pancyclic or $G \backsimeq K_{n/2,n/2}$.*

The following result, due to Chvátal, uses the degree sequence of a graph to establish its hamiltonicity.

Theorem 84. *[56] Let G be a graph with degree sequence $d_1 \leq d_2 \leq ... \leq d_n$. If for all integers i with $1 \leq i < \frac{n}{2}$, $d_i \leq i$ implies $d_{n-i} \geq n - i$, then $circum(G) = n$.*

The following theorem of Hoáng is relevant :

Theorem 85. *[95] Let $t \in \{1, 2, 3\}$ and let G be a t-tough graph with degree sequence $d_1 \leq d_2 \leq ... \leq d_n$. If for all integers i with $t \leq i < \frac{n}{2}$, $d_i \leq i$ implies $d_{n-i+t} \geq n - i$, then $circum(G) = n$.*

For the next two theorems we need the definition of path-tough graphs.

Definition 28. A graph G is *path-tough* if for every nonempty set S of vertices, the graph $G - S$ can be covered by at most $|S|$ vertex disjoint paths.

It turns out that being path-tough is a necessary condition for a graph to be hamiltonian. A graph that is path-tough is also 1-tough. An interesting discussion on the notion of path-tough graphs can be found in [23]. A number of results on the subject can also be found in [61]. One of the results, was that it is NP-complete to determine if a graph is path-tough. In the same work, the authors proved the following :

Theorem 86. *Let G be a path-tough graph on $n \geq 3$ vertices. If $\delta(G) \geq \frac{3n}{6+\sqrt{3}}$, then $circum(G) = n$.*

Schiermeyer gave the minimum degree sum version of the theorem 86 :

Theorem 87. *[128] Let G be a path-tough graph on $n \geq 3$ vertices with $\sigma_2(G) > \frac{4(n-\frac{6}{5})}{5}$. Then $circum(G) = n$.*

The next result involves a new type of sufficient degree condition for a graph to be hamiltonian. It was introduced by Fan and has led to many new and interesting results in hamiltonian graph theory.

Theorem 88. *[74] Let G be a 2-connected graph on n vertices. If for all vertices $v, u \in V(G)$, $dist_G(v,u) = 2$ implies $\max\{deg_G(v), deg_G(u)\} \geq \frac{n}{2}$, then $circum(G) = n$.*

Replacing the 2-connectivity of the graph in the precondition of theorem 88 with the condition that the graph be 1-tough can improve the lower bound on the degree condition.

Theorem 89. *[10] Let G be a 1-tough graph on $n \geq 3$ vertices such that $\sigma_3(G) \geq n$. If for all vertices $v, u \in V(G)$, $dist_G(v,u) = 2$ implies $\max\{deg_G(v), deg_G(u)\} \geq \frac{n-4}{2}$, then $circum(G) = n$.*

If we further require that the graph is 3-connected, then the condition on $\sigma_3(G)$ can be dropped.

Theorem 90. *[10] Let G be a 3-connected 1-tough graph on $n \geq 35$ vertices. If for all vertices $v, u \in V(G)$, $dist_G(v,u) = 2$ implies $\max\{deg_G(v), deg_G(u)\} \geq \frac{n-4}{2}$, then $circum(G) = n$.*

We now turn to results, concerning the circumference of t-tough graphs, making no assumptions regarding vertex degrees or neighborhoods unions as we did in the previous results of this section. It is well known that if G is a k-connected graph on $n \geq 2k$ vertices, then $circum(G) \geq 2k$. This bound is tight, as is shown by graph $K_{n,n-k}$ with $n \geq 2k \geq 4$, regardless of the size of n. In proposition 8 we stated that $\tau(K_{n,n-k}) = \min\{\frac{n}{n-k}, \frac{n-k}{n}\}$. If k is a constant independent from n, then $\lim_{n\to\infty} \tau(K_{n,n-k}) = 0$. So asymptotically $K_{n,n-k}$ is 0-tough. When $t > 0$, the situation is different for t-tough graphs. Let

$$\gamma_k(t,n) = \min\{circum(G) \mid G \text{ is a } k\text{-connected 1-tough graph on } n \text{ vertices}\}$$

Theorem 91. *[49] Let $t > 0$ be fixed. Then $\gamma_2(t,n)\log(\gamma_2(t,n)) \geq (2 - o(1))\log n$ as $n \to \infty$.*

In [49] the authors also give examples that show that for $0 < t \leq 1$, $\gamma_2(t,n) = O(\log n)$. Then next result is a corollary of theorem 91.

Corollary 21. *Let $t > 0$ be fixed. Then $\lim_{n\to\infty} \gamma_2(t,n) = \infty$.*

In the case of 3-connected graphs, we have the following stronger than theorem 91 result :

Theorem 92. *[49] Let $t > 0$ be fixed. Then as $n \to \infty$*

$$\gamma_3(t,n) \geq \left(\frac{4}{5\log((1/t)+1)} - o(1)\right)\log n$$

In the same work, it is shown that for $t \leq 1$, theorem 92 is best possible. The authors also made the following conjecture.

Conjecture 8. *Let $t > 0$ be fixed. There exists a constant $A > 0$, depending only on t such that $\gamma_2(t,n) \geq A\log n$.*

Some progress has been made in the case of planar graphs. The following results present some bounds on the circumference of a planar graph given some conditions.

Theorem 93. *[134] Let G be a 4-connected planar graph on n vertices. Then $circum(G) = n$.*

Since every non-hamiltonian planar graph G must have $\kappa(G) \leq 3$ and $\tau(G) \leq \frac{\kappa(G)}{2}$, G must be at most $\frac{3}{2}$-tough. Hence, we have the following corollary :

Corollary 22. *Let G be a $(\frac{3}{2}+\epsilon)$-tough planar graph, with $\epsilon > 0$. Then G is hamiltonian.*

Theorem 94. *[97] Let G be a 3-connected planar graph on n vertices. Then there exist constants $\rho, \theta > 0$ such that $circum(G) \geq \rho n^{\theta}$.*

Theorem 95. *[38] Let G be a planar graph on n vertices, with $\kappa(G) = 2$ such that $c(G-S) \leq \xi$, for every $S \subseteq V(G)$, with $|S| = 2$. Then $circum(G) \geq \psi\left(\frac{1}{\xi-1}\right)^{0.4}\ln n$, where $\psi \approx 0.10$.*

The following result is a corollary of theorem 95

Corollary 23. *Let G be a planar graph on n vertices, with $\kappa(G) = 2$. Then $circum(G) \geq \psi\left(\frac{\tau(G)}{2-\tau(G)}\right)^{0.4}\ln n$, where $\psi \approx 0.10$.*

2.8 The disproof of the 2-tough conjecture

In [57] Chvátal conjectured that there exists a finite constant $t_0 \geq 1$ such that every t_0-tough graph is hamiltonian (we already know that being 1-tough is a necessary condition for a graph to be hamiltonain). In the same paper, Chvátal established the existence of $\frac{3}{2}$-tough nonhamiltonian graphs. It was later shown in [31] that there exist t-tough nonhamiltonian graphs with $t > \frac{3}{2}$. Later Enomoto et al. ([72]) showed that there exist $(2-\epsilon)$-tough graphs having no 2-factor for arbitrary $\epsilon > 0$.

The research on this area focused for many years mainly on determining whether the conjecture was true for $t_0 = 2$. If the 2-tough conjecture held true, that would imply that a number of important conjectures would also be true. For some time, it was believed that $t_0 = 2$ might be the threshold for a graph to be hamiltonian ([72]). The 2-tough conjecture, if true, would also imply a well known result from Fleischner :

Theorem 96. *[80] Let H be a 2-connected graph and let $G = H^2$. Then G is hamiltonian.*

The following two conjectures are examples of implications of the 2-tough conjecture. We first need the definition of a line graph.

Definition 29. Let G be a graph. The *line graph* of G, denoted $L(G)$, is the graph whose vertices are the edges of G, and if $e_1, e_2 \in E(G)$ then $\{e_1, e_2\} \in E(L(G))$ if $e_1 \cap e_2 \neq \emptyset$. A graph H is called a *line graph* if $H = L(G)$ for some graph G.

Conjecture 9. *[131] Let G be a 4-connected line graph. Then G is hamiltonian.*

Another important conjecture is the following :

Conjecture 10. *[114] Let G be a 4-connected $K_{1,3}$-free graph. Then G is hamiltonian.*

Since for every claw-free graph G, $\tau(G) = \frac{\kappa(G)}{2}$, conjecture 10, if true, would imply that every 2-tough claw-free graph is hamiltonian. These conjectures were shown by Ryjáček to be equivalent in [127]. In spite of all these efforts, it was proven in [11] that not all 2-tough graphs are hamiltonian. More accurately, the initial result in [11] was that not all 2-tough graphs are traceable. However, the proof of this can be slightly changed to a proof of 2-tough nonhamiltonian graphs. The statement was proven by counterexamples, using a family of graphs constructed using a specific graph L as a building block. This graph has at least two vertices that are not connected via a Hamilton path, and the graph constructed based on graph L, conditioned on the values of some parameters, is nontraceable. The toughness of this constructed graph, with the same restrictions on the values of it's parameters, gives us a $(\frac{9}{4} - \epsilon)$-tough graph. Combining these two observations, we can construct a $(\frac{9}{4} - \epsilon)$-tough graph that is nontraceable. Most of the ingredients used in these counterexamples were already present in [8]. The counterexamples were also inspired by constructions in [21]. We now give a brief outline of the construction of this family of graphs, in which we can find our counterexamples for the 2-tough conjecture.

Let H be a graph on at least two vertices and $x, y \in V(H)$. We define the graph $G(H, x, y, l, m)$ as follows. Take m disjoint copies $H_1, ..., H_m$ of H, with x_i, y_i being the vertices in H_i corresponding to the vertices x and y

in H (for all $i = 1...m$). Let F_m be the graph obtained from $H_1 \cup ... \cup H_m$ by adding all possible edges between pairs of vertices in $\{x_1, ..., x_m, y_1, ..., y_m\}$. Finally, set $G(H, x, y, l, m) = K_l * F_m$. We have the following result, with respect to graph $G(H, x, y, l, m)$:

Theorem 97. *[11] Let H be a graph and x, y two vertices of H which are not connected by a Hamilton path in H. If $m \geq 2l + 3$, then $G(H, x, y, l, m)$ is nontraceable.*

In order to get advantage of theorem 97, we need a graph H having two vertices $u, v \in V(H)$, not connected by a Hamilton path, and for $m \geq 2l+3$ we want $\tau(G(H, x, y, l, m)) \geq 2$. Consider the graph L of figure 5. It is clear that there does not exist a Hamilton path between v and u in graph L.

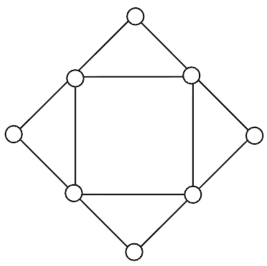

Figure 5: The graph L. Vertices u, v are not connected via a Hamilton path

Thus we have the following corollary of theorem 97 :

Corollary 24. *If $m \geq 2l + 3$, then $G(L, u, v, l, m)$ is nontraceable.*

It remains to show that for $m \geq 2l + 3$, $\tau(G(L, u, v, l, m)) \geq 2$. The following theorem establishes the toughness of $G(L, u, v, l, m)$.

Theorem 98. *[11] For $l \geq 2$ and $m \geq 1$,*

$$\tau(G(L, u, v, l, m)) = \frac{l + 4m}{2m + 1}.$$

Combining theorems 97 and 98 for sufficiently large m and l we get the following result :

Theorem 99. *[11] For every $\epsilon > 0$, there exists a $(\frac{9}{4} - \epsilon)$-tough nontraceable graph.*

The proof of theorem 99 in [11] can be turned into a proof of the following theorem by replacing $m \geq 2l + 3$ with $m \geq 2l + 1$ and "nontraceable" with "nonhamiltonian.

Theorem 100. *For every $\epsilon > 0$, there exists a $(\frac{9}{4} - \epsilon)$-tough nonhamiltonian graph.*

Consider the graph $G(L, u, v, 2, 5)$ of figure 6. From theorem 98, the graph $G(L, u, v, 2, 5)$ has toughness 2, and from theorem 100 it is nonhamiltonian. Graph L has 8 vertices, thus this counterexample for the 2-tough conjecture has 42 vertices. The smallest graph 2-tough $G(L, u, v, l, m)$ which is nontraceable is for $l = 2$ and $m = 7$ and has order 58.

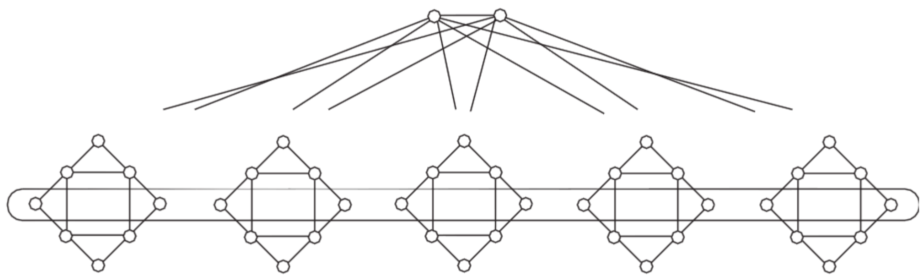

Figure 6: The graph $G(L, u, v, 2, 5)$ is nonhamiltonian with toughness $\tau(G(L, u, v, 2, 5)) = 2$

Chvátal also stated a weaker version of the 2-tough conjecture, that is based on the following definition.

Definition 30. A graph G is *neighborhood connected* if the neighborhood of each vertex of G induces a connected subgraph of G.

Chvátals conjecture was the following.

Conjecture 11. *[57] Let G be a 2-tough neighborhood connected graph. Then G is hamiltonian.*

The graph $G(L, u, v, 2, 5)$ is neighborhood connected and we already know that it is 2-tough. Therefore $G(L, u, v, 2, 5)$ is also a counterexample for conjecture 11. The following result is also due to Chvátal :

Theorem 101. *[57] For every $\epsilon > 0$, there exists a $(\frac{3}{2} - \epsilon)$-tough graph without a 2-factor.*

The examples on which the theorem is based, are all chordal. It was later shown that :

Theorem 102. *[17] Let G be a $\frac{3}{2}$-tough chordal graph. Then G has a 2-factor.*

Later, Kratsch asked whether every $\frac{3}{2}$-tough chordal graph is hamiltonian, but it was falsified in [11] using theorem 97. The graphs that were used as counterexamples followed almost the same stages of construction, but instead of using graph L as a "building block", they used graph M that is shown in figure 7.

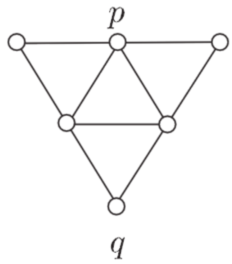

Figure 7: The graph M.

The graph M is chordal and there is no Hamilton path connecting vertices p and q. It turns out that the graphs $G(M, p, q, l, m)$ are also chordal, and from theorem 97, they are nontraceable whenever $m \geq 2l + 3$. By arguments similar to the ones used in the proof of theorem 98, we can compute the toughness of $G(M, p, q, l, m)$:

Theorem 103. *For $l \geq 2$ and $m \geq 1$,*

$$\tau(G(M, p, q, l, m)) = \frac{l + 3m}{2m + 1}.$$

Therefore, the graphs $G(M, p, q, l, 2l+3)$ for $l \geq 2$, are chordal nontraceable graphs with toughness $\frac{7l+9}{4l+7}$. We have the following result :

Theorem 104. *[11] For every $\epsilon > 0$, there exists a $(\frac{7}{4} - \epsilon)$-tough chordal nontraceable graph.*

For the next two results we need some definitions first.

Definition 31. A *closed walk* is a walk whose endpoints coincide.

Definition 32. A *spanning walk* is a walk that covers all vertices of the graph.

Definition 33. A *k-walk* in a graph G is a closed spanning walk of G that visits every vertex of G at most k times.

Clearly, a Hamilton cycle is a 1-walk. The same concept of a construction of a family of graphs was used by Ellingham and Zha to show that there exist graphs of relatively high toughness without a k-walk. They established the following results :

Theorem 105. *[68] Every 4-tough graph has a 2-walk.*

Theorem 106. *[68] For every $\epsilon > 0$ and every $k \geq 1$ there exists a $(\frac{8k+1}{4k(2k-1)} - \epsilon)$-tough graph with no k-walk.*

2.9 Toughness and factors

Chvátal in [57] conjectured that every k-tough graph on $n \geq k+1$ vertices and kn even contains a k-factor. Enomoto et al. ([72]) then proved that k-toughness is a threshold for this property, as we can see in the following two theorems.

Theorem 107. *[72] Let G be a k-tough graph on n vertices with $n \geq k+1$ and kn even. Then G has a k-factor.*

Theorem 108. *[72] Let $k \geq 1$. For every $\epsilon > 0$ there exists a $(k-\epsilon)$-tough graph G on n vertices, with $n \geq k+1$ and kn even which has no k-factor.*

The next two corollaries follow easily from theorems 107 and 108.

Corollary 25. *Every 2-tough graph has a 2-factor.*

Corollary 26. *There exist infinitely many $(2-\epsilon)$-tough graphs with no 2-factor.*

A stronger version of theorem 107 was given by Enomoto :

Theorem 109. *[69] Let k be a positive integer and G be a graph on n vertices with $n \geq k+1$ and kn even. Suppose $|S| \geq kc(G-S) - \frac{7k}{8}$ for all cutsets $S \subseteq V(G)$. Then G has a k-factor.*

Later, Enomoto improved his result in the case of $k = 1$ and $k = 2$. We need the following definition.

Definition 34. Let G be a graph and let

$$\tau'(G) = \max\{t \mid |S| \geq tc(G-S) - t \text{ for all cutsets } S \subset V(G)\}$$

$$= \min\left\{\tfrac{|S|}{c(G-S)-1} \mid c(G-S) > 1\right\}$$

if G is not complete, otherwise set $\tau'(G) = \infty$.

Theorem 110. *[70] Let G be a graph on n vertices, where n is even. If $\tau'(G) \geq 1$, then G has a 1-factor.*

Theorem 111. *[70] Let G be a graph on $n \geq 3$ vertices. If $\tau'(G) \geq 2$ then G has 2-factor.*

A generalization of theorem 111 came later by Enomoto and Hagita. This result is also a stronger version of theorem 107 for graphs with a sufficiently large number of vertices.

Theorem 112. *[71] Let k be a positive integer and G be a graph on k^2-1 vertices with kn even. If $\tau'(G) \geq k$, then G has a k-factor.*

The next results involve some lower bounds on the minimum degree that a t-graph can have, where $1 \leq t \leq 2$, in order to contain a 2-factor.

Theorem 113. *[21] Let G be a t-tough graph on $n \geq 3$ vertices, where $1 \leq t \leq 2$. If $\delta(G) > (\frac{2-t}{1+t})n$, then G has a 2-factor.*

In the same paper, Bauer and Schmeichel showed that for any $t \in [1, \frac{3}{2}]$ there are infinitely many t-tough graphs having no 2-factor and with $\delta(G) \geq (\frac{2-t}{1+t})n - \frac{5}{2}$. An improvement of the bound on $\delta(G)$ was also given for $\frac{3}{2} < t < 2$.

Theorem 114. *[21] Let G be a t-tough graph on $n \geq 3$ vertices, where $\frac{3}{2} < t < 2$. If $\delta(G) \geq (\frac{2-t}{1+t})(\frac{t^2-1}{7t-7-t^2})n$, then G has a 2-factor.*

The authors in [21] give examples that show that theorem 114 is asymptotically tight if $t = \frac{2r-1}{r}$, for any integer $r \geq 2$. In [28] can be found similar results concering minimum degree conditions for a t-tough graph, with $1 \leq t < 3$, regarding the existence of a 3-factor.

An improvement of theorem 107 came also from Chen, in [52], where he showed that under similar conditions it is possible to find a k-factor containing a specified edge and a k-factor not containing a specified edge. Another improvement of theorem 107 was obtained by Katerinis in [102].

For the results that follow we need the next definition.

Definition 35. An $[a, b]$-*factor* of a graph G is a spanning subgraph H of G, such that $a \leq deg_H(v) \leq b$, for all $v \in V(G)$.

Theorem 115. *[102] Let $a \leq b$ and G be a graph on n vertices such that $a < b$ or bn is even. If $\tau(G) \geq a + \frac{a}{b} - 1$, then G has an $[a, b]$-factor.*

Chen improved the latter result for $a = 2 < b$.

Theorem 116. *[53] Let $b > 2$ and G be a graph on $n \geq 3$ vertices. If $\tau(G) \geq 1 + \frac{1}{b}$, then G has a $[2, b]$-factor.*

This result has been extended to connected factors by Ellingham et al. in [67]. The next result, due to Katerinis, involves the existence of 2-factors in 1-tough bipartite graphs.

Theorem 117. *[101] Every 1-tough bipartite graph on $n \geq 3$ vertices has a 2-factor.*

Next, we present a generalization of theorem 54, made by Faudree et al. in [76].

Theorem 118. *[76] There exists an integer n_0 such that every 1-tough graph on $n \geq n_0$ vertices with $\delta(G) \geq \frac{n-4}{2}$, has a 2-factor with k cycles, for all k such that $1 \leq k \leq \frac{n-10}{4}$.*

We will present some results on factors that relate toughness to (r,k)-factor-critical graphs.

Definition 36. A graph G is (r,k)-*factor-critical* if for all $X \subseteq V(G)$ with $|X| = k$, $G - X$ contains an r-factor .

In the case of $r \geq 2$, Liu and Yu in [112] established some results on (r,k)-factor-critical graphs, although the term used was not that of definition 36 but (r,k)-extendable graphs.

Theorem 119. *[112] Let G be a graph on n vertices with $\tau(G) \geq 3$. Then G is $(2,k)$-factor-critical, for every integer k such that $3 \leq k \leq \tau(G)$ and $k \leq n-3$.*

Liu and Yu also made the following conjecture :

Conjecture 12. *Let G be a graph on n vertices with $\tau(G) \geq q$ and $n \geq 2q+1$ for some integer $q \geq 1$. Then G is $(2, 2q-2)$-factor-critical.*

For $q = 1$ conjecture 12 states that every graph G on $n \geq 3$ vertices with $\tau(G) \geq 1$ has a 2-factor. From theorem 108 we have that for every $\epsilon > 0$, there exist $(2-\epsilon)$-graphs with no 2-factor, so the conjecture clearly does not hold for $q = 1$. However, Cai et al. in [51] and independetly Enomoto in [70] showed that the conjecture is true for all integers $q \geq 2$.

Theorem 120. *Let G be a graph on n vertices with $\tau(G) \geq 2$. Then G is $(2,k)$-factor-critical, for every non-negative integer $k \leq \min\{2\tau(G)-2, n-3\}$.*

Cai et.al in [51], also showed that the bound $2\tau(G) - 2$ is sharp. The next two theorems concern the relationship between toughness and (r,k)-factor-critical graphs. The first is by Favaron in [77], where he examined this relationship in the case of $r = 1$.

Theorem 121. *[77] Let G be a graph on n vertices and k be an integer with $2 \leq k < n$ and $n+k$ even. If $\tau(G) > \frac{k}{2}$, then G is $(1,k)$-factor-critical.*

The bound on $\tau(G)$ was also shown to be tight. The case of $r = 3$ was then examined by Shi et al. in [130].

Theorem 122. *[130] Let G be a graph on n vertices with $\tau(G) \geq 4$. Then G is $(3,k)$-factor-critical for every non-negative integer k such that $n+k$ is even, $k < 2\tau(G) - 2$ and $k \leq n-7$.*

The upper bounds in theorem 122 on k are the best possible. We need some definitions for the rest of this section.

Let G be a graph, $X \subseteq V(G)$ and $Y \subseteq E(G-X)$. Let $G(Y)$ denote the graph induced by the edges in Y, i.e $V(G(Y)) = \bigcup_{e \in Y} e$ and $E(G(Y)) = \{\{v,u\} \in E(G) \mid v,u \in V(G(Y))\}$. Notice that there may be edges in $E(G(Y))$ that are not in Y. Let $C_1, C_2, ..., C_k$ be the components of the edge-induced subgraph $G(Y)$ induced by edge set Y. Let H be a graph and B be a subgraph of H, then the *boundary* of B in H is defined as :

$$bd_H(B) = \{v \in V(B) \mid \exists u \in N_H(v)\ [u \notin B]\}$$

$$in_H(B) = V(B) - bd_H(B)$$

Thus the boundary of B is two sets, the first that contains all vertices of B that have a neighbor outside B and the second that contains all vertices whose neighborhood is a subset of B. Therefore, the boundary of a component C_i, with $1 \leq i \leq k$ in $G-X$ is

$$bd_{G-X}(C_i) = \{v \in V(C_i) \mid \exists u \in N_{G-X}(v)\ [u \notin C_i]\}$$

$$in_{G-X}(C_i) = V(C_i) - bd_{G-X}(C_i)$$

Finally, $G-X-Y-\bigcup_{i=1}^{k} in_{G-X}(C_i)$ denotes the graph which is obtained from G by deleting vertices X and $in_{G-X}(C_i)$ for every $1 \leq i \leq k$ and the edges Y (without deleting the corresponding endpoints). Then we have the following definition of t-edge-tough graphs :

Definition 37. Let G be a graph and let $t > 0$. Then G is *t-edge-tough* if G is connected and if

$$c\left(G - X - Y - \bigcup_{i=1}^{k} in_{G-X}(C_i)\right) \leq \frac{|X| + \sum_{i=1}^{k} \lfloor \frac{|bd_{G-X}(Q_i)|}{2} \rfloor}{t} := \frac{s(X,Y;G)}{t}$$

holds for every $X \subseteq V(G)$ and $Y \subseteq E(G-X)$ satisfying $c(G-X-Y-\cup_{i=1}^{k} in_{G-X}(C_i)) > 1$.

Definition 38. The *edge-toughness* of a graph G, denoted by $\tau_e(G)$, is the maximum value of t_e for which G is t_e-edge-tough. Set $\tau_e(K_n) = \infty$.

The notion of t-edge-toughness was introduced by Katona in [103]. Katona also established the following results :

Theorem 123. *[103] If G is a hamiltonian graph, then G is 1-edge-tough.*

Theorem 124. *[103] If G is a t-edge-tough graph, then G is t-tough.*

Theorem 125. *[103] If G is a $2t$-tough graph, then G is t-edge-tough.*

We already know from theorem 107 that every 2-tough graph has a 2-factor. Setting $t = 1$ in theorem 125, we get that every 2-tough graph is 1-edge-tough. A natural question, answered affirmatively, is whether every 1-edge-tough graph has a 2-factor.

Theorem 126. *[104] Let G be 1-edge-tough graph on $n \geq 3$ vertices. Then G has a 2-factor.*

Katona also made the following conjecture :

Conjecture 13. *Let t be a positive integer and G be a t-edge-tough graph on $n \geq 2t + 1$ vertices. Then G has a $2t$-factor.*

2.10 Computational complexity of toughness

Chvátal first raised the problem of determining the computational complexity of recognizing t-tough graphs in [55], and then it appeared also in [32] and in [58]. Consider the following decision problem :

t-Tough
Instance: Graph G
Question: Is $\tau(G) \geq t$?

The following result was shown in [15] :

Theorem 127. *For any positive rational number t, t-Tough is NP-hard.*

The proof in [15] used a reduction from a well-known NP-hard variant of Independent Set problem to 1-Tough. The 1-Tough problem is easily reduced to the general t-Tough problem. Analogous arguments to those used in the first reduction, can be used to give a reduction from Independent Set to 1-Tough, as is shown in [22].

An interesting question is whether there are subclasses of graphs in which determining the toughness of the graphs they contain becomes tractable. In proposition 2 we saw a result from Matthews and Sumner, stating that the toughness of a claw-free graph is equal to half its connectivity. The connectivity of a graph can be computed in polynomial time, since for any graph G the maximum number of disjoint paths between two vertices can be computed with a max-flow min-cut algorithm and by Menger's theorem the connectivity of a graph equals the minimum over all pairs of vertices of the maximum number of vertex disjoint paths between them. Hence the toughness of claw-free graphs can be determined in polynomial time. Line graphs are clearly a subclass of claw-free graphs (every edge can be intersected by at most two edges without them intesecting as well), thus their toughness can be computed in polynomial time as well. It is well known that deciding whether a line graph is hamiltonian, is NP-complete [33]. However it is polynomial time computable to determine if it is 1-tough. This holds for split graphs as well, as it is NP-complete to decide whether a split graph is hamiltonian [60], but it is polynomial to decide whether a split graph is 1-tough.

Theorem 128. *[107] The class of 1-tough split graphs can be recognized in polynomial time.*

This result was extended in the case of t-tough split graphs by Woeginger, where t is a nonnegative rational number.

Theorem 129. *[137] For any rational number $t \geq 0$, the class of t-tough split graphs can be recognized in polynomial time.*

We have presented some subclasses of graphs for which 1-toughness is polynomial time decidable and others for which the general t-Tough problem is polynomial time decidable. However, there are other subclasses of graphs for which it is NP-hard to determine their toughness. Such a case is the class of graphs having minimum degree almost high enough to ensure that the graph is t-tough.

Theorem 130. *[18] Let $t \geq 1$ be a rational number. If $\delta(G) \geq (\frac{t}{t+1})n$, then G is t-tough. On the other hand, for any fixed $\epsilon > 0$, it is NP-hard to determine if G is t-tough for graphs with $\delta(G) \geq (\frac{t}{t+1} - \epsilon)n$.*

Notice that using Dirac's theorem, if $\delta(G) \geq \frac{n}{2}$ then G is hamiltonian and hence it is 1-tough. So for these graphs, it is polynomial time decidable to test for 1-toughness. An interesting case is that of graphs with $\delta(G) \geq \frac{n}{2} - 2$. It turns out that testing for 1-toughness in this class is also polynomially decidable. This is a consequence of a result of Häggvist in [89], which states that if $\delta(G) \geq \frac{n}{2} - 2$, there is a polynomial time algorithm that decides whether G is hamiltonian, and Jung's theorem (theorem 54). Jung's theorem states that every 1-tough graph on $n \geq 11$ vertices with $\delta(G) \geq \frac{n}{2} - 2$ is hamiltonian. Every hamiltonian graph is necessarily 1-tough, so a graph G on $n \geq 11$ vertices with $\delta(G) \geq \frac{n}{2} - 2$ is hamiltonian if and only if is 1-tough. Combining Häggvist's and Jung's resuls, testing for 1-toughness becomes tractable when the minimum degree condition is satisfied.

An interesting class of graphs is that of bipartite graphs. Notice that from proposition 8, the complete bipartite graph $K_{m,n}$ with $m \leq n$ has toughness $\tau(K_{m,n}) = \frac{m}{n} \leq 1$. Also from proposition 5, any spanning subgraph H of $K_{m,n}$ will have toughness $\tau(H) \leq \tau(K_{m,n})$. So we have the following corollary :

Corollary 27. *Let G be a bipartite graph. Then $\tau(G) \leq 1$.*

In [107], Kratsch et al. reduced the 1-Tough problem for general graphs to 1-Tough for bipartite graphs.

Theorem 131. *[107] 1-Tough remains NP-hard for bipartite graphs.*

Since bipartite graphs are also triangle-free, we have the following corollary of theorem 131

Corollary 28. *1-Tough is NP-hard for the class of triangle-free graphs.*

Another interesting class of graphs is that of r-regular graphs. First notice that since the maximum number of vertex disjoint paths in an r-regular graph is at most r, the connectivity of an r-regular graph G is $\kappa(G) \leq r$. Hence, if a graph G is r-regular, then $\tau(G) \leq \frac{\kappa}{2} \leq \frac{r}{2}$. In [57], Chvátal asked for which values of r and $n > r+1$ there exists an r-regular, $\frac{r}{2}$-tough graph on n vertices. He observed that when r is even, there is always such a graph. He also made the following conjecture :

Conjecture 14. *[57] If G is an r-regular $\frac{r}{2}$-tough graph on $n > r+1$ vertices with r odd, then $n \equiv 0 \mod r$.*

Chvátal also verified the conjecture for $r = 3$, but later Doty in [66] and Jackson and Katerinis in [96], independently constructed an infinite family of r-regular $\frac{r}{2}$-tough graphs on n vertices with $n \not\equiv 0 \mod r$.

Jackson and Katerinis in [96], gave a characterization of cubic $\frac{3}{2}$-tough graphs which gives us a way of recognizing them in polynomial time. This characterization used Cvátal's concept of inflation in [57]. In particular,

Definition 39. The *inflation of a graph G* is the graph obtained from G by replacing all vertices $v_1, v_2, ..., v_n$ of G by disjoint complete graphs on $deg_G(v_i)$ vertices $v_{i,1}, ..., v_{i,deg_G(v_i)}$, and all edges $\{v_i, v_j\}$ by disjoint edges $\{v_{i,p}, v_{i,q}\}$, with $i, j \in \{1, 2, ..., n\}$, $p \in \{1, ..., deg_G(v_i)\}$ and $q \in \{1, ..., deg_G(v_j)\}$. We use the term *inflation* for a graph that is isomorphic to the inflation of some graph.

Theorem 132. *[96] Let G be a cubic graph. Then G is $\frac{3}{2}$-tough if and only if $G = K_4$, $G = K_2 \times K_3$, or G is the inflation of a 3-connected cubic graph.*

An analogous characterization of r-regular $\frac{r}{2}$-tough graphs for $r \geq 1$ was conjectured by Goddard and Swart in [86]. This characterization, would also allow such graphs to be recognized in polynomial time. Altough $\frac{3}{2}$-tough cubic graphs are polynomial time recognizable, the situation is different in the case of 1-tough cubic graphs, as was shown by Bauer et al. in [24].

Theorem 133. *1-Tough remains NP-hard for cubic graphs.*

This result was generalized as follows :

Theorem 134. *[25] For any integer $t \geq 1$ and any fixed $r \geq 3t$, it is NP-hard to recognize r-regular, t-tough graphs.*

Bauer et al. also made the following conjecture in [25] :

Conjecture 15. *For any rational $t \geq 1$ and any fixed integer $r \geq 1$, t-Tough remains NP-hard for the class of r-regular graphs if and only if $r > 2t$.*

Conjecture 15 appears quite difficult, particularly when r is slightly larger than $2t$.

A number of graph classes have unknown complexity with respect to recognizing if they are t-tough. As Dillencourt pointed out in [63, 64], we still do not know the complexity of recognizing 1-tough planar graphs or 1-tough maximal planar graphs.

In the case of 2-connected graphs, we can take advantage of a result of Bauer et al. in [16], in order to give better lower bound for their circumference. The result states that 2-connected graphs with $circum(G) \in \{\sigma_2(G), \sigma_2(G)+1\}$, constitute a family $\mathcal{H}$ of eight easily-recognized classes of graphs. Note that by theorem 52, a 2-connected graph on $n \geq 3$ vertices satisfies $circum(G) \geq \min\{n, \sigma_2(G)\}$. Hence, the next result follows easily.

Theorem 135. *Let G be a 2-connected graph on $n \geq 3$ vertices. Then $circum(G) \geq \min\{n, \sigma_2(G)+2\}$ unless $G \in \mathcal{H}$.*

Note also that, by theorem 56, if a graph G is 1-tough graph on $n \geq 3$ vertices, then $circum(G) \geq \min\{n, \sigma_2(G)+2\}$. Hence, theorem 135 is an improvement of theorem 56.

In several rusults in hamiltonian graph theory, an NP-hard property of graphs implies an NP-hard cycle structure property. Two such theorems are the well-known theorems of Chvátal and Erdös in [59], and of Jung in [99]. In [58], Chvátal gave a proof of the Chvátal-Erdös theorem in [59], which given a graph G, it constructs in polynomial time either a Hamilton cycle in G, or and independent set of more than k vertices in G. A similar type of polynomial time constructive proof was given by Bauer et al. in [7] for Jung's theorem in [99] on graphs with at least 16 vertices.

Theorem 136. *[7] Let G be a graph on $n \geq 16$ vertices with $\sigma_2(G) \geq n-4$. Then we can construct in polynomial time either a Hamilton cycle in G or a set $X \subseteq V(G)$ with $c(G-X) > |X|$.*

2.11 Toughness and matchings

In this section we present some results involving matchings in graphs. We need some definitions first :

Definition 40. Given a graph $G = (V, E)$, a *matching* M in G is a set of pairwise disjoint edges (i.e no two edges share a common vertex). A vertex is *matched* if it is an endpoint of one of the edges in the matching. Otherwise the vertex is *unmatched*.

Definition 41. A *maximal matching* is a matching M of a graph G with the property that if any edge not in M is added to M, it is no longer a matching, (i.e M is maximal if it is not a proper subset of any other matching in graph G). A *maximum matching* is a matching that contains the largest possible number of edges. The *matching number*, denoted by $\nu(G)$, is the size of the maximum matching in G.

Definition 42. A *perfect matching* is a matching which matches all vertices of the graph (i.e every vertex of $V(G)$ is incident to exactly one edge of the matching).

Note that not all graphs have a perfect matching, as is shown by graphs K_3, K_5. Also notice that the edges of a specific perfect matching form a 1-factor of that graph.

Definition 43. Let m and n be positive integers with $m \leq \frac{n}{2} - 1$ and let G be a graph on n vertices with a perfect matching. A graph G is *m-extendable* if every matching of size m extends to a perfect matching.

Plummer investigated in [123] the relationship between the toughness of a graph and whether a given matching in a graph can be extended to a perfect matching. In [57] Chvátal, showed the following :

Theorem 137. *[57] Every 1-tough graph on an even number of vertices has a perfect matching.*

In [123] Plummer proved the following result :

Theorem 138. *[123] Suppose G is a graph on n vertices, with n even. Let m be a positive integer with $m \leq \frac{n}{2}-1$. If $\tau(G) > m$, then G is m-extendable. Moreover, the lower bound on $\tau(G)$ is tight for all m.*

An improvement of theorem 137 was shown by Lovász and Plummer, using the notion of elementary graphs.

Definition 44. A graph G is called *elementary* if G has a perfect matching, and if the edges of G which occur in a perfect matching of G induce a connected subgraph of G.

They proved the following result.

Theorem 139. *[113] Let G be a 1-tough graph on an even number of vertices. Then G is elementary.*

In the case of 1-tough graphs, Bauer et al. in [6] have established results involving the notion of factor-critical graphs, a term introduced in [113].

Definition 45. A graph G is called *factor-critical* if $G - v$ has a perfect matching, for all $v \in V(G)$.

Theorem 140. *[6] Let G be a 1-tough graph on an odd number of vertices. Then G is factor-critical.*

The size of a maximum matching can be found in polynomial time, using a result involving maximum Tutte sets.

Definition 46. Let $c_{odd}(G)$ denote the number of odd components of a graph G. A set $T \subseteq V(G)$ is called a *Tutte set* for G if

$$c_{odd}(G - T) - |T| = \max_{X \subseteq V(G)} \{c_{odd}(G - X) - |X|\}$$

A *maximum Tutte set* in a graph G is a Tutte set for G. The quantity $\max_{X \subseteq V(G)}\{c_{odd}(G - X) - |X|\}$ is called the *deficiency* of G, and it can be shown that it equals the number of vertices unmatched by a maximum matching in G.

An important result in matching theory is due to Tutte :

Theorem 141. *[133] A graph G has a perfect matching if and only if $c_{odd}(G - X) - |T| \leq |X|$ for all $X \subseteq V(G)$.*

In 1958, Berge extended Tutte's Theorem to give the exact size of a maximum matching in a graph G. This result reveals the importance of Tutte sets with respect to maximum matchings.

Theorem 142. *[29] Let G be a graph and T be a Tutte set for G. Then the matching number of G is given by*

$$\nu(G) = \frac{1}{2}(|V(G)| - (c_{odd}(G - T) - |T|))$$

In [6], it was shown that finding maximum Tutte sets in general graphs is NP-hard. However, the situation is different in the case of 1-tough graphs.

Theorem 143. *[113] Maximum Tutte sets can be found in polynomial time for the class of 1-tough graphs.*

2.12 Other toughness results

In this section we cover some remaining results on graph toughness. We begin with polyhedral graphs.

Definition 47. A graph is *polyhedral* if it is planar and 3-connected.

By a well-known theorem of Tutte (theorem 93), 4-connected planar graphs are hamiltonian. Hence, any nonhamiltonian planar graph G has $\kappa(G) \leq 3$ and since $\tau(G) \leq \frac{\kappa(G)}{2}$, G is at most $\frac{3}{2}$-tough. This bound is actually tight, since Harant in [92], constructed nonhamiltonian regular polyhedral graphs of degree 3,4 and 5 with maximum toughness $\frac{3}{2}$.

Definition 48. A planar graph G is called *maximal planar* graph if the addition of any edge to G results in a nonplanar graph.

An alternative way to define a maximal planar graphs is as the triangulation of a planar graph. In [91], Harant and Owens constructed nonhamiltonian maximal planar graphs with toughness $\frac{5}{4}$. This result was improved by Owens in [120], where he constructed nonhamiltonian maximal planar graphs with toughness $\frac{3}{2} - \epsilon$, for any $\epsilon > 0$, without a 2-factor.

Theorem 144. *[120] For any $\epsilon > 0$, there exist $(\frac{3}{2} - \epsilon)$-tough nonhamiltonian planar graphs.*

By corollary 22, every $(\frac{3}{2} + \epsilon)$-tough planar graph is hamiltonian. A question that comes natural, is whether $\frac{3}{2}$-tough planar and maximal planar graphs are hamiltonian or if they even contain a 2-factor.

Chvátal had raised the question whether there exist 1-tough nonhamiltonian maximal planar graphs. The answer came from Nishizeki in [118], where he constructed a nonhamiltonian 1-tough maximal planar graph on 19 vertices. Later, Dillencourt in [63] found such a graph with 15 vertices. The best such result was due to Tkáč in [132]. He found a nonhamiltonian 1-tough maximal planar graph on 13 vertices and showed that there is no such graph with fewer vertices.

In [79], Ferland investigated the toughness of generalized Petersen graphs.

Definition 49. For each $n \geq 3$ and $0 < k < n$, the *generalized Petersen graph* $G(n,k)$ has vertex set $V = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ and edge set $E = \{\{u_i, u_{i+1}\} \mid 1 \leq i \leq n\} \cup \{\{u_i, v_i\} \mid 1 \leq 1 \leq n\} \cup \{\{v_i, v_{i+k}\} \mid 1 \leq i \leq n\}$, where all the indices are modulo n.

The Petersen graph then is $G(5,2)$. Ferland in [79, 78] was interested in bounds on the toughness of $G(n,k)$. In particular, he was interested in asymptotic bounds for $\tau(G(n,k))$. He defined a real number to be an *asymptotic upper bound* for $\tau(G(n,k))$ if $\lim_{n\to\infty} \tau(G(n,k)) \leq b$. Similarly, we can define asymptotic lower bounds. In [121], the value of $\tau(G(n,1))$ was determined, and $\tau(G(n,1))$ has an asymptotic upper bound of 1. In [78], it was found that $\frac{5}{4}$ was both a lower and upper asymptotic bound for $\tau(G(n,2))$. In [78], they were given upper and lower asymptotic bounds for $\tau(G(n,k))$ for $n \geq 3$ and $0 < k < n$.

2.13 Conclusions

The research on toughness has its origins in a work of Chvátal in 1973 ([57]). A great part of this research concerns the relation between toughness and the extistence of cycle structures. This research was motivated by conjectures in [57], which if true, would have a number of implications, as other conjectures would also be true. The most intriguing of Chvátal's conjectures, which is still open, asks if there exists a finite t_0 such that every t_0-tough graph is hamiltonian.

If the conjecture is true, then by theorem 100, $t_0 \geq \frac{9}{4}$. Although the question is open for general graphs, we now know of graph classes for which the conjecture holds. In this thesis, we have presented results showing that the conjecture holds for claw-free graphs, planar graphs and chordal graphs. In particular, we know that for chordal graphs $t_0 \leq 18$ (theorem 46) and that $t_0 \geq \frac{7}{4}$ (theorem 104), for claw-free graphs $t_0 \leq \frac{7}{2}$ (corollary 14) and for planar graphs $t_0 > \frac{3}{2}$ (corollary 22), which is the best we know. We remind the reader that there exist nonhamiltonian $(\frac{3}{2} - \epsilon)$-tough maximal planar graphs (theorem 144). It is still open whether all $\frac{7}{4}$-tough chordal graphs are hamiltonian, and is open even for 2-tough chordal graphs. The t_0-tough conjecture is open in the case of triangle-free graphs. We know that if it were true, then $t_0 > \frac{5}{4}$ (theorem 47). Finally, we now know that Chvátal's t_0-tough conjecture is true within the class of graphs on n vertices satisfying $\delta(G) \geq \epsilon n$, for any fixed $\epsilon > 0$ (theorem 75).

Another direction in characterizing hamiltonian graphs is by imposing minimum degree conditions, combined with a specified level of toughness. By corollary 18, every 2-tough graph G on n vertices with $\delta(G) \geq \frac{n}{3}$ is hamiltonian. In the disproof of the 2-tough conjecture, they were used graphs having minimum degree 4. So, what is the minimum degree condition in $5 \leq \delta(G) \leq \frac{n}{3}$ for which we can guarantee hamiltonicity for 2-tough graphs? This is another, yet unresolved problem. Similarly, it is intriguing to find the minimum degree condition to ensure that 1-tough triangle-free graphs are hamiltonian. Theorems 31, 34 suggest that the best possible minimum degree guaranteeing that a 1-tough triangle free graph is hamiltonian lies somewhere between $\frac{n+2}{4}$ and $\frac{n}{3}$.

An area that has drawn a lot of research is about finding toughness conditions for the existence of certain factors in graphs. A problem of major interest is to determine whether every $\frac{3}{2}$-tough maximal planar graph has a 2-factor, and if it holds, whether they are hamiltonian. This problem is still open, and so is it for planar graphs.

On the computational complexity part of the research, we now know that it recognizing t-tough graphs is NP-hard for general graphs. However, the same problem on claw-free graphs and split graphs has polynomial time complexity. The complexity of recognizing t-tough graphs is unknown for planar graphs, maximal planar graphs and chordal graphs, just to mention a few. In the case of r-regular graphs G with $r \geq 3\tau(G)$, it has been shown that recognizing t-tough graphs is NP-hard, but for $r < 2t$ the problem has polynomial time complexity. It is still open for $2t \leq r < 3t$, and has drawn a lot of interest in the case where $r = 2t + 1$.

3 On the relation between Treewidth and Toughness

3.1 Relating Treewidth to Toughness

In this section we present some of our own results concerning the relationship between the treewidth of a graph G, $tw(G)$, and its toughness, $\tau(G)$. Researching on this topic was mostly motivated by a vague intuition that big toughness obstructs small treewidth. As we shall see, this is indeed the case, as treewidth bounds toughness from above. In particular, we will show that $\tau(G) \leq \frac{tw(G)}{2}$ for any graph G. Moreover, we will present examples of graphs for which this relation is also satisfied by equality. It is reasonable to ask, how close are treewidth and tougness in general graphs. It turns out that one can construct an infinite sequence of graphs, $G_1, G_2, ...$, such that $tw(G_n) \to \infty$ and $\tau(G_n) \to 0$ as $n \to \infty$. Hence there exist graphs having arbitrarily large treewidth and arbitrarily small toughness. We also present some generalizations of the initial upper bound.

We begin with some lemmata bounding toughness from above, using the separation properties of tree decompositions. The following lemma uses the edge separation property of tree decompositions.

Lemma 24. *Let G be a noncomplete graph and $(\{X_i \mid i \in I\}, T = (I, F))$ be a nonredundant tree decomposition of G with at least two nodes. Then*

$$\tau(G) \leq \min_{\{i,j\} \in F} \frac{|X_i \cap X_j|}{2}$$

Proof. Let $i, j \in I$ be any two adjacent nodes of T. Then by lemma 3, the vertex set $X_i \cap X_j$ is a cutset of G. Thus, $\tau(G) \leq \frac{|X_i \cap X_j|}{2}$. □

The next lemma shows a bound on toughness using the vertex separation property of tree decompositions,

Lemma 25. *Let G be a noncomplete graph and $(\{X_i \mid i \in I\}, T = (I, F))$ be a nonredundant tree decomposition of G with at least two nodes. Then*

$$\tau(G) \leq \min_{i \in I} \frac{|X_i|}{deg_T(i)} \tag{1}$$

Proof. Let $i \in I$ be a node of the tree decomposition. If $deg_T(i) = 1$, let j be its adjacent node. Then by lemma 24, $\tau(G) \leq \frac{|X_i \cap X_j|}{2}$, and since $|X_i \cap X_j| \leq |X_i|$, $\tau(G) \leq \frac{|X_i|}{2} \leq |X_i|$. Suppose now that $deg_T(i) > 1$. Then by corollary 1, X_i is a cutset of G. In particular, $G - X_i$ has components $G - V_{T_1}, ..., G - V_{T_{deg_T(i)}}$, where $T_1, ...T_{deg_T(i)}$ are the components of $T - \{i\}$. Hence, $c(G - X_i) = c(T - \{i\}) = deg_T(i)$ and consequently $\tau(G) \leq \frac{|X_i|}{deg_T(i)}$. □

The following theorem is one of our main results. It shows that the toughness of a graph can be at most half its treewidth.

Theorem 145. *Let G be a noncomplete graph. Then*

$$\tau(G) \leq \frac{tw(G)}{2} \tag{2}$$

Proof. Let $(\{X_i \mid i \in I\}, T = (I, F))$ be an optimal tree decomposition of G (i.e $\max_{i \in I} X_i - 1 = tw(G)$). Also assume w.l.o.g that there do not exist $i, j \in I$ such that $\{i, j\} \in F$ and $X_i \subseteq X_j$, that is the tree decomposition is nonredundant. If there are such nodes, we can run algorithm 1 and obtain an optimal nonredundant tree decomposition with the desired property. (note that algorithm 1 does not increase the size of the bags).

First of all, notice that since G is noncomplete then by lemma 8, $tw(G) \leq n-2$ and consequently the tree decomposition must have at least two nodes (otherwise the maximum bag would have n elements). By lemma 24, $\tau(G) \leq \min_{\{i,j\} \in F} \frac{|X_i \cap X_j|}{2}$. Furthermore it holds that $|X_i \cap X_j| \leq \max_{i \in I} |X_i| - 1 = tw(G)$ (since none of X_i, X_j is a subset of the other, they must have an element not in their intersection). Hence, $\tau(G) \leq \min_{\{i,j\} \in F} \frac{|X_i \cap X_j|}{2} \leq \frac{\max_{i \in I} |X_i| - 1}{2} = \frac{tw(G)}{2}$. □

A natural question is whether there exists a graph G for which relation 2 is satisfied by equality. It turns out that it does. Consider the graph G shown in figure 8. It is easy to verify that any cutset of this graph has exactly 4 vertices, whereas the 2 remaining vertices are the components that arise after the removal of the corresponding cutset. Alternatively, we can use the known bound $\frac{\kappa(G)}{\alpha(G)} \leq \tau(G) \leq \frac{n-\alpha(G)}{\alpha(G)}$ (theorem 11) and observe that in this case $n = 6$, $\kappa(G) = 4$ and $\alpha(G) = 2$. We could also use the bound $\frac{\kappa(G)}{\alpha(G)} \leq \tau(G) \leq \frac{\kappa(G)}{2}$ and compute the same value for $\tau(G)$. Thus $\tau(G) = 2$. Now, set $X_1 = \{1, 2, 3, 5, 6\}$ and $X_2 = \{1, 3, 4, 5, 6\}$. It is also easy to verify that the $(\{X_1, X_2\}, T = (\{1, 2\}, \{\{1, 2\}\}))$ is a valid tree decomposition for the same graph. Actually, this is an optimal tree decomposition, as there can be no tree decomposition with smaller maximum bag (notice also that we must have $tw(G) \geq 2\tau(G) = 4$). Therefore, $tw(G) = 4$ and this graph becomes an exaple in which $\tau(G) = \frac{tw(G)}{2}$. So the upper bound in theorem 145 is tight.

Lemma 25 has an interesting corollary.

Corollary 29. *Let G be a noncomplete graph and let $(\{X_i \mid i \in I\}, T = (I, F))$ be a nonredundant optimal tree decomposition of G. Then*

$$\tau(G) \leq \frac{tw(G) + 1}{\Delta(T)} \tag{3}$$

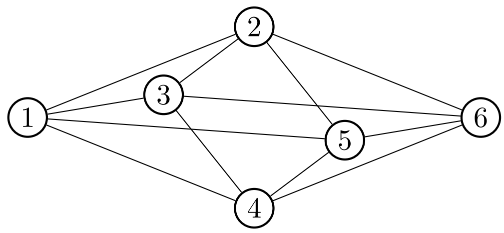

Figure 8: Graph G.

Proof. Since G is noncomplete and the tree decomposition is optimal, it must have at least two nodes. Then by lemma 25 (and using the nonredundancy of the tree decomposition), $\tau(G) \leq \min_{i\in I} \frac{|X_i|}{deg_T(i)}$. Let i^* be any node with $deg_T(i^*) = \Delta(T)$. Then $\tau(G) \leq \min_{i\in I} \frac{|X_i|}{deg_T(i)} \leq \frac{|X_{i^*}|}{\Delta(T)} \leq \frac{\max_{i\in I}|X_i|}{\Delta(T)}$. Consequently, $\tau(G) \leq \frac{tw(G)+1}{\Delta(T)}$. $\square$

It is easy to verify that when $tw(G), \Delta(T) \geq 3$, we can obtain much better bounds by using corollary 29 instead of theorem 145. Moreover, we can choose an optimal tree decomposition that has the greatest maximum degree to obtain even tighter upper bounds.

We have presented upper bounds on graph toughness as a function of treewidth. A reasonable question is whether there exist lower bounds on toughness as a function of treewidth. The next result shows that there is no such bound for general graphs.

Theorem 146. *There exists a sequence of graphs $G_1, G_2, ...$ such that $\tau(G_n) \to 0$ and $tw(G_n) \to \infty$ as $n \to \infty$.*

Proof. Let k, m be integers with $k, m > 1$ and let $G_{k,m}$ be the following graph : Take m disjoint copies of K_k, $H_1, ..., H_m$, and let $v_1 \in H_1, ..., v_m \in H_m$.

Finally let v be a new vertex and connect v with $v_1, ...v_m$. So $G_{k,m}$ consists of a vertex that is connected to disjoint cliques $H_1, ..., H_m$ of the same size k, only through vertices $v_1 \in H_1, ..., v_m \in H_m$. Now, each H_i has treewidth $k-1$ and since H_i is a subgraph of $G_{k,m}$, we get $tw(G_{k,m}) \geq k-1$. Removing v from $G_{k,m}$ results in a disconnected graph with m components (namely the graphs $H_1, ..., H_m$). Hence, $\tau(G_{k,m}) \leq \frac{1}{m}$. Therefore $tw(G_{k,m}) \to \infty$ as $k \to \infty$ and $\tau(G_{k,m}) \to 0$ as $m \to \infty$.

$\square$

The proof of theorem 146 shows that we can find graphs having arbitrarily large treewidth and arbitrarily small toughness.

Consider now the graph $G_{k,m}$ of theorem 146. Graph $G_{k,m}$ is chordal with $\omega(G_{k,m}) = k$. Hence $tw(G_{k,m}) = k - 1$. Since $\tau(G_{k,m}) \leq \frac{1}{m}$,we conclude that for any fixed value of k we can find a graph with treewidth k and arbitrarily small toughness (namely graph $G_{k+1,m}$).

3.2 Conclusions

In this section we have presented upper bounds on graph toughness as a function of treewidth. For general graphs, the bound $\tau(G) \leq \frac{tw(G)}{2}$ is tight, as we have presented an example in which the bound is also satisfied by equality. We have also shown that for the more restricted class of graphs G that satisfy $tw(G), \Delta(T) \geq 3$, where T is the tree of an optimal tree decomposition of G, the better bound $\tau(G) \leq \frac{tw(G)+1}{\Delta(T)}$ can be achieved. Both bounds establish the fact that large toughness obstructs small treewidth. Finally, we constructed a sequence of graphs G_n that satisfy $tw(G_n) \to \infty$ and $\tau(G_n) \to 0$ as $n \to \infty$, showing that we can find graphs that have arbitrarily large treewidth and at the same time, arbitrarily small toughness.

However it could be the case that these examples are pathological, and toughness could be closer to treewidth in most graphs. It would be interesting to examine the relation between treewidth and toughness in special graph classes. We know that the treewidth of the $(n \times n)$-grid is n, whereas its toughness is at most 1 (asymptotically is exactly 1). Thus, even in the class of planar graphs we can find examples of graphs that have bounded toughness and unbounded treewidth. The $(n \times n)$-grid is also triangle-free, hence the same holds for this class of graphs too. It would also be interesting to examine what graphs achieve equality in the bound $\tau(G) \leq \frac{tw(G)}{2}$ (we know that they must satisfy $tw(G) < 3$ or $\Delta(T) < 3$ for any optimal tree decomposition with T as tree).

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