Large cardinals and elementary embeddings of V

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To the memory of my parents, Ioannis Koulakis and
Anastasia Chrysochoidou

Contents

Introduction

This thesis is a short survey of the theory of large cardinals. The notion of elementary embeddings is crucial in this study, since it is used to provide a quite general way of defining large cardinals. The survey concludes with a reference to Kunen's theorem, which sets some limitations to the existence of specific elementary embeddings, and an attempt of P. Corazza to surmount those limitations. Most of the material used through the survey is defined in chapter 1, thus the text is accessible to all readers having knowledge of basic set theory and logic.

Chapter 1 serves as an extensive introduction to some notions needed for the rest of the text. Mostowski's collapsing theorem provides us with a way of transforming a lot of models of set theory, into standard transitive ones. Some results concerning Lévi's hierarchy illuminate the notion of absoluteness, i.e. the case in which a formula or a term behaves the same way in V as in a standard transitive model. The important cumulative hierarchies V_α and L_α are introduced, along with some of their basic properties. Finally, there is a quick reference to some model theoretical constructions we will use later on.

On chapter 2, the first large cardinals appear, though they are not strong enough to yield a connection with elementary embeddings. The first section of this chapter is somehow a continuation of the first chapter, as it provides some fundamental knowledge on cardinal arithmetic. Afterwards, we define inaccessible cardinals, then go on to Mahlo cardinals and finally introduce weakly compact and Erdős cardinals. There is an additional reference to Ramsey cardinals, which have a combinatorial nature similar to that of Erdős cardinals, even though they are essentially stronger than the other large cardinals we have mentioned here.

On chapter 3 things seem to get more interesting, as we introduce measurable cardinals. They emerge in a very natural way from measure theory and their study moves from analysis to set theory. They are connected with the elementary embeddings of the form $j : V \prec M$, and out of this connection we get the result, due to Scott, that if there is a measurable cardinal, then $V \neq L$. Concluding, we define normal measures, which become invaluable in the study of elementary embeddings.

Chapter 4 goes a little bit backwards, since we study a weaker large cardinal hypothesis, the existence of $0^\sharp$. Some effort is needed in order to define $0^\sharp$ but it pays back, as $0^\sharp$ is very close to the critical question of whether L is a good approach of V. Silver's theorem and Jensen's covering theorem give a more than satisfactory answer to this question which had been roughly approached by measuble and Ramsey cardinals. We finally see, through a theorem of Kunen, that $0^\sharp$ is connected with the elementary embeddings of the form $j : L \prec L$.

Chapter 5 goes on from the point where chapter 3 ended, as it presents the use of elementary embeddings of the form $j : V \prec M$, in order to obtain stronger large cardinals. This way, strong, Woodin, strongly compact, supercompact and extendible cardinals appear. We come up with even stronger hypotheses, such as Vopěnka's principle, the existence of huge cardinals and I0-I3. The latter, seem to lie near the existence of an embedding $j : V \prec V$, which cannot exist due to Kunen's result, thus there is a probability that they could be disproved from ZFC. Finally, we examine the case of an elementary embedding $j : V \prec V$. Kunen's theorem states that there is not such an embedding. In the process, we examine the consequences of the wholeness axiom, proposed by Corazza, which asserts the existence of an embedding $j : V \prec V$ that is not weakly definable. The wholeness axiom may weaken the hypotheses of Kunen's result, but it implies the existence of a super-n-huge cardinal, thus it is very near inconsistency and it should be further investigated.

Chapter 1

Preliminaries

In this chapter, we present some basic notions of set theory and model theory which will be necessary for our study. On the same time, we try to give some motives for the introduction of large cardinals. Our first step is to present the well known axiomatic system ZFC. Most of the axioms are presented as the closure of V under certain operations, in order to provide a more vivid image of the structure of V. Afterwards, the most powerful of those operations, the powerset operation, is used in order to introduce the notion of cumulative hierarchies. Cumulative hierarchies are roughly sequences of levels U_α, which expand, by the addition of more complex sets, as α gets bigger. Two very common hierarchies are then introduced; von Newmann's hierarchy V_α, and the hierarchy of the constructible sets L_α. Moving towards a model theoretical study of set theory, Mostowski's collapsing theorem becomes very useful, as it gives us the opportunity to work, in almost all cases studied here, with standard transitive models. Since standard models share the same relation with V, we are interested in the cases where some formulas have the same truth value in a stander transitive model as in V. The notion that expresses those cases is absoluteness, and an easy way to study it is the introduction of Lévi's hierarchy of formulas. Next, we meet again the hierarchies V_α, L_α and this time prove some of their basic properties. Finally, we state some model theoretic concepts, such as Skolem functions, ultraproducts and reflection principles.

For a more extensive treatment of the material covered in this chapter, the reader is referred to [7], [11] and [21]. For an elementary introduction to set theory, he is referred to [15] and [12]. A classical book on model theory is [2] and much more information on ultraproducts can be found in [1].

1.1 Cumulative hierarchies

The axioms of ZFC

The first two axioms describe the nature of sets:

- **A1. Extensionality axiom**

$$\forall z\ (z \in x \leftrightarrow z \in y) \rightarrow x = y$$

- **A2. Foundation axiom**

$$\exists y\ y \in x \rightarrow \exists y \in x\ \forall z \in x\ z \notin y$$

The next 7 axioms express that the universe V is closed under certain operations, and are written in the form

$$x \in V \rightarrow Q(x) \in V$$

where Q denotes an operation. These are the following:

- **A3$_\phi$. Separation axiom**

$$x \in V \rightarrow \{y \in x : \phi(y)\} \in V$$

- **A4. Empty set axiom**

$$\rightarrow \emptyset \in V$$

- **A5. Pairing axiom**

$$x, y \in V \rightarrow \{x, y\} \in V$$

- **A6. Power set axiom**

$$x \in V \rightarrow P(x) \in V$$

- **A7. Union axiom**

$$x \in V \rightarrow \cup x \in V$$

- **A8. Infinity axiom** (inf)

$$\emptyset \in V \rightarrow \omega \in V$$

- **A9$_\phi$. Replacement axiom** (F)

$$x \in V \rightarrow F''_\phi(x) \in V$$

  Where $F''_\phi(x) = y \leftrightarrow \forall z\ (z \in y \leftrightarrow \exists w \in x\ \phi(w, z))$

  for ϕ which define functions, i.e. $\forall x\ \exists! y\ \phi(x, y)$.

  Finally, the tenth axiom, the axiom of choice, demands the existence of a well-ordering for every set.

- **A10. Choice axiom** (AC)

$$\forall x\ \exists y\ y\ \textit{wellorders}\ x$$

An immediate consequence of the axiom of foundation is the fact that there can not exist infinite $\ni$ sequences. If we combine this with the axiom of extensionality, which states that every set is defined exactly by its elements, we can imagine a construction of V, beginning with the empty set and recursively moving to new sets using axioms A4-A10. We could also add new sets which do not appear in this construction, but may be contained in sets we have already been constructed.

We should note here that by $ZFC - Ai$ we mean the axiom system ZFC without the axiom Ai, where $1 \leq i \leq 10$. We also denote, $ZF = ZFC - AC$ and $Z = ZF - F$.

Cumulative hierarchies

The above approach is quite vivid and reveals the structure that V should have, in order to satisfy ZFC. Unfortunately, it is quite complicated, especially when dealing with sets of infinite cardinality. What could be done to simplify it, is to begin with the empty set and produce an $\subset$ sequence $\langle U_\alpha : \alpha \in On\rangle$ of sets with increasing complexity, relative to $\in$. Using this method, we approach V by constructing bigger and bigger sets of its elements instead of adding them one by one. This idea is captured by the notion of cumulative hierarchies.

Definition 1.1. A transfinite sequence $\langle U_\alpha : \alpha \in On\rangle$ is called a *cumulative hierarchy* if the following are true:

(i) $U_0 = \emptyset$;

(ii) $U_\alpha \subset U_{\alpha+1} \subset P(U_\alpha)$;

(iii) $U_\alpha = \bigcup_{\beta<\alpha} U_\beta,\ limit(\alpha)$.

The sets U_α of the hierarchy are all transitive.

Reaching the end of this section, we are going to introduce two important cumulative hierarchies and some of their trivial properties. More information about them will be unveiled in section 1.4 where we are going to treat them as models of ZFC. The fact that they are indeed cumulative hierarchies can be easily checked in all cases using transfinite induction.

Von Neumann's hierarchy

Definition 1.2. We call *von Neumann's hierarchy* the transfinite sequence $\langle V_\alpha : \alpha \in On\rangle$ with the following properties:

(i) $V_0 = \emptyset$;

(ii) $V_{\alpha+1} = P(V_\alpha)$;

(iii) $V_\alpha = \bigcup_{\beta<\alpha} V_\beta,\ limit(\alpha)$.

Von Neumann's universe is the class $R = \bigcup\{V_\alpha : \alpha \in On\}$.

- $R = V$. The power set operation turns out to be so powerful that in ZF every set x belongs to $V_{\alpha+1}$ for some $\alpha \in On$. The least α satisfying this property is called the rank of x and is written as $rank(x)$ (We will use the symbolization $rank_U(x)$ for the same notion concerning another hierarchy U).

- For every ordinal α, $rank(\alpha) = \alpha$.

- $|V_{\omega+\alpha}| = \beth_\alpha$ and for every $n \in \omega$, V_n is finite[1].

The hierarchy of constructible sets

In order to introduce constructible sets, we first need the notion of definability.

Definition 1.3.

(i) Suppose M is a model and $X, I \subset M$. X is *definable in M from I*, if there is a formula $\phi(v, \bar{u})$ and a n-tuple $\bar{a}$ of I, such that:

$$\forall x \in M\ (x \in X \leftrightarrow M \vDash \phi[x, \bar{a}]).$$

In the case $I = M$ we say that X is *definable over M*. If $x \in M$, then x is definable over M (or in M from I) if $\{x\}$ is respectively definable over M (or in M from I).

(ii) $def_M(I) = \{X \subset M : X\ definable\ in\ M\ from\ I\}$
$def(M) = \{X \subset M : X\ definable\ over\ M\}$.

Definition 1.4. The hierarchy $\langle L_\alpha : \alpha \in On\rangle$ of the *constructible sets* is the one which satisfies the following:

(i) $L_0 = \emptyset$;

(ii) $L_{\alpha+1} = def(L_\alpha)$;

(iii) $L_\alpha = \bigcup_{\beta<\alpha} L_\beta,\ limit(\alpha)$.

The universe of constructible sets is $L = \bigcup\{L_\alpha : \alpha \in On\}$.

- $\bigcup_{\alpha\in On} L_\alpha$ does not necessarily contain all sets. Moreover it cannot be proved within ZFC weather $V = L$ or not. An interesting result to come is that this question is tightly connected with the existence of $0^\sharp$, which is a large cardinal property.

[1]The sequence $\beth_\alpha$ is recursively defined as follows:

$$\beth_0 = \aleph_0$$
$$\beth_{\alpha+1} = 2^{\beth_\alpha}$$
$$\beth_\alpha = \bigcup_{\beta<\alpha} \beth_\beta,\ limit(\alpha)$$

- For every ordinal α, $rank_L(\alpha) = \alpha$.
- $|L_\alpha| = |\alpha|$.
- $L_\omega = V_\omega$ and $L_n = V_n$ for every $n \in \omega$.

1.2 The Mostowski collapse

In the previous section we introduced cumulative hierarchies in order to get a grasp of how a model of ZFC should look like. In general, a model of set theory is a couple (M, E) where M is a class and E is a binary relation interpreting $\in$. The models of ZFC may be quite complicated but we can reduce many of them to simpler ones using Mostowski's theorem. This theorem states that, under certain assumptions, an E structure is isomorphic to a transitive $\in$ class.

We give a definition concerning relations and afterwards the proof of Mostowski's theorem. The reason for this is to illustrate the use of transfinite recursion and provide a clearer image of the connection of some E relations to $\in$.

Definition 1.5. Suppose (N, E) is a structure of the language $\{\in\}$.

(i) The *E-extension* of $x \in N$ is the class $ext_E(x) = \{y \in N : yEx\}$.

(ii) E is *well-founded* on N if every non empty subset of N has an E-minimal element.

(iii) E is *set-like* if for all $x \in N$, $ext_E(x)$ is a set.

(iv) E is extensional relation on N if

$$\forall x, y \in N \ (ext_E(x) = ext_E(y) \to x = y)$$

Theorem 1.6. ***(Mostowski's collapsing theorem)***

(i) If E is well-founded set-like and extensional on N, then there exists a transitive class M and an isomorphism F between (N, E) and $(M, \in)$.

(ii) M and F are unique.

Proof. (i) We define F by well-founded recursion so that

$$F(x) = F''(ext_E(x)) = \{F(y) : yEx\}.$$

This is possible because E is set-like so $ext_E(x)$ is a set, which means by the axiom of replacement that $\{F(y) : yEx\}$ is also a set, and additionally E is well-founded. Let $M = F''(N)$.

- M is transitive:

$$y' \in x' \in M \to \exists x \in N \ x' = F(x)$$

and

$$F(x) = \{F(z) : zEx\} \to \exists yEx \ y' = F(y) \to y' \in M.$$

– F is 1-1: If it wasn't, there would exist an x', of least rank, with the property $x' = F(x) = F(y)$ for some $y \neq x$.

$$y \neq x \rightarrow ext_E(x) \neq ext_E(y) \rightarrow \exists zEx\ \neg zEy$$

but since $F(x) = F(y)$

$$\exists uEy\ F(z) = F(u) \rightarrow z = u \rightarrow zEy$$

which is a contradiction.

– $yEx \leftrightarrow F(y) \in F(x)$: If yEx, then $F(y) \in F''(ext_E(x)) = F(x)$. Additionally, if $F(y) \in F(x)$, then $\exists z \in ext_E(x)\ F(z) = F(y)$ and since F is 1-1 we have $z = y$ which means that yEx.

(ii) Let F be the above isomorphism and G another one having K as its range. If $x \in N$ is an element of least rank for which $F(x) \neq G(x)$, then since G is an isomorphism

$$G(x) = \{G(y) : yEx\} = \{F(y) : yEx\} = F(x)$$

which can not be true. As a consequence of this

$$F = G \rightarrow K = G''N = F''N = M$$

so $K = M$.

□

In the cases we are interested in, (M, E) will be a model of the axiom of extensionality. Using the following lemma we will show that if this is true, then it is enough to check that $ext_E(x)$ is a set for every $x \in M$ and that E is well-founded, in order to apply Mostowski's theorem.

Lemma 1.7. *E is extensional on M iff $(M, E) \vDash$ "extensionality axiom".*

Proof. $M \vDash$ "*extensionality axiom*" is equivalent to

$$\begin{aligned} &M \vDash \forall x, y\ (\forall z \in x\ z \in y \wedge \forall z \in y\ z \in x \rightarrow x = y) \leftrightarrow \\ \leftrightarrow &\forall x, y \in M\ (\forall z \in x\ z \in y \wedge \forall z \in y\ z \in x \rightarrow x = y)^M \\ \leftrightarrow &\forall x, y \in M\ (\forall zEx\ zEy \wedge \forall zEy\ zEx \rightarrow x = y) \end{aligned}$$

which is equivalent to the extensionality of M. □

1.3 Absoluteness

As we have mentioned, we will only work with transitive models $(M, \in)$, where M can either be a set or a class[2]. Since the interpretation of $\in$ coincides with $\in$,

[2]Here one should be careful using the relation $\vDash$, which is definable in ZFC for models that are sets, but not for models that are classes.

it would be interesting to study the cases where $(\phi)^M$ is equivalent to ϕ for all valuations of ϕ. Those formulas are called *absolute for* M. If moreover, one of them is absolute for every transitive model M, it is simply called *absolute*. Our aim in this section is to present methods of proving that a formula is absolute, and using them provide a list of absolute formulas.

Absoluteness

Definition 1.8. Let M and N be two structures such that $M \subset N$:

(i) A formula $\phi(\bar{x})$ is *preserved under the extension (restriction) from* M *to* N *(from* N *to* M*)* if

$$\forall \bar{a} \in M^n (M \vDash \phi[\bar{a}] \rightarrow N \vDash \phi[\bar{a}])$$

$$[\forall \bar{a} \in M^n (N \vDash \phi[\bar{a}] \rightarrow M \vDash \phi[\bar{a}])].$$

(ii) A fomula $\phi(\bar{x})$ is *absolute for* M, N if it is preserved both under the extension from M to N and the restriction from N to M.

(iii) A term $t(\bar{x})$ is *absolute for* M, N if $y = t(\bar{x})$ is absolute for M, N.

(iv) M is an *elementary substructure* of N, $M \prec N$, if every formula is absolute for M, N.

(v) We will call a formula ϕ *absolute for a theory* T if, for every model M of T, ϕ is absolute for M, V.

Lévy's hierarchy

One way we can show that a formula is absolute is, by checking its position in Lévy's hierarchy. We will give a brief definition of it and a theorem which explains the link between absoluteness and the first levels of this hierarchy.

Definition 1.9. Let T be a theory. *Lévy's hierarchy* and *Lévy's hierarchy relative to* T are defined as follows:

(i)
- $\Delta_0 = \Sigma_0 = \Pi_0$ is the set of all formulas with bounded quantifiers (of the form $\exists x \in y$ or $\forall x \in y$).
- $\Sigma_{n+1} = \{\exists x\ \phi(x) : \phi \in \Pi_n\}$.
- $\Pi_{n+1} = \{\forall x\ \phi(x) : \phi \in \Sigma_n\}$.

(ii)
- $\phi \in \Sigma_n^T$ iff $\exists \psi \in \Sigma_n\ T \vdash \phi \leftrightarrow \psi$.
- $\phi \in \Pi_n^T$ iff $\exists \psi \in \Pi_n\ T \vdash \phi \leftrightarrow \psi$.
- $\Delta_n^T = \Sigma_n^T \cap \Pi_n^T$.

Theorem 1.10. *Let* $M \subset N$ *be transitive models of* T*, then:*

(i) If $T \vdash \phi \leftrightarrow \psi$ *then* ψ *is absolute for* M, N *iff* ϕ *is absolute for* M, N.

(ii) *Every* Δ_0^T *sentence is absolute for* M, N.

(iii) *Every* Σ_1^T *sentence is preserved under the extension from* M *to* N.

(iv) *Every* Π_1^T *sentence is preserved under the restriction from* N *to* M.

(v) *Every* Δ_1^T *sentence is absolute for* M, N.

In order to use the above theorem effectively, the lemma below is very useful. It helps us classify formulas in Lévy's hierarchy.

Lemma 1.11.

(i) *Logical connectives (for* $T \vdash Z - inf$*)*

- *If* ϕ *and* ψ *are both* Σ_n^T *or* Π_n^T, *then so are* $\phi \wedge \psi$ *and* $\phi \vee \psi$.
- *If* $\phi \in \Sigma_n^T$ *and* $\psi \in \Pi_n^T$ *then* $\phi, \psi \in \Delta_{n+1}^T$, $\phi \to \psi \in \Pi_n^T$ *and* $\psi \to \phi \in \Sigma_n^T$.
- $\phi \in \Sigma_n^T \to \neg\phi \in \Pi_n^T$ *and* $\phi \in \Pi_n^T \to \neg\phi \in \Sigma_n^T$.

(ii) *Quantifiers (for* $T \vdash Z - inf$*)*

- $\phi \in \Sigma_n^T \to \forall x\ \phi \in \Pi_n^T$
- $\phi \in \Pi_n^T \to \exists x\ \phi \in \Sigma_n^T$
- *If* $n > 0$ *then* $\phi \in \Sigma_n^T \to \exists x\ \phi \in \Sigma_n^T$ *and* $\phi \in \Pi_n^T \to \forall x\ \phi \in \Pi_n^T$.

(iii) *Bounded quantifiers (for* $T \vdash ZF$*)*
If ϕ *is* Σ_n^T *or* Π_n^T *then so are* $\exists x \in y\ \phi$ *and* $\forall x \in y\ \phi$.

(iv) *Terms (if* $n > 0$ *and* $T \vdash ZF$*)*

- *If* $\phi(x) \in \Delta_n^T$ *then* $\{x : \phi(x)\} \in \Pi_n^T$ *and* $\{x \in z : \phi(x)\} \in \Delta_n^T$.
- *If* $t \in \Sigma_n^T$ *and* $T \vdash \exists x\ x = t$ *then* $t \in \Delta_n^T$.
- *Suppose* $T \vdash \exists x\ x = t$. *Then if* $\phi(y)$, $s(y)$ *and* t *are* Δ_n^T *then so are* $\phi(t)$, $s(t)$, $\exists x \in t\ \phi$, $\forall x \in t\ \phi$ *and* $\{x \in t : \phi\}$.

Now that we are equipped with theorem 1.10 and lemma 1.11 we are ready to give a list of terms and formulas in set theory which are Δ_1^{ZF} and thus absolute. The notions described by those terms and formulas, such as the ordinals, tend to be easier in their manipulation. In contrast, notions which are not Δ_1^{ZF}, such as cardinals or the power set operation, are more vague and a variety of hard (many times unsolvable in ZFC) problems arise.

Lemma 1.12.

(i) The following terms and formulas are Δ_0^{ZF} *thus absolute:*

(1) $x \in y$	(9) $x \setminus y$	(17) $dom(R)$	(25) $succord(x)$
(2) $x = y$	(10) $s(x)$	(18) $rang(R)$	(26) $finord(x)$
(3) $x \subset y$	(11) $trans(x)$	(19) $funct(f)$	(27) ω
(4) $\{x, y\}$	(12) $\cup x$	(20) $f(x)$	(28) $n,\ n \in \omega$
(5) $\langle x, y \rangle$	(13) $\cap x$	(21) $f \restriction x$	(29) A^n
(6) $\emptyset$	(14) $ordpair(z)$	(22) $f\ is\ 1-1$	(30) $A^{<\omega}$
(7) $x \cup y$	(15) $A \times B$	(23) $ordinal(x)$	
(8) $x \cap y$	(16) $rel(R)$	(24) $limord(x)$	

(ii) If $G(x, z, \bar{y})$ *is a* Σ_n^{ZF} *term, and* $F(x, \bar{y})$ *is the term obtained by transfinite* $\in$ *or* $<$ *induction then,* $F(x, \bar{y})$ *is* Δ_n^{ZF}*.*

(iii) The following terms and formulas are Δ_1^{ZF} *thus absolute:*

(1) $\alpha - 1$	(4) α^β	(7) V_ω
(2) $\alpha + \beta$	(5) $rank(x)$	
(3) $\alpha \cdot \beta$	(6) $TC(x)$	

(iv) The following terms and formulas are Σ_1^{ZF} *thus absolute for extensions:*

(1) $x =_c y$
(2) $x \leq_c y$
(3) $cf(\alpha)$

(v) The following terms and formulas are Π_1^{ZF} *thus absolute for restrictions:*

(1) $card(\alpha)$	(4) $inaccessible(\alpha)$	(7) V_α
(2) $regular(\alpha)$	(5) $<\ wellorders\ x$	
(3) $limcard(\alpha)$	(6) $P(x)$	

Applications

This section is quite technical but the reader should reach a level of familiarity with it. Absoluteness is a powerful tool when dealing with models of ZF. Imagine a set M which is a model of set theory. If we are interested in whether $M \vDash \phi$, where ϕ is absolute, we only have to check if it is true in V instead of checking if $(\phi)^M$ is true in V, which could be difficult. Absoluteness also comes in handy when somebody wants to show that a set M is a model of ZFC, since many of its axioms are Δ_0 thus absolute. We will close this section with two applications. In the first one, we find some axioms of ZFC which are absolute for all transitive models and in the second one, we prove that there exists a Δ_1^{ZF} formula which expresses the notion $M \vDash ZF$.

Application 1.13. *The following formulas and terms related to the corresponding axioms of ZF are absolute for every transitive model* M*:*

*A*1. *Extensionality*	*A*5. $\{x, y\}$
*A*2. *Foundation*	*A*7. $\cup x$
*A*4. $\emptyset$	*A*8. ω

The following formulas and terms related to the corresponding axioms of ZF are preserved by restriction to M, for every transitive structure M:

A6. $P(x)$
A10. $<$ *wellorders* x

Additionally, for every transitive model M, $P^M(x) = P(x) \cap M$.

The above application is useful when we have to show that a given transitive structure satisfies A4, A5, A7 and A8. It is transitive thus it satisfies A1,A2 and in order to show that it satisfies A4, A5, A7, A8, we only have to check that it contains $\emptyset$ and it is closed under the respective operations. A6 and A10 may need some more work, but at least this theorem covers the one direction of the proof.

For the next application we need some definitions first. In order to express the notion of $\vDash$ within the language of set theory we must initially do this for its formulas. This is the idea of Gödel numbering and it can be implemented in many ways.

Definition 1.14. Each formula is represented by an element of V_ω using the function $\ulcorner\urcorner$ which is defined by the following recursion:

(i) For atomic formulas:

- $\ulcorner x_i = x_j \urcorner = \langle 0, i, j \rangle$
- $\ulcorner x_i \in x_j \urcorner = \langle 1, i, j \rangle$

(ii) For non-atomic:

- $\ulcorner \phi \vee \psi \urcorner = \langle 2, \ulcorner \phi \urcorner, \ulcorner \psi \urcorner \rangle$
- $\ulcorner \neg \phi \urcorner = \langle 3, \ulcorner \phi \urcorner \rangle$
- $\ulcorner \exists x_i \ \phi \urcorner = \langle 4, i, \ulcorner \phi \urcorner \rangle$

The subset of V_ω, which consists exactly of its elements that express a formula, will be denoted $Form$.

Definition 1.15. We give the following definitions:

(1) $fm(u, f, n)$: "$u = \ulcorner \phi \urcorner$ where ϕ is the sentence whose structure is described by the function f in n steps".

(2) $fmla(u)$: "u is the Gödel set of a formula".

(3) $s(m, g, r, f, M)$: "when f, g are functions, f describes the construction of a sentence of rank r and $f(k) = \ulcorner \phi_m \urcorner$ then, $g(m)$ contains all the r-tuples, a, of M^r for which $\phi_m[a]$ takes its truth value according to Tarski's definition of truth for M".

(4) $sat(u, M, \bar{b})$: "u is the Gödel set of a formula ϕ for which $M \vDash \phi[\bar{b}]$".

(5) $ax^{ZFC}(u)$: "u is a formula and it is one of the axioms of ZFC".

(6) $M \vDash ZF$: "M is a model of ZF".

Application 1.16. *The above notions are all Δ_1^{ZF} thus absolute for all models of ZF. In particular $M \vDash ZF$ is absolute for models of ZF.*

Proof.

(1) Writing down the definition of fm we get the following formula:

$$\begin{aligned} fm(u,f,n) \equiv\ & funct(f) \wedge finord(n) \wedge dom(f) = n+1 \wedge f(n) = u \wedge \\ & \wedge \forall m < n+1\ [\exists i,j < \omega (f(m) = \langle 0,i,j\rangle \vee f(m) = \langle 1,i,j\rangle) \vee \\ & \vee \exists k,l < m\ (f(m) = \langle 2, f(k), f(l)\rangle) \vee f(m) = \langle 3, f(k)\rangle) \vee \\ & \vee \exists k < m\ \exists i < \omega\ f(m) = \langle 4, i, f(k)\rangle] \end{aligned}$$

which by lemma 2 can be easily proved to be Δ_1^{ZF}.

(2) $fmla(u)$ is equivalent to the formula

$$\exists n < \omega\ \exists f \in V_\omega\ fm(u,f,n)$$

which is Δ_1^{ZF} by lemma 2 and (1).

(3) $s(m,g,r,f,M)$ is equivalent to the following formula (this is a coding of Tarski's truth definition)

$$\begin{aligned} & \exists i,j < \omega\ [(f(m) = \langle 0,i,j\rangle \wedge g(m) = \{a \in M^r : a(i) = a(j)\}) \vee \\ & \qquad \vee (f(m) = \langle 1,i,j\rangle \wedge g(m) = \{a \in M^r : a(i) \in a(j)\})] \vee \\ & \vee \exists k,l < m\ [(f(m) = \langle 2, f(k), f(l)\rangle \wedge g(m) = g(k) \cup g(l)) \vee \\ & \qquad \vee (f(m) = \langle 3, f(k)\rangle \wedge g(m) = M^r \setminus g(k))] \vee \\ & \vee \exists i < \omega\ \exists k < m\ (f(m) = \langle 4, i, f(k)\rangle \wedge g(m) = \{a \in M^r : \exists x \in M\ a(i|x) \in g(k)\}) \end{aligned}$$

which is Δ_1^{ZF} by lemma 2 and the fact that $\alpha(i|x)$ is an abbreviation for

$$(a \setminus \{\langle i, a(i)\rangle\}) \cup \{\langle i, x\rangle\}.$$

(4) $sat(u,M,b)$ is equivalent to the formula

$$\begin{aligned} & \exists f,n,r \in V_\omega\ [fm(u,f,n) \wedge r = rank(u) \wedge \\ & \wedge \exists g\ (funct(g) \wedge dom(g) = n+1 \wedge \forall m < n+1\ s(m,g,r,f,M) \to b \in t(n))]. \end{aligned}$$

Since whenever a function g having above properties exists, it is unique, the formula

$$\exists g\ (funct(g) \wedge dom(g) = n+1 \wedge \forall m < n+1\ s(m,g,r,f,M) \to b \in t(n))$$

is equivalent to

$$\forall g\ (funct(g) \wedge dom(g) = n+1 \wedge \forall m < n+1\ s(m,g,r,f,M) \to b \in t(n))$$

so it is Δ_1^{ZF}. Thus $sat(u,M,b)$ is also Δ_1^{ZF}.

(5) $ax^{ZF}(u)$ is equivalent to the formula

$$fmla(u) \wedge [u = \ulcorner A1 \urcorner \vee \ldots \vee \ulcorner A8 \urcorner \vee \exists v \in V_\omega \ (fmla(v) \wedge u = \ulcorner v \urcorner)]$$

which is Δ_1^{ZF}. What needs explanation here is $\ulcorner A9 \urcorner(v)$ which is the set representing $A9_\phi$, where ϕ is the formula represented by v. To prove that $\ulcorner Ai \urcorner$ can be written as a Δ_1^{ZF} sentence is immediate, as the sentence just describes the procedure by which we construct the set representing axiom i. For example for $A4 \equiv \exists x_0 \ \neg \exists x_1 \ (x_1 \in x_0)$

$$\ulcorner A4 \urcorner = \langle 4, 0, \langle 3, \langle 4, 1, \langle 1, 1, 0 \rangle \rangle \rangle \rangle$$

In the case of the axiom of replacement we must additionally replace ϕ with v wherever occuring in the axiom.

(6) $M \vDash ZF$ is equivalent to the formula

$$\forall u \in V_\omega \ [ax^{ZF}(u) \to \forall b \in M^{rank(u)} \ sat(u, M, b)]$$

which is Δ_1^{ZF}.

□

It is easy to see that the above procedure can be done for every set of axioms of set theory which is defined by a Δ_1^{ZF} formula. The reader who has some knowledge of computability theory will realize that every set of axioms S is of this kind, if and only if there exists an algorithm which decides whether a given element of $Form$ belongs to S. A special case is when we have a finite extension of ZF.

1.4 V_α *and* L_α

In this section we study models of set theory created by considering the sets V_α, L_α for different kinds of α, and their unions which are classes. Some of those are models of ZF or ZFC and some others satisfy part of it but disagree in one of its axioms. This way we will get some results on the independence of the axioms of ZF and ZFC. For more on this subject see [21]. Our intention in this chapter is to give some more information on the structure of a universe of sets satisfying ZFC or part of it. We also want to raise some questions referring to models of ZFC and which will be correlated with the existence of specific large cardinals in later chapters.

V_α

We have already introduced $V_\alpha, \ \alpha \in On$ as the sets forming von Neumann's hierarchy. In the case α is a successor cardinal, they don't present much interest as models because they fail to be closed under most of the operations appearing in ZF. On the other hand, if α is a limit ordinal V_α is nearly a model of ZFC. The following lemma provides some information on this case.

Lemma 1.17.

(i) $V_\omega \models ZF(C) - inf + \neg inf$.

(ii) *If* $\alpha > \omega$ *and* $limord(\alpha)$ *then* $V_\alpha \models ZC$.

(iii) $V_{\omega+\omega} \models ZC + \neg F$.

Finally, we have to notice that $R = \cup_{\alpha \in On} V_\alpha$ is a model of ZFC, something we already knew since $R = V$.

L_α

The sets L_α have been defined in section 1.2. They are similar to the sets V_α but the notion of definability provides us with a much clearer view of the universe L. This is also the reason why many more axioms can be proved to be valid in L, such as AC and GCH, by accepting only ZF.

Theorem 1.18.

(i) $L_\omega \models ZFC - inf + \neg inf$.

(ii) $L_\alpha \models ZC$, *if* α *is a limit ordinal.*

(iii) $L \models ZF$

(iv) $L \models V = L$

Proof.

(i)-(iii) The proofs are like the ones for V_α because all the notions appearing in the axioms of ZFC are definable.

(iv) The formula $y = def(M)$ is equivalent to

$$\forall x \in M \ \forall u \in V_\omega \ (x \in y \leftrightarrow fmla(u) \wedge M \models u[x])$$

which is Δ_1^{ZF} so $def(M)$ is a Δ_1^{ZF} term.

The sets L_a are defined by transitive $<$ induction using the term

$$G(f,z) = \begin{cases} \emptyset & z = 0 \\ def(f(z-1)) & succord(z) \\ \cup_{\alpha < z} f(\alpha) & limord(z) \end{cases}$$

which is Δ_1^{ZF} so L_α is also Δ_1^{ZF}.

As a result of that

$$L \models V = L \leftrightarrow \forall x \in L \ \exists \alpha \in L \ (x \in L_\alpha)^L \leftrightarrow \forall x \in L \ \exists \alpha \in L \ x \in L_\alpha$$

which is true.

□

$V = L$ is a very crucial property of L. Along with the fact that L_α is absolute for transitive models of ZF, it will assist us in the proof of the next theorems.

Definition 1.19. An *inner model* of ZF is a transitive class $(M, \in)$, such that $M \models ZF$ and $On \subset M$.

Theorem 1.20. *L is the smallest inner model of ZF.*

Proof.

- We have proved that $L \models ZF$ and that $\forall\alpha \in On\ \alpha \in L_\alpha$ so L is an inner model of ZF.
- Since M is a model of ZF, $M \models \forall\alpha\ \exists x\ x = L_\alpha$, but L_α is absolute and $\alpha \in M$ for every ordinal, so $\forall\alpha\ L_\alpha \in M \to L \subset M$.

□

Lemma 1.21. ***(Gödel's condensation lemma)*** *If $M \prec L_\alpha$, $limord(\alpha)$, then the Mostowski collapse of M is L_β for some $\beta \leq \alpha$.*

Proof. The main idea is to prove first, that there exists a sentence ϕ_L such that for every transitive model N satisfying enough of the axioms of ZF

$$N \models \phi_L \leftrightarrow \exists\alpha\ (limord(\alpha) \wedge N = L_\alpha).$$

After a step by step examination of the proof of the absoluteness of L_α we can see L_α is not only Δ_1^{ZF} but Δ_1^T where T is an appropreate, finite part of ZF. Thus if N is a transitive model satisfying T then

$$N \models \forall x\ \exists\alpha\ x \in L_\alpha \leftrightarrow \forall x \in N\ \exists\alpha \in N\ x \in L_\alpha \leftrightarrow \exists\alpha\ (limord(\alpha) \wedge N = L_\alpha).$$

If $M \prec L_\omega$ then $M = L_\omega$. Else, if $\alpha > \omega$ we must notice that the axiom of replacement was only needed in order to define some sets by ω-transitive recursion and thus show that L_α is absolute. For those cases needed it is also true in L_α. Thus since $L_\alpha \models Z$ we have that $L_\alpha \models T$ so $L_\alpha \models \phi_L$. If $M \prec L_\alpha$ then its transitive collapse, N, $N \models \phi_L \to \exists\beta\ N = L_\beta$. Since $M \subset L_\alpha$ the ordinals of M, and thus of N, belong to L_α, hence $\beta \leq \alpha$. □

Theorem 1.22. *$L \models AC$*

Proof. In order to prove that $L \models AC$ we only have to show, in ZF, that every set of L can be well-ordered, because then from theorem 1.18, (iv)

$$L \models (ZF + V = L) \to L \models \forall x\ \exists R\ (R\ well-orders\ x) \to L \models AC.$$

The proof is done using transitive $<$ induction.

- $L_0 = \emptyset$ is well-ordered by the empty relation.

- Let $<_\alpha$ be a well-ordering of L_α, $<^n_\alpha$ the lexicographical order, derived from $<_\alpha$, of ordered n-tuples, $<_{form}$ a well-ordering of the set of formulas (encoded inside V_ω) and ϕ_x the $<_{form}$-least formula defining $x \in L_{\alpha+1}$ so that $x = \{y \in L_{\alpha+1} : L_\alpha \vDash \phi_x[y, \bar{a}_x]\}$. Then, we can define the following well-ordering, $<_{\alpha+1}$, of $L_{\alpha+1}$:

$$x <_{\alpha+1} y \leftrightarrow \phi_x <_{form} \phi_y \vee (\phi_x = \phi_y \wedge \bar{a}_x <^n_\alpha \bar{a}_y).$$

- If α is a limit ordinal and $rank_L(x)$ is the rank of x in the L_α hierarchy then we can define the following well-ordering of L_α:

$$x <_\alpha y \leftrightarrow rank_L(x) < rank_L(y) \vee (rank_L(x) = rank_L(y) \wedge x <_{rank_L(x)} y).$$

□

A well-ordering $<_L$ of L emerges from the above proof. We will call this ordering the *canonical well-ordering* of L.

Theorem 1.23. $L \vDash GCH$

Proof. [3] We will show that if $x \in P(\aleph_\alpha)$, then there is a $\beta < \aleph_{\alpha+1}$ such that $x \in L_\beta$. So $P(\aleph_\alpha) \subset L_{\aleph_{\alpha+1}}$ thus, since $|L_{\aleph_{\alpha+1}}| = \aleph_{\alpha+1}$,

$$|P(\aleph_\alpha)| \leq \aleph_{\alpha+1} \rightarrow |P(\aleph_\alpha)| = \aleph_{\alpha+1}.$$

Provided that the above holds for every ordinal α, GCH will be true.

Let $x \in P(\aleph_\alpha)$ and $x \in L_\gamma$. By the Lövenheim-Skolem theorem, there is a model $M \prec L_\gamma$ which satisfies $\aleph_\alpha \cup \{x\} \subset M$ and $|M| = \aleph_\alpha$. By lemma 1.21, there exists a limit ordinal $\beta \leq \gamma$ such that the Mostowski collapse N of M is L_β.

We also have that

$$|\beta| = |L_\beta| = |N| = |M| = \aleph_\alpha < \aleph_{\alpha+1}.$$

Finally, $\aleph_\alpha \subset M$, so if F is Mostowski's collapsing function of M, $F \upharpoonright \aleph_\alpha$ is the identity function. But $x \subset \aleph_\alpha$ thus $x = F(x) \in N = L_\beta$. □

The conception of the universe L of the constructible sets and the above theorems are due to Gödel. They imply that $ZF + AC$ and $ZF + AC + GCH$ are consistent provided that ZF is. Exploring L, one can prove the relative consistency of many other principles, such as $\diamondsuit$ and $\diamondsuit^+$.

[3] We note that in this proof we are working inside L. Thus $P(x)$ is $(P(x))^L$, $\aleph_\alpha$ is $(\aleph_\alpha)^L$ etc.

Constructibility relative to a set A

Finally, we briefly state some facts about constructibility relative to a set A. This kind of consructibility can lead to two inner models, $L[A]$ and $L(A)$.

The model $L[A]$ is constructed as L, with the only difference that we allow the set A to participate in the procedure of defining new sets. Thus, we alter the notion of definable sets in the following way:

Definition 1.24.

$$def_A(M) = \{X \subset M : X \textit{ is definable over } (M, \in, A \cap M)\}.$$

Definition 1.25. The class $L[A]$ of the sets *constructible from* A is equal to $\bigcup_{\alpha \in On} L_\alpha[A]$, where

(i) $L_0[A] = \emptyset$;

(ii) $L_{\alpha+1}[A] = def_A(L_\alpha[A])$;

(iii) $L_\alpha[A] = \bigcup_{\beta<\alpha} L_\beta[A]$ if $limord(\alpha)$.

The following theorem contains some results relative to those we have proved for L.

Theorem 1.26. *For every set A,*

(i) $L[A] \models ZFC$;

(ii) $L[A] \models \exists X \; V = L[X]$;

(iii) If M is an inner model and $A \cap M \in M$, then $L[A] \subset M$;

(iv) If $limord(\alpha)$ and $M \prec (L_\alpha[A], \in, A \cap L_\alpha[A])$, then the Mostowski collapse of M is L_β for some $\beta \leq \alpha$;

(v) $\exists \alpha \; \forall \beta \geq \alpha \; L[A] \models 2^{\aleph_\beta} = \aleph_{\beta+1}$;

(vi) If $A \subset \aleph_1$, then $L[A] \models GCH$.

The model $L(A)$ is different from $L[A]$, as it contains A. Actually, it is the smallest inner model which contains A.

Definition 1.27.

(i) $L_0(A) = TC(A)$;

(ii) $L_{\alpha+1}(A) = def(L_\alpha(A))$;

(iii) $L_\alpha(A) = \bigcup_{\beta<\alpha} L_\beta(A)$ if $limord(\alpha)$.

$L(A) = \bigcup_{\alpha \in On} L_\alpha(A)$.

1.5 Model theory

In this section we sketch some model theoretical concepts which will become useful later on. The reader can refer to [2] and [21] for more details and some of the proofs which have been omitted.

Skolem functions

We have already met the notion of elementary substructures while introducing absoluteness. It is quite clear that all the logical and nonlogical symbols, except from the quantifiers, are interpreted the same way in both structures $M \subset N$. Hence, in order to check that $M \prec N$, we must examine whether the quantifiers are interpreted the same way between M, N. Since $\forall$ is equivalent to $\neg\exists\neg$, we only have to work with $\exists$. Thus the following holds:

Lemma 1.28. *$M \prec N$ iff for every $\phi \in Form$ and every n-tuple $\bar{b} \in M$:*

$$\exists a \in N \; N \vDash \phi[a, \bar{b}] \rightarrow \exists a \in M \; N \vDash \phi[a, \bar{b}].$$

The lemma above can be used in constructing elementary substructures of a given structure M. Specifically, we can define for every formula $\phi(v, u_1, \ldots, u_n)$ a function $h_\phi : M^n \to M$ such that for all $a_1, \ldots, a_n \in M$

$$\begin{aligned} M \vDash \phi[h_\phi(a_1, \ldots, a_n), a_1, \ldots, a_n] \quad & \textit{if } \exists a \in M \; M \vDash \phi[a, a_1, \ldots, a_n]; \\ h_\phi(a_1, \ldots, a_n) = a_0 \in M \quad & \textit{else.} \end{aligned}$$

The functions satisfying those properties are called *Skolem functions*. The closure $\mathcal{H}(X)$ of a set $X \subset M$ under those functions is called the *Skolem hull* of X and by lemma 1.28, $\mathcal{H}(X) \prec M$. Many times Skolem functions can be defined inside a structure but, even if not, we can always prove the existence of external Skolem functions for every structure M, using AC. This fact leads to the following theorem:

Theorem 1.29. *(Löwenhein-Skolem) Suppose $\mathcal{L}$ is a language and M a structure for $\mathcal{L}$. Then,*

(i) (downward) for every cardinal $|\mathcal{L}| \leq \kappa \leq |M|$, there is a stucture $N \prec M$, such that $|N| = \kappa$;

(ii) (upward) for every cardinal $\kappa \geq |M|$, there is a structure $N \succ M$, such that $|N| = \kappa$.

We define now the notion of elementary embeddings, which will play a crucial role in our study.

Definition 1.30. An *elementary embedding* of M into N, is a function $f : M \to N$ such that:

(i) f is an isomorphism between M and $f''M$;

(ii) $M \prec N$.

We will use the notation $f : M \prec N$ to express the fact that f is an elementary embedding of M into N.

Elementary chains

Definition 1.31. Every sequence of structures $\{M_\beta\}_{\beta<\alpha}$, such that for all $\beta_1 < \beta_2 < \alpha$, $M_{\beta_1} \prec M_{\beta_2}$, is called an *elementary chain*.

For every elementary chain we may define its union $M = \bigcup_{\beta<\alpha} M_\beta$:

(i) $M = \bigcup_{\beta<\alpha} M_\beta$;

(ii) $P^M = \bigcup_{\beta<\alpha} P^{M_\beta}$;

(iii) $F^M = \bigcup_{\beta<\alpha} F^{M_\beta}$;

(iv) $c^M = c^{M_\beta}$ for all $\beta < \alpha$.

It is straightforward to check that M is well defined and that it is a structure. What is also true is the following:

Theorem 1.32. *(Elementary chain theorem) For every chain $\{M_\beta\}_{\beta<\alpha}$ and every $\gamma < \alpha$, $M_\gamma \prec \bigcup_{\beta<\alpha} M_\beta$.*

Ultraproducts

A very useful method in constructing structures is the one using ultraproducts. In order to understand it we must first define filters and ultrafilters. A filter on a set S is a subset of $P(S)$, which contains those subsets of S that are in some sense "large" and can be considered to be nearly the same as S. The aspect of what is large, depends on the nature of each specific filter.

Definition 1.33. Let S be a set and $F \subset P(S)$. F is a *filter on* S if:

(i) $\emptyset \notin F$ and $S \in F$;

(ii) $X, Y \in S \to X \cap Y \in S$;

(iii) $X \in S \wedge X \subset Y \subset S \to Y \in S$.

Some examples of filters are the Fréchet filter on $\mathbb{N}$, which contains the subsets of $\mathbb{N}$ that have finite complements, and the filter on $\mathbb{R}$, which contains all the subsets of the reals having Lebesque measure 1. It is up to the reader to establish why those two are filters and understand the meaning of "large" in both cases.

Definition 1.34. A filter U on S is called an *ultrafilter*, if for every $X \subset S$ $X \in U$ or $S \setminus X \in U$.

Our intention is to use the notion of ultrafilter to define some sort of product of structures, the same way we construct products of groups or topological spaces. Suppose $\{M_x : x \in S\}$ is a family of structures and U an ultrafilter. We define the equivalence relation $=_U$ on $\prod_{x \in S} M_x$ so that, $f =_U g \leftrightarrow \{x \in S : f(x) = g(x)\} \in U$. The ultraproduct of the structures M_x is then defined the following way:

Definition 1.35. For every family of structures $\{M_x : x \in S\}$ and every ultrafilter U on S, $M = \prod_U M_x$ is the structure with the following properties:

(i) $M = \prod_{x \in S} M_x / =_U$;

(ii) $P^M([f_1], \dots, [f_n]) \leftrightarrow \{x \in S : P^{M_x}(f_1(x), \dots, f_n(x)\} \in U$ for every predicate symbol P;

(iii) $F^M([f_1], \dots, [f_n]) = [f] \leftrightarrow \{x \in S : F^{M_x}(f_1(x), \dots, f_n(x)) = f(x)\} \in U$, for every function symbol F;

(iv) $c^M = [f]$, $\forall x \in S\ f(x) = c^{M_x}$, for all constant symbols c.

It is not difficult to check that the relation $=_U$ is an equivalence relation and that M is indeed a structure, as those concepts depend solely on the properties of ultrafilters. Again by applying those properties, we can get the following theorem, which gives us a way of calculating $\vDash_M$ from $\vDash_{M_x}, x \in S$.

Theorem 1.36. *(Łoś) Let $M = \prod_U M_x$ be as above. Then, for every formula ϕ and $[f_1], \dots, [f_n] \in M$,*

$$M \vDash \phi[[f_1], \dots, [f_n]] \leftrightarrow \{x \in S : M_x \vDash \phi[f_1(x), \dots, f_n(x)]\} \in U.$$

If for every $x \in S$ $M_x = N$, where N is a specific structure, we call the ultraproduct $\prod_U M_x$ an *ultrapower* and denote it by $Ult(N, U)$. It is straightforward by the theorem above that $N \equiv Ult(N, U)$, i.e. they satisfy the same sentences. Moreover we can define the *canonical embedding* $j : N \to Ult(N, U)$, $j(x) = [c_x]$, where c_x is a constant function such that $\forall y\ c_x(y) = x$. This -again by theorem 1.36- is an elementary embedding.

Reflection principles

A reflection principle, in general, asserts that a certain property of the universe V is absolute for V, M, where M is a model of ZFC. The property can be expressed by a first order formula, a higher order formula, a type, or anything else we could imagine. Most of the times the validity of reflection principles, for some specific kind of properties, cannot be proved from ZFC, hence they could be proposed as new axioms. This is the case of Π^n_m-indiscernibles, which we will study in the next chapter. For the time being, we state the following theorem, that contains three similar reflection principles, provable from ZF or ZFC.

Theorem 1.37. *For every formula ϕ and every set X:*

(i) There is an ordinal α, such that $X \subset V_\alpha$ and ϕ is absolute for V, V_α.

(ii) There is a structure M, such that $X \subset M$, $|M| \leq |X| \cdot \aleph_0$ and ϕ is absolute for V, M.

(iii) There is a transitive structure M such that $|M| \leq \aleph_0$ and ϕ is absolute for V, M.

Proof.

(i) We define a function for ϕ using Scott's trick, which looks like a Skolem function, and by repeatingly applying it to sets of von Newmann's hierarchy we get V_α.

(ii) It's the same as before, though we use Skolem's functions this time, which exist by AC.

(iii) We apply Mostowski's collapsing lemma to (ii) and get the needed structure.

For more details, see [11] and [8]. $\square$

Chapter 2

Some smaller large cardinals

As we have seen, V can be approached by von Newmann's cumulative hierarchy, thus we could say that it is fully described by the class On and the powerset operation[1]. This way, we can imagine that the size of On determines the "length" of V and the size of P (for several x) determines its "width". In this chapter we are going to see some examples of large cardinals. Although there is not a certain definition for this concept, we could say that large cardinals are cardinals whose existence cannot be proved from ZFC and they extend the universe V in length. Throughout the text, we will notice that the categories of large cardinals we are going to introduce, appear in a sequence of increasing strength (a large cardinal property a is stronger than b if $ZFC + \exists\kappa\ a(\kappa) \vdash \exists\kappa\ b(\kappa)$, e.g. $ZFC + \exists\kappa\ Mahlo(\kappa) \vdash \exists\kappa\ inaccessible(\kappa)$).

We begin by citing some aspects of cardinal arithmetic, necessary for the definitions we are going to give and for some of the proofs. We define then inaccessible cardinals, which cannot be defined using the operations described in the first chapter, thus in some way they cannot be approached by the means of ZFC. We will see that their existence is connected to the existence of models of ZFC inside V. By requiring the existence of more and more inaccessible cardinals, we will come up with Mahlo cardinals which demand the existence of a stationary set of inaccessible cardinals. From this point on, we will need new methods for finding even stronger categories of large cardinals. The Π^n_m-inaccessibility will provide us with the weakly compact cardinals (Π^1_1-inaccessible), along with some useful characterizations or properties of other categories of large cardinals. Weakly compact cardinals, owe their name to their connection with the

[1]Those two notions are not fully determined by ZFC, therefore V could vary in length and width. Using forcing, we can show that the cardinality of $P(X)$ may take various different values greater or equal to $|X|$. Hence, we can not even count the increase in size the powerset operation imposes. As for On, the more large cardinals we require to exist, the larger On gets. The only difference here from the case of forcing, is that we can not prove in ZFC that the existence of large cardinals is consistent with ZFC.

property of weak compactness of an infinitary logic $\mathcal{L}_{\kappa,\omega}$, where κ is an infinite cardinal. Finally we introduce Erdős and Ramsey cardinals, which are defined using combinatorial partition properties.

We call the cardinals defined in this section (except from Ramsey cardinals) "small", since their existence is relatively consistent with the assumption $V = L$. Hence, they are not strong enough to cause drastic changes to the structure of V. In the contrary, the existence of Ramsey cardinals or of the large cardinals introduced in the next chapters, will have $V \neq L$ as a consequence. Even more, if one of them exists, V will be much more complicated than L. For more on this check chapter 4.

2.1 Cardinal arithmetic

This section contains some basic knowledge on cardinal arithmetic. All the material covered here can be found in any introductive book on set theory.

$+, \cdot, \sum$ and $\prod$

$+, \cdot, \sum$ and $\prod$ are some of the functions defined on cardinals that will be used many times in our study. Below we give their definitions, ways of calculating them and some of their properties.

Definition 2.1.

(i) The sum of two cardinals κ and λ, $\kappa + \lambda$, is defined by:

$$\kappa + \lambda = |\kappa \times \{0\} \cup \lambda \times \{1\}|.$$

(ii) The product of two cardinals κ and λ, $\kappa \cdot \lambda$, is defined by:

$$\kappa \cdot \lambda = |\kappa \times \lambda|.$$

Theorem 2.2. *The class $On \times On$ can be well-ordered by the following relation:*

$$\begin{aligned}\langle \alpha_1, \alpha_2 \rangle \lhd \langle \beta_1, \beta_2 \rangle \leftrightarrow \max\{\alpha_1, \alpha_2\} &< \max\{\beta_1, \beta_2\} \vee \\ &\vee [\max\{\alpha_1, \alpha_2\} = \max\{\beta_1, \beta_2\} \wedge \\ &\wedge (\alpha_1 < \alpha_2 \vee (\alpha_1 = \alpha_2 \wedge \beta_1 < \beta_2))].\end{aligned}$$

Moreover, for every cardinal κ, $\langle \kappa \times \kappa, \lhd \rangle =_o \kappa$.

Proof. $\lhd$ is a well-ordering of $On \times On$. If A is a subclass of it, then the set

$$\{\alpha \in On : \exists \langle \alpha_1, \alpha_2 \rangle \in On \times On \ \min\{\alpha_1, \alpha_2\} = \alpha\}$$

has a minimum element β. The minimum of the set

$$\{\langle \alpha_1, \alpha_2 \rangle \in On \times On : \min\{\alpha_1, \alpha_2\} = \beta\}$$

ordered by the lexicographical ordering, is a minimum for A.

The reduction of $\triangleleft$ to $\kappa \times \kappa$ is a well-ordering for $\kappa \times \kappa$ and $\kappa \leq_o \langle \kappa \times \kappa, \triangleleft \rangle$. Thus we have to show that $\langle \kappa \times \kappa, \triangleleft \rangle \leq_o \kappa$. If this is not true and κ is the least cardinal with the property $\kappa <_o \langle \kappa \times \kappa, \triangleleft \rangle$ then

$$\exists \alpha, \beta \in \kappa \ \kappa =_o seg_{\triangleleft}(\alpha, \beta)$$

and if $\gamma = \max\{\alpha, \beta\}$

$$seg_{\triangleleft}(\alpha, \beta) \leq_o \langle \gamma \times \gamma, \triangleleft \rangle =_o \langle \gamma, < \rangle <_o \kappa.$$

This means that $\kappa < \kappa$ which is a contradiction. □

Corollary 2.3. *For all cardinals κ, λ and $\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$.*

Proof.

$$\max\{\kappa, \lambda\} \leq \kappa + \lambda \leq \kappa \cdot \lambda$$

so we only need to prove that $\kappa \cdot \lambda \leq \max\{\kappa, \lambda\}$, but this is true since
$\kappa \cdot \lambda \leq \max\{\kappa, \lambda\} \cdot \max\{\kappa, \lambda\} \leq_o \max\{\kappa, \lambda\}$. □

The above corollary reveals that the computation of $\kappa+\lambda$ and $\kappa\times\lambda$ is trivial.

Definition 2.4.

(i) The sum of the cardinals belonging to the set $\{\kappa_i : i \in I\}$ is defined as:

$$\sum_{i \in I} \kappa_i = | \cup_{i \in I} (\kappa_i \times \{i\})|.$$

(ii) The product of the cardinals belonging to the set $\{\kappa_i : i \in I\}$ is defined as:

$$\prod_{i \in I} \kappa_i = | \prod_{i \in I} \kappa_i|$$

(the second product is the set product and it is different from the first one)

We have to note here that the definition of $\prod$ requires in most cases the axiom of choice in order to be used. This is because every sequence $\{\alpha_i\}_{i \in I}$, $a_i \in \kappa_i$ is a choice function on the collection of the cardinals κ_i. From now on we will take AC as for granted and we will just keep in mind that it is essential in most of the results that follow.

Some basic properties of $\sum$ and $\prod$ are stated below:

Lemma 2.5.

(i) $\sum_{\alpha < \lambda} \kappa = \lambda \cdot \kappa$;

(ii) $\prod_{\alpha < \lambda} \kappa = \kappa^{\lambda}$;

(iii) $\sum_{i \in I} \kappa_i = \sum_{j \in J} (\sum_{i \in A_j} \kappa_i)$, *where I is the disjoint union* $\cup_{j \in J} A_j$.

(iv) $\prod_{\alpha<\mu} \kappa_\alpha^\lambda = (\prod_{\alpha<\mu} \kappa_\alpha)^\lambda$*;*

(v) $\prod_{\alpha<\mu} \kappa^{\lambda_\alpha} = \kappa^{\sum_{\alpha<\mu} \lambda_\alpha}$*;*

(vi) $\prod_{i\in I} \kappa_i = \prod_{j\in J} (\prod_{i\in A_j} \kappa_i)$*, where* I *is the disjoint union* $\cup_{j\in J} A_j$*.*

The computation of $\sum_{i\in I} \kappa_i$ is also quite trivial due to the following theorem:

Theorem 2.6. *If* $\omega < \lambda$ *and* $\forall\alpha < \lambda\ 0 < \kappa_\alpha$ *then*

$$\sum_{\alpha<\lambda} \kappa_\alpha = \lambda \cdot \sup_{\alpha<\lambda} \kappa_\alpha.$$

The computation of $\prod$ is not at all trivial, because it is related to the exponentiation function. The next theorem reveals this connection.

Theorem 2.7. *If* λ *is an infinite cardinal and* $\langle \kappa_\alpha : \alpha < \lambda \rangle$ *is nondecreasing sequence of nonzero cardinals, then*

$$\prod_{\alpha<\lambda} \kappa_\alpha = (\sup_{\alpha<\lambda} \kappa_\alpha)^\lambda$$

Since $\prod$ is related to exponentiation and $\sum$ to multiplication, which is much weaker, it is natural to search for a result generalizing Cantor's theorem $\aleph_\alpha < 2^{\aleph_\alpha}$. König's theorem is such a result.

Theorem 2.8. *(König) Suppose that* $\forall\alpha < \mu\ \kappa_\alpha < \lambda_\alpha$*. Then*

$$\sum_{\alpha<\mu} \kappa_\alpha < \prod_{\alpha<\mu} \lambda_\alpha.$$

Cofinality

A useful notion in the study of cardinals is cofinality. The cofinality of a cardinal is the smallest cardinal which can be used to approach it. Below we formally define it and state some of its properties.

Definition 2.9. Suppose κ is a cardinal and $\alpha \leq \kappa$ is an ordinal.

(i) A function $f : \alpha \to \kappa$ is *cofinal*, if $\sup(f''\alpha) = \kappa$.

(ii) The *cofinality* of κ, is the least ordinal α, such that there is a cofinal function $f : \alpha \to \kappa$. We denote it by $cf(\kappa)$.

Another characterization of cofinality is the following:

Lemma 2.10. *The cofinality of a cardinal* κ *is the least ordinal* α*, such that there is a family of cardinals* $\{\kappa_\beta : \beta < \alpha\}$*,* $\forall\beta < \alpha\ \kappa_\beta < \kappa$*, having the property* $\sum_{\beta<\alpha} \kappa_\beta = \kappa$*.*

It is easy to check that the cofinality of a cardinal is a cardinal and that $cf(cf(\kappa)) = cf(\kappa)$.

The following lemma is useful in calculating the cofinality of a cardinal.

Lemma 2.11. *Suppose that κ, λ are cardinals. If there is a nondecreasing cofinal function $f : \kappa \to \lambda$, then $cf(\kappa) = cf(\lambda)$.*

As an example of the above, we see that $cf(\aleph_0) = \aleph_0$ since it must be infinite. Thus, taking under consideration that $f : \aleph_0 \to \aleph_{\omega_0}$, $f(n) = \aleph_n$ is increasing and cofinal, $cf(\aleph_{\omega_0}) = cf(\aleph_0) = \aleph_0$.

We end our discussion on cofinality, by stating the following result which is a corollary of König's theorem.

Corollary 2.12. *For all cardinals κ, λ:*

(i) $cf(\kappa^\lambda) > \lambda$;

(ii) $\kappa^{cf(\lambda)} > \kappa$.

$[A]^\lambda$, $[A]^{<\lambda}$ and SCH

We finally define the two following two notions and state the singular cardinal hypothesis (SCH).

Definition 2.13.

(i) $[A]^\kappa = \{X \subset A : |X| = \kappa\}$.

(ii) $[A]^{<\kappa} = \{X \subset A : |X| < \kappa\}$.

Many times we denote $[A]^{<\kappa}$ by $P_\kappa(A)$.

Lemma 2.14.

$$|A| \geq \lambda \to |[A]^\kappa| = |A|^\kappa.$$

Definition 2.15. *SGH*: For every singular cardinal κ

$$2^{cf(\kappa)} < \kappa \to \kappa^{cf(\kappa)} = \kappa^+.$$

The *SCH* is a weaker version of *GCH* and by the following theorem it is clear that it simplifies the calculation of the exponential function.

Theorem 2.16. *Let SCH hold.*

(i) If κ is singular, then

- $2^\kappa = 2^{<\kappa}$, *if* $2^{<\kappa} = 2^\lambda$, $\lambda < \kappa$;
- $2^\kappa = (2^{<\kappa})^+$, *else.*

(ii) If $\kappa, \lambda \geq \omega$, then

- $\kappa^\lambda = 2^\lambda$, *if* $\kappa \leq 2^\lambda$;

– $\kappa^\lambda = \kappa$, *if* $2^\lambda < \kappa$ *and* $\lambda < cf(\kappa)$*;*
– $\kappa^\lambda = \kappa^+$, *if* $2^\lambda < \kappa$ *and* $cf(\kappa) \leq \lambda$*.*

On the other hand, GCH provides us with a specific form of the exponential function, hence we can compute it in this case.

Theorem 2.17. *Let* GCH *hold. Then,*

(i) $\kappa^\lambda = \lambda^+$, *if* $\kappa \leq \lambda$*;*

(ii) $\kappa^\lambda = \kappa^+$, *if* $cf(\kappa) \leq \lambda < \kappa$*;*

(iii) $\kappa^\lambda = \kappa$, *if* $\lambda < cf(\kappa)$*.*

2.2 Inaccessible cardinals

Definition 2.18.

(i) A cardinal κ is called *regular*, if $cf(\kappa) = \kappa$.

(ii) A cardinal κ is *strong limit*, if for every $\lambda < \kappa$, $2^\lambda < \kappa$.

Definition 2.19. The cardinal κ is called *inaccessible* if it is strong limit and regular.

By its definition, if there exists an inaccessible cardinal κ, then it ought to be quite a big cardinal. One easy way to see this, is by noticing that $\beth_\kappa = \kappa$ and that the smallest cardinal having this property is the union of the following sequence:

$$\alpha_0 = \beth_0$$
$$\alpha_{n+1} = \beth_{\alpha_n}$$

Inaccessible cardinals and models of ZFC

The main property that inaccessible cardinals satisfy, is that they define models of ZFC in the cumulative hierarchy V_α.

Theorem 2.20. *If* κ *is an inaccessible cardinal, then* $V_\kappa \vDash ZFC$*.*

Proof. Since κ is a cardinal, it is also a limit ordinal. We know from chapter 1 that in this case $V_\kappa \vDash ZC$, thus we only need to show that $V_\kappa \vDash F$.

Let us assume that x is an arbitrary set and that ϕ is a formula which defines a function. Every element of $F_\phi^{V_\kappa}{}''x$ belongs to V_α for some $\alpha < \kappa$ so

$$F_\phi^{V_\kappa}{}''x \subset V_\beta$$

$$\text{where } \beta = \sup\{\alpha : \exists y \in F_\phi^{V_\kappa}{}''x \; y \in V_\alpha\}.$$

κ is a strong limit cardinal thus we can show using transfinite induction that $|x| < \kappa$ for every $x \in V_\kappa$. As a result of that $|F_\phi^{V_\kappa}{}''x| < \kappa$ and provided that κ is regular, $\beta < \kappa$ which means that $F_\phi^{V_\kappa}{}''x \in V_\kappa$. ∎

The inverse of this theorem is not true. This can be justified by considering the case where there exists at least one inaccessible cardinal. If κ is the least inaccessible cardinal, then we can construct the following elementary chain $\{A_n\}_{n<\omega}$ of models:

$$A_0 = V_\omega$$
$$A_{n+1} = V_\beta,\ \beta = \inf\{\alpha < \kappa : \mathcal{H}(A_n) \subset V_\alpha\}$$

where $\mathcal{H}(x)$ is the Skolem hull of x in the model V_κ. By induction, $A_n \subset V_{\alpha_n}$, $\alpha_n < \kappa$, so the sequence is well defined and additionally $\cup_{n<\omega} A_n = V_\alpha,\ \alpha < \kappa$. By the elementary chain theorem, $V_\alpha \prec V_\kappa$. This means that $V_\alpha \vDash ZFC$ and α is not inaccessible.

Despite of that, if we formulate the axioms of ZFC in second order logic, so that the axiom of replacement becomes

$$\forall F\ \forall x\ \exists y\ y = F''x$$

then the inverse of 2.20 is also true.

Theorem 2.21. *The cardinal κ is inaccessible iff $V_\kappa \vDash ZFC^2$.*

Proof.

→ This can be done as in the proof of theorem 1.4, since we only used there the fact that $F_\phi^{V_\kappa}$ is a function.

← Suppose $V_\kappa \vDash ZFC^2$. κ is regular because if it was not, then there would exist a cardinal $\lambda < \kappa$ and an increasing function $f : \lambda \mapsto \kappa$ such that $\cup f''\lambda = \kappa$. But then, by second-order replacement

$$\cup f''\lambda \in V_\kappa \to \kappa \in V_\kappa$$

which is a contradiction.

κ is also strong limit because if not, then there would be a cardinal $\lambda < \kappa$ such that $\kappa \leq 2^\lambda$. This means that there is function $f : P(\lambda) \to \kappa$ onto κ. We already know that $P(\lambda) \in V_\kappa$ so, by second-order replacement, $\kappa = f''P(\lambda) \in V_\kappa$ which is again a contradiction.

□

This connection of inaccessible cardinals with the existence of models of ZFC^2 in the cumulative hierarchy, provides us with a way of understanding those cardinals. Suppose that we have a universe V, where none of the sets V_α are models of ZFC^2. This is the smallest, in length, closure of the empty set, under the operations described in 1.1 for ZFC^2. If we rename V so that $V = V_\kappa$ ($\kappa \notin V$) and consider it to be a set, we can produce the smallest in length closure V' of $V \cup \{V\}$ under the operations described above. V' will be a new universe where there will exist exactly one inaccessible cardinal. Thus the existence of inaccessible cardinals is a way to express the number of times we have iterated this closure to the empty set.

An analogue of this situation is in the study of the models of Peano arithmetic, where the standard model is the smallest one in length, but we can add more elements, using Skolem functions, and produce nonstandard models which are the one extension of the other. Of course, the difference here is that the existence of those models is granted by ZFC but in the universe V the existence of the closures is based just on intuition.

As it is expected, the existence of inaccessible cardinals is not provable in ZFC. Even more, the relative consistency of their existence with ZFC can not be proved in ZFC. We will denote by IC the statement that claims the existence of an inaccessible cardinal.

Theorem 2.22.

(i) $ZFC \nvdash IC$.

(ii) $ZFC \nvdash [cons(ZFC) \to cons(ZFC + IC)]$.

Proof.

(i) If the opposite was true, then

$$\begin{aligned} ZFC \vdash IC \to & ZFC \vdash \exists\kappa\ (V_\kappa \vDash ZFC) \to \\ \to & ZFC \vdash cons(ZFC) \end{aligned}$$

which is impossible according to Gödel's second incompleteness theorem.

(ii) If ZFC is consistent, and we could prove that

$$cons(ZFC) \to cons(ZFC + IC)$$

then $ZFC + IC$ would also be consistent. We would have

$$ZFC + IC \vdash [cons(ZFC) \to cons(ZFC + IC)]$$

and since $ZFC + IC \vdash cons(ZFC)$ we would be able to prove that $ZFC + IC \vdash cons(ZFC+IC)$ which is again a contradiction due to Gödel's second incompleteness theorem.

□

Inaccessible cardinals as Σ^1_1-indescribables

Finally, another characterization of inaccessible cardinals is the following:

Theorem 2.23. *κ is inaccessible iff it is Σ^1_1-indescribable.*

Before proving this theorem, we state a characterization of inaccessible cardinals by elementary embeddings.

Lemma 2.24. *κ is inaccessible iff for any $R \subset V_\kappa$ there is an α such that*

$$(V_\alpha, \in, R \cap V_\alpha) \prec (V_\kappa, \in, R).$$

Proof.

$\rightarrow$ We define Skolem functions on the model $(V_\kappa, \in, R)$ and form the following elementary chain:

$$\begin{aligned} A_0 =& (V_\omega, \in, R \cap V_\omega) \\ A_{n+1} =& (V_\beta, \in, R \cap V_\beta),\ \beta = \inf\{\alpha < \kappa : \mathcal{H}(A_n) \subset V_\alpha\}. \end{aligned}$$

Every A_n is a subset of V_κ, because κ is inaccessible and moreover the union of the chain is V_α for some $\alpha < \kappa$. By the elementary chain theorem $(V_\alpha, \in, R \cap V_\alpha) \prec (V_\kappa, \in, R)$.

$\leftarrow$
- κ is regular because in the opposite case there would exist an ordinal β and an increasing function $f : \beta \mapsto \kappa$ onto κ. If $R = \{\beta\} \cup f$, then $V_\alpha \vDash \exists x \in R\ ordinal(x)$ which means that $\beta \in V_\alpha$ and this is a contradiction because $(f \cap V_\alpha)(\beta) = \kappa \in V_\alpha$.
- κ is also strong limit. On the contrary, there would exist a $\lambda < \kappa$ such that $\kappa \leq 2^\lambda$ thus also a function $f : P(\lambda) \mapsto \kappa$ onto κ. If $R = \{\lambda + 1\} \cup f$, then again $\lambda + 1 \in V_\alpha \rightarrow \lambda < \alpha$ thus $P(\lambda) \in V_\alpha$ which is a contradiction.

□

Proof. (theorem 2.23)

By lemma 2.24, if κ is inaccessible then it is Π^1_0-indescribable. The inverse is also true. In order to show this, one must go through the second part of the proof of lemma 2.24 and notice that instead of the elementary submodel, we could just use a submodel of $(V_\kappa, \in, R)$ which satisfies a specific formula.

Thus we only need to show that Π^1_0-indescribability is equivalent to Σ^1_1-indescribability:

- If a formula ϕ is Σ^1_1 then it is also Π^1_0 so the one direction is obvious.
- Suppose that κ is Π^1_0-indescribable and let $(V_\kappa, \in, R) \vDash \exists X\ \phi(X)$. Then for some $S \subset V_\kappa$:

$$\begin{aligned} (V_\kappa, \in, R) \vDash \phi(S) \rightarrow & (V_\kappa, \in, \langle R, S\rangle) \vDash \phi(S) \rightarrow \\ \rightarrow & \exists \alpha < \kappa\ (V_\alpha, \in, \langle R \cap V_\alpha, S \cap V_\alpha\rangle) \vDash \phi(S) \rightarrow \\ \rightarrow & \exists \alpha < \kappa\ (V_\alpha, \in, R \cap V_\alpha) \vDash \exists X\ \phi(X) \end{aligned}$$

thus κ is Σ^1_1-indescribable.

□

2.3 Mahlo cardinals

In the previous section we introduced inaccessible cardinals and described the way they extend the universe of sets. Since we can extend the universe by adding one inaccessible cardinal to it, we could do the same by adding another one and repeat this procedure many times.

Up to this point, the universe has been extended only by the use of inaccessible cardinals. Thus, in order to get one step further, we have to introduce a new kind of cardinals, such that the set of inaccessible cardinals below them is unbounded. This means that if κ is one of those cardinals, it must be an inaccessible which is a strong limit of inaccessible cardinals. Again, by repeating the new procedure we can create new cardinals which are strong limits of the ones of the previous kind. This way we get the following sequence of classes:

$$\begin{aligned} P_0 =& \{\kappa : inaccessible(\kappa)\} \\ P_{\alpha+1} =& \{\kappa : \kappa \text{ strong limit of } P_\alpha\} \\ P_\alpha =& \cap_{\beta<\alpha} P_\beta \ if \ limord(\alpha) \end{aligned}$$

The cardinals belonging to P_α are called α-inaccessible and π_α is the least element of P_α. Someone may notice that we have not defined a set where the ordinal α belongs. This was done on purpose because while extending the universe with new cardinals, new ordinals are being created thus we can further extend this sequence. As before we may introduce a new kind of cardinals κ, such that $\kappa \in P_\kappa$, and build inaccessible cardinals over them and go on this way.

In an attempt of finding an upper bound to this way of extending the universe, Mahlo cardinals were introduced. We will define those large cardinals below, but before we do that we will give some information on the closed unbounded (club) filter and on stationary sets.

Definition 2.25. Let κ be a regular uncountable cardinal and $X \subset \kappa$.

(i) X is *unbounded* in κ, if $\sup(X) = \kappa$.

(ii) X is closed, if for every increasing sequence $\{x_\beta\}_{\beta<\alpha}$ of elements of X, $\bigcup_{\beta<\alpha} x_\beta \in X$.

It is not hard to prove that for any given regular uncountable cardinal, the closed and unbounded subsets of κ form a κ-complete[2] filter on κ which is closed under diagonal intersections[3]. This filter is called the *closed undounded (club)* filter on κ. Let F be the club filter on κ. Since it is not an ultrafilter, F divides $P(\kappa)$ into three categories; the subsets of κ that belong to F (big sets), their

[2] Closed under α-intersections, $\alpha < \kappa$.

[3] The diagonal intersection of a family of sets of ordinals $\{X_\beta\}_{\beta<\alpha}$ is defined as follows:

$$\triangle_{\beta<\alpha} X_\beta = \{\gamma < \alpha : \gamma \in \bigcap_{\beta<\gamma} X_\beta\}$$

complements which belong to I (small sets), and the rest of $P(\kappa)$ (sets which are not small or big). The latter, together with the sets of F, are called stationary. We formally define them in the following way:

Definition 2.26. Let κ be as in 2.25. A set S is a *stationary* subset of κ, if for every X belonging to the club filter of κ, $S \cap X \neq \emptyset$.

There are two important theorems on stationary sets which should be stated, in order to present the richness of their structure.

Definition 2.27. A function $f : X \to \bigcup X$, where X is a set of ordinals, is *regressive* if for every $\alpha \in X$ $f(\alpha) < \alpha$.

Theorem 2.28. *(Fodor) Suppose f is a regressive function on a stationary subset S of κ. Then, there is a stationary set $T \subset S$, such that f is constant on T.*

Theorem 2.29. *(Solovay) Every stationary subset of κ, is equal to the union of κ disjoint stationary subsets of κ.*

Definition 2.30. κ is a *Mahlo cardinal* if the set $\{\lambda < \kappa : inaccessible(\lambda)\}$ is stationary in κ.

This is a quite reasonable definition for this notion. One could require that the set of inaccessible cardinals in κ is club, in order to ensure that they are unbounded in κ and on the same time, that if we consider one of the sequences mentioned before, its limit is inside κ. Unfortunately, this way we would have inaccessible cardinals with cofinality $\aleph_0$, which is a contradiction, so instead we require that the set of inaccessible cardinals in κ intersects all the club sets of κ. This way the notion of stationary sets is introduced and it is sufficient to ensure the preceding facts. As an example we prove the following:

Theorem 2.31. *If κ is a Mahlo cardinal, then κ is κ-inaccessible.*

Proof. Let S be the set $\kappa \cap P_0$ and C_α the sets defined by transfinite recursion:

$$\begin{aligned} C_0 &= \kappa \\ C_{\alpha+1} &= limitpoints(C_\alpha \cap S) \setminus \{\kappa\} \\ C_\alpha &= \cap_{\beta<\alpha} C_\alpha \end{aligned}$$

κ is a club set. If C_α is club then $C_{\alpha+1} \cap S$ is stationary so the set of its limit points is club. If $\alpha < \kappa$ then, since the club filter is κ-complete, $\cap_{\beta<\alpha} C_\alpha$ is also club. By transfinite induction,

- $C_0 \cap S = \kappa \cap P_0$
- $C_{\alpha+1} \cap S = limitpoints(\kappa \cap P_\alpha \setminus \{\kappa\}) \cap S = \kappa \cap P_{\alpha+1}$
- $C_\alpha \cap S = \cap_{\beta<\alpha} P_\beta \cap S = \cap_{\beta<\alpha} P_\beta \cap \kappa = \kappa \cap P_\alpha$

□

Just as with inaccessible cardinals, we can express the existence of a Mahlo cardinal using elementary embeddings.

Theorem 2.32. *κ is Mahlo iff for every $R \subset V_\kappa$ there is an inaccessible $\lambda < \kappa$ such that $(V_\lambda, \in, R \cap V_\lambda) \prec (V_\kappa, \in, R)$.*

Proof. If not, there would be a C belonging to the club filter of κ, containing no inaccessible cardinals. By setting $R = C$, we get that $(C \cap \lambda) \cap R = C \cap \lambda$ is unbounded thus $\lambda \in C$ which is not true. □

Another important fact is that the existence of a Mahlo cardinal is not implied by the existence of an inaccessible cardinal.

Theorem 2.33.

(i) $ZFC + IC \nvdash \exists\kappa\ Mahlo(\kappa)$

(ii) If there exist Mahlo cardinals, then the least of them is greater than the least inaccessible cardinal.

(iii) Every Mahlo cardinal is inaccessible.

Proof. (i) Let κ be the least Mahlo cardinal. V_κ is a model of $ZFC + IC$ where no Mahlo cardinals exist.

(ii)-(iii) They are straightforward from the definition of Mahlo cardinals.

□

We could even further iterate the methods used before and create bigger cardinals but unfortunately we can not create all the known large cardinals this way. We will give a last example of this method by introducing the sequence of κ-Mahlo cardinals:

$$\begin{aligned} R_0 =& \{\kappa : inaccessible(\kappa)\} \\ R_{\alpha+1} =& \{\kappa : \{\lambda < \kappa : \lambda \text{ is Mahlo}\} \text{ is stationary in } R\} \\ R_\alpha =& \{\kappa : \forall\beta < \alpha\ \kappa \in R_\beta\ if\ limord(\alpha)\} \end{aligned}$$

We denote the least element of R_α by ρ_α.

Finally, we note that by dropping the request that the inaccessible or Mahlo cardinals are strong limits, we get their weak forms, i.e.

Definition 2.34.

(i) A cardinal is called *weakly inaccessible* if it is limit and regular.

(ii) A cardinal is called *weakly Mahlo* if the set of weakly inaccessible cardinals below it is stationary.

2.4 Weakly compact, Erdős and Ramsey cardinals

In this section we are going to give a brief introduction to weakly compact, Erdős cardinals and Ramsey cardinals. Their common feature, is that they can be defined using the same kind of partition properties. We define below this kind of properties and provide some results describing some cases where they are true or they are not true.

Definition 2.35.

(i) A *partition* of a set X into λ pieces is a function $f : X \to \lambda$.[4] If f is a partition of $[\kappa]^n$ into m pieces, then a set $Y \subset \kappa$ is called homogeneous if f is constant on Y.

(ii) $\kappa \to (\alpha)^n_\lambda$ is true, if every partition of $[\kappa]^n$ into λ pieces has a homogeneous set of order type α.

(iii) $\kappa \to (\alpha)^{<\omega}_\lambda$ is true if every partition of $[\kappa]^{<\omega}$ into λ pieces has a homogeneous set of order type α.

In every notation above, we drop the subscript λ whenever it is equal to 2.

Theorem 2.36. *(Ramsey) For all integers m, n, $\aleph_0 \to (\aleph_0)^n_m$.*

Lemma 2.37. *For all cardinals κ:*

(i) $2^\kappa \not\to (\aleph_0)^2_\kappa$;

(ii) $2^\kappa \not\to (\kappa^+)^2$.

Theorem 2.38. *(Erdős-Rado)*

$$\beth_n^+ \to (\aleph_1)^{n+1}_\omega.$$

Weakly compact cardinals

Definition 2.39. A cardinal, κ, is called *weakly compact* if it satisfies the weak compactness theorem in the language $\mathcal{L}_{\kappa,\kappa}$[5]. Specifically, for every set of sentences Σ such that $|\Sigma| \leq \kappa$, if every $S \subset \Sigma$, $|S| < \kappa$ has a model, then Σ has a model.

We are going to give three equivalent definitions of weakly compact cardinals. The proof of their equivalence can be found in [8] and [10].

Theorem 2.40. *The following are equivalent:*

(i) κ *is weakly compact.*

---

[4]This is equivalent with the usual definition of a partition, since f divides X to the λ parts $f^{-1}(0), f^{-1}(1), \ldots$.

[5]For a definition and more information on infinitary languages see [8] or [1].

(ii) $\omega < \kappa$ *and* $\kappa \to (\kappa)^2$.

(iii) κ *is* Π^1_1*-indescribable.*

(iv) *For every* $R \subset V_\kappa$ *there is an transitive set* $X \neq V_\kappa$ *and an* $S \subset X$ *such that*
$(V_\kappa, \in, R) \prec (X, \in, S)$.

Weakly compact cardinals are stronger than Mahlo cardinals in the following sense:

Theorem 2.41. *The set of Mahlo cardinals below a weakly compact cardinal* κ *is stationary thus:*

(i) $ZFC + \exists\kappa\ Mahlo(\kappa) \nvdash \exists\lambda\ w.compact(\lambda)$

(ii) *If there exist weakly compact cardinals, the least of them is greater than the least Mahlo cardinal.*

(iii) *Every weakly compact cardinal is Mahlo.*

Proof. κ is Π^1_1 thus Π^1_0, which means that it is inaccessible. If C is in the club filter of κ, then $(V_\kappa, \in, C)$ satisfies the sentence

$$ZFC^2 \wedge unbounded(C).$$

By the hypothesis, there is an $\alpha < \kappa$ satisfying the same sentence for $C \cap V_\alpha$. Since $V_\alpha \vDash ZFC$, α is inaccessible and additionally C is unbounded in α, so $C \cap \alpha \neq \emptyset$. Thus, κ is Mahlo. Using the same argument and the sentence

$$\forall X(closed(X) \wedge unbounded(X) \to \exists\lambda \in X\ inaccessible(\lambda))$$

we can prove that the set of Mahlo cardinals below κ is stationary.

(i),(ii) and (iii), are immediate consequences of the above result (also see theorem 2.33). □

Erdős cardinals

Definition 2.42.

(i) $H_\alpha = \{\kappa : \kappa \to (\alpha)^{<\omega}\}$;

(ii) The Erdős cardinal η_α, $\alpha \in On$, is the least cardinal in H_α.

There is an interesting connection between Erdős cardinals and the axiom of constructability. In particular, the existence of η_ω is consistent with $V = L$ though η_{ω_1} is not. We shall see in the next chapter that this is related with the elementary embeddings of L.

Below we state, without proof, some properties of Erdős cardinals. For the proof see [8] or [10].

Theorem 2.43.

(i) $\eta_\alpha \to (\alpha)^{<\omega}_\beta$ *for every* $\beta < \eta_\alpha$

(ii) $\eta_\alpha < \eta_\beta$ *for all* $\alpha < \beta$ *in On*

The next theorem settles the relation between weakly compact and Erdős cardinals.

Theorem 2.44.

(i) *If* η_ω *exists, then there is a weakly compact cardinal below* η_ω*.*

(ii) $ZFC + \exists\kappa$ *w.compact*$(\kappa) \nvdash \exists\lambda$ *Erdős*(λ)

(iii) *Every Erdős cardinal is weakly compact.*

We left for the end the following result which implies that η_ω is consistent with $V = L$.

Theorem 2.45. $\kappa \in H_\omega \to L \models \kappa \in H_\omega$ *for all cardinals* κ*. In particular,*

$$\exists\kappa\ \kappa = \eta_\omega \to L \models \exists\kappa\ \kappa = \eta_\omega.$$

On the other hand $H_{\omega_1} \neq \emptyset \to V \neq L$, thus the point where the universe no longer has chance of being constructible, is between η_ω and η_{ω_1}.

Ramsey cardinals

Ramsey cardinals can be viewed as a result of diagonalizing the sequence of Erdős cardinals. Specifically,

Definition 2.46. A cardinal κ is called a *Ramsey cardinal* if $\kappa \to (\kappa)^{<\omega}$.

The existence of a Ramsey cardinal κ, implies the existence of κ Erdős cardinals below κ. Hence, Ramsey cardinals are stronger than Erdős. Ramsey cardinals are the strongest ones we have encountered up to now. In order to get further up we need a new notion capable of producing even stronger large cardinals. The notion we are going to use is that of elementary embeddings of the form $V \to M$. In the next chapter we will use it to define measurable cardinals, and in the last two chapters we will extend it as far as possible and try to reach its limitations.

Chapter 3

Measurable cardinals and elementary embeddings of V

3.1 Aspects of measurability

This section serves as a brief reference to the foundations of the theory of measurability, introduced by Lebesque. Lebesque's attempt to define a translation-invariant measure on $\mathbb{R}$ was unsuccessful and later work, by Vitali, showed that such a measure can not exist (provided that AC holds). A generalization of the notion of measure, by Banach, raised new possibilities for the existence of such a measure, which is not restricted by translation-invariance. Finally, Ulam realized that there are essentially two types of measures: the ones taking values on a dense subset of $[0,1]$, called *atomless*, and the ones whose valuation is restricted on $\{0,1\}$, called *two-valued*. The existence of measurable cardinals is equivalent to the existence of two-valued measures.

The measure problem

A measure, on a set S, as it was defined by Banach, is a function $m : P(S) \to [0,1]$ having the following properties:

(i) $m(S) = 1$;

(ii) $m(\{x\}) = 0$, forall $x \in S$ (nontriviality);

(iii) $m(\bigcup_n X_n) = \sum_n m(X_n)$, $\forall n \in \omega\ X_n \subset S$ *and* $\forall i, j \in \omega\ X_i \cap X_j = \emptyset$.

Such a measure is called *σ-additive*, because of property (iii). If the same property holds for all subsets of $P(S)$ with cardinality less than κ, then the corresponding measure is called *κ-additive*.

We are now going to give some definitions. They are going to assist us in separating the σ-*additive* measures into two categories.

Definition 3.1.

(i) A measure m is called *two valued* if $m''S = \{0, 1\}$.

(ii) A set $A \subset S$ is called an *atom* if $m(A) > 0$ and $\forall B \subset A$ $(m(B) = 0 \vee m(B) = m(A))$.

(iii) A measure is called *atomless* if it has no atoms.

A simple but useful lemma in measure theory is the following:

Lemma 3.2. *Let S be a set and m a σ-additive measure on it. Every set of pairwise disjoint sets $\{X_\alpha : \alpha < \gamma\} \subset P(S)$, such that $m(X_\alpha) > 0$ for every $\alpha < \gamma$, is countable.*

Proof. If not, there would be an n such that the set $A = \{X_\alpha : \alpha < \gamma \wedge \frac{1}{n} < m(X_\alpha)\}$ is uncountable. By σ-additivity, the union of more than n elements of A has measure more than 1 which is a contradiction. □

Lemma 3.3. *If there exists a measure m on S, then S can be partitioned into, at most, a set B and ω sets $A_n, n < \omega$ with the following properties:*

(i) the measure $m_B : P(B) \to [0, 1]$, $m_B(X) = \frac{m(X)}{m(B)}$ is atomless;

(ii) the measures $m_{A_n} : P(A_n) \to [0, 1], m_{A_n}(X) = \frac{m(X)}{m(A_n)}$ are two-valued.

Proof. We define, by transfinite induction and DC, the sequence $C_\alpha \subset S$, $\alpha < \omega_1$,

$$C_\alpha = \text{an atom on the set } B_\alpha = S \setminus \bigcup_{\alpha<\omega_1} C_\alpha \text{ with measure } m_{B_\alpha};$$
$$\text{or } \emptyset \text{ if there exists none .}$$

By lemma 3.2, only countable sets of the sequence $C_{\alpha_n} = A_n$, $n < \omega$ are nonempty thus $B = S \setminus \bigcup_{n<\omega} A_n$ contains no atoms of S as subsets. It is straightforward that the preceding sets have the required properties. □

The above lemma suggests that we only have to deal with atomless and two-valued measures. We will restrict ourselves on measures defined on a cardinal κ and investigate both cases. In order to do this though, our measure must be κ-additive on κ. This is guaranteed by the following lemma which states that if there exists a σ-additive measure on a cardinal κ, then there exists a cardinal λ having a λ-additive measure.

Lemma 3.4. *The least cardinal $\omega < \kappa$ having a σ-additive measure, is such that every measure on κ is κ-additive.*

Proof. Suppose m is a measure on κ which is not κ additive and $\{X_\alpha : \alpha < \gamma < \kappa\}$ is a set of pairwise disjoint subsets of S with the property $m(\bigcup_{\alpha<\gamma} X_\alpha) \neq \sum_{\alpha<\gamma} m(X_\alpha)$. Since m is σ-additive, we can remove the -at most countable by 3.2- sets with positive measure and get a set $\{Y_\alpha : \alpha < \gamma\}$ where $m(Y_\alpha) = 0$ and $0 < m(\bigcup_{\alpha<\gamma} Y_\alpha) = s$. The measure

$$m^* : P(\gamma) \to [0,1],\ m^*(X) = \frac{m(\bigcup_{\alpha\in X} Y_\alpha)}{s}$$

is σ-additive on $\gamma < \kappa$ which implies that κ cannot be the smallest carrying having a measure. □

Thus being interested, for our purpose, only in the existence of a measure and not in the size of the set which has this measure, we will restrict our study to the following type of measures:

Definition 3.5. A cardinal $\omega < \kappa$ will be called *real-valued measurable* if there exists a κ-additive measure on κ.

We are going to show that real valued cardinals are weakly inaccessible. This will be carried out by the use Ulam's matrices.

Definition 3.6. An *Ulam* (λ, μ) *matrix* is a collection $\{A_{\alpha,\xi} : \alpha < \lambda,\ \xi < \mu\}$ of subsets of λ, with the properties:

(i) $\alpha \neq \beta \to \forall \xi < \mu\ A_{\alpha,\xi} \cap A_{\beta,\xi} = \emptyset$;

(ii) $\forall \alpha < \lambda\ |\lambda \setminus \bigcup_{\xi<\mu} A_{\alpha,\xi}| \leq \mu$.

Theorem 3.7. *(Ulam) An Ulam* (λ^+, λ) *matrix exists.*

Proof. Since λ^+ is the successor cardinal of λ, there is a family of functions $f_\alpha : \lambda \to \lambda^+,\ \alpha < \lambda^+$ such that $\alpha \subset rang(f_\alpha)$. We let $A_{\alpha,\xi} = \{\gamma : f_\gamma(\xi) = \alpha\}$, hence

- $A_{\alpha,\xi} \cap A_{\beta,\xi} = \{\gamma : \alpha = f_\gamma(\xi) = \beta\} = \emptyset\ if\ \alpha \neq \beta$;
- $|\lambda^+ \setminus \bigcup_{\xi<\mu} A_{\alpha,\xi}| = |\{\beta : \alpha \notin rng(f_\beta)\}| < \lambda^+$.

Thus, the set containing $A_{\alpha,\xi},\ \alpha < \lambda,\ \xi < \mu$ is a (λ^+, λ) Ulam matrix. □

Corollary 3.8. *Every real-valued measurable cardinal* κ *is weakly inaccessible.*

Proof. Suppose m is a κ-additive measure on κ. First of all, we should notice that if $X \subset \kappa \wedge |X| < \kappa$, then

$$m(X) = m(\bigcup\{\{x\} : x \in X\}) = \sum_{x\in X} m(\{x\}) = 0.$$

If κ is not regular then $\kappa = \sum_{\alpha<\mu} \lambda_\alpha,\ \mu, \lambda_\alpha < \kappa$ thus $|\kappa| = \sum_{\alpha<\mu} |\lambda_\alpha| = 0$ which is a contradiction.

Suppose that $\kappa = \lambda^+$. Let us consider an Ulam (λ^+, λ) matrix, just as the one in the above theorem. By (ii) of theorem 3.7, for every $\alpha < \lambda$ there is some ξ_α such that $m(A_{\alpha,\xi_\alpha}) > 0$. Since $\lambda < \lambda^+$ there is a ξ for which there are λ^+ pairwise disjoint sets $A_{\alpha,\xi}$ of the matrix, such that $|A_{\alpha,\xi}| > 0$, which is a contradiction by lemma 3.2. Hence, κ must be a limit cardinal. □

Atomic measures

In order to deal with the case where an atomless measure exists, we quote a lemma concerning partition properties of atomless measures and afterwards a theorem, which clarifies this case.

Lemma 3.9. *Let m be an atomless κ-additive measure over κ. Then:*

(i) For any $\epsilon \in [0,1]$ and $X \subset \kappa$, $0 < m(X)$ there is a $Y \subset X$ such that $0 < m(Y) < \epsilon$.

(ii) Every set $X \subset \kappa$ can be partitioned into two sets A, B satisfying $m(A) = m(B) = \frac{1}{2}m(X)$.

Proof.

(i) Since m is atomless, X can be divided into two sets A, B satisfying $0 < m(A) < m(X), 0 < m(B) < m(X)$. A and B are disjoint, thus one of them has measure at most $\frac{1}{2}m(X)$. By repeating this procedure recursively, for sets such that $0 < m(X_{i+1}) \leq \frac{1}{2}m(X_i)$, more than $\log_2 \frac{1}{\epsilon}$ times, we get a set having the required properties.

(ii) We define a sequence $X_\alpha, \alpha < \omega_1$ as follows:

- $X_0 = X$;
- $\frac{1}{2}m(X) < m(X_{\alpha+1}) \leq m(X_\alpha)$, $X_{\alpha+1} \subset X_\alpha$;
- $X_\alpha = \bigcap_{\beta<\alpha} X_\beta$ if *limord*(α).

It is easy to see by induction that every X_α has measure greater or equal to $\frac{1}{2}m(X)$. If those measures were all greater than $\frac{1}{2}m(X)$, then the set $\{X_\alpha \setminus X_{\alpha+1} : \alpha < \omega_1\}$ would be an uncountable collection of sets of nonzero measure, which contradicts lemma 3.2.

□

Theorem 3.10. *(Ulam) If there exists an atomless κ-additive measure m over κ, then:*

(i) $\kappa \leq 2^{\aleph_0}$;

(ii) there is a measure μ over $\mathbb{R}$ extending Lebesque's measure (for a definition and more information on measure theory see [16]).

Proof.

(i) The idea behind this proof is that the partition property of lemma 3.9 (ii) leads to the construction of a partition of κ into $2^{\aleph_0}$ sets of measure 0, which means that m is not $(2^{\aleph_0})^+$-additive, thus $\kappa \leq 2^{\aleph_0}$.

The sets $X_s \subset \kappa,\ s \in \omega^{<\omega}$ are the ones who generate this partition. We define

– $X_\emptyset = \kappa$;

– $X_s = \bigcup_{n<\omega} X_{s^\frown\langle n\rangle}$, $m(X_{s^\frown\langle n\rangle}) = \frac{1}{2^{i+1}} m(X_s)$;
(partition property (ii) applied ω times)

where the sets $X_{s^\frown\langle n\rangle},\ n < \omega$ are pairwise disjoint. Finally, we define $X_f = \bigcap_{n<\omega} X_{f\restriction n},\ f \in \omega^\omega$ and it is straightforward that $m(X_f) = 0$, because for every $n < \omega$ $m(X_f) < \frac{1}{2^n}$.

(ii) We first define the measure m^* on $A = \{X_f : f \in \omega^\omega\}$ with the property $m^*(Y) = m(\bigcup Y)$ for every $Y \subset A$. The transformation $F : A \to [k, k+1],\ k \in \mathbb{Z}$, where $F(X_f) = k + \sum_{n<\omega} m(X_{f\restriction n})$, is one to one, onto $[k, k+1]$ and thus $[k, k+1]$ inherits, in a natural way, a measure $\mu_{[k,k+1]}$. Those measures, for different $k \in \mathcal{Z}$, generate the measure μ, $\mu(X) = \sum_{k\in\mathcal{Z}} m(X \cap [k, k+1]), X \subset \mathbb{R}$. μ can be easily checked to be an extension of Lebesque's measure.

□

If we consider that κ, in the above theorem is weakly inaccessible, we come to the conclusion that the existence of an atomless measure results to the failure of the CH. Even more, the following holds:

Theorem 3.11. *(Banach-Kuratowski) $CH \to$ there is no measure on $2^{\aleph_0}$.*

Proof. See [8]. □

Two valued measures

We now turn to the case of two-valued measures.

Lemma 3.12. *A two-valued κ-additive measure m on a set S defines a κ-complete nonprincipal ultrafilter U such that:*

$$U = \{X \subset S : m(X) = 1\}.$$

Conversely, a κ-complete nonprincipal ultrafilter U on S defines a two-valued κ-additive measure m on S such that:

$$m(X) = \begin{cases} 0 & X \notin U \\ 1 & X \in U \end{cases}$$

Proof. It is straightforward from the corresponding definitions. □

If the measure is also real valued on κ, it defines a κ-complete nonprincipal ultrafilter on κ. This way the notion of measurable cardinals, which plays an important role on the theory of large cardinals, turns up.

Definition 3.13. A cardinal $\omega < \kappa$, is called measurable if there exists a κ-complete nonprincipal ultrafilter on κ.

Lemma 3.14. *Every measurable cardinal κ is inaccessible.*

Proof. κ is also real-valued measurable thus it is regular. Let $\lambda < \kappa$ and $\kappa \leq 2^{\lambda}$. Let $f : \kappa \to 2^{\lambda}$ be a 1-1 function and U a nonprincipal κ-complete ultrafilter on κ. Then, there is a sequence of sets X_α and $d_\alpha \in \{0, 1\}$, $\alpha < \lambda$ such that $X_\alpha = \{\xi < \kappa : f(\xi)(\alpha) = d_\alpha\} \in U$. The cut of those sets belongs to U since it is κ-complete but it has one member which opposes to the nonprincipality of the ultrafilter. Hence, κ is strong limit. □

Combining the results we have mentioned up to now in this section we get the following theorem:

Theorem 3.15. *Suppose there is a σ-additive measure on S. Then, the following are true:*

(i) If there exists a set, such that A m_A is two-valued, in the partition described in lemma 3.3, then there exists a measurable cardinal.

(ii) If there exists a set B, such that m_B is atomless, in the partition mentioned above, then there is a real-valued atomless measure on a weakly inaccessible cardinal $\kappa \leq 2^{\aleph_0}$.

Remark 3.16. We have to notice here that the existence of a real-valued atomless measure is not only inconsistent with the CH but it also implies that the gap between $\aleph_1$ and $2^{\aleph_0}$ is wide. On the other hand, the existence of a measurable cardinal does not seem to effect CH, as it is inaccessible thus greater than $2^{\aleph_0}$. In fact, using mild forcing extensions, one can prove that assuming the existence of a measurable cardinal the GCH can either be true or false (see [9] or [8]).

3.2 Elementary embeddings of V

The elementary embeddings of V are tightly related to measurable cardinals. In particular, the existence of a nontrivial elementary embedding[1] is equivalent to the existence of a measurable cardinal. In this section we are going to prove this result, perform a quick investigation on elementary embeddings and see some of the consequences of their existence. One of those will be the fact that $V \neq L$ if a measurable cardinal exists. This is an important result due to Scott.

[1]Different from the identity function.

Elementary embeddings and ultrapowers of inner models

We have defined elementary embeddings and ultrapowers for models which are sets, but here we have to deal with inner models which are classes. The easiest way to redefine those notions, would be by working on a higher order set theory such as Morse-Kelley, but it can still be done in ZFC and thus maintain the simplicity of first order logic. From now on we will work on the extended language $\mathcal{L} = \{\in, j\}$, where j is a function symbol. We will also add new axioms which state that j is an elementary embedding and that the axiom of replacement holds in the extended language.

Lemma 3.17. *Let M, N be inner models such that $M \prec N$, then:*

(i) $ordinal(\alpha) \to ordinal(j(\alpha)) \wedge \alpha \leq j(\alpha)$;

(ii) if j is nontrivial and $N \subset M$, then $\exists \delta \in On\ \delta < j(\delta)$.

Proof.

(i) The term $ordinal(x)$ is preserved by j, thus $ordinal(j(\alpha))$. By induction,

- $j(0) = 0$;
- $\alpha \leq j(\alpha) \to j(\alpha + 1) = j(\alpha) + 1 \geq \alpha + 1$;
- if $\forall \beta < \alpha\ \beta \leq j(\beta)$ then $\forall \beta < \alpha\ j(\alpha) \geq j(\beta) \geq \beta \to j(\alpha) \geq \alpha$;

hence for all ordinals α, $\alpha \leq j(\alpha)$.

(ii) Suppose x is a set of least rank such that $j(x) \neq x$ and $rank(x) = \alpha$. Suppose $j(\alpha) = \alpha$,then

$$rank(x) = \alpha = j(\alpha) = j(rank(x)) = rank(j(x)).$$

We also have that $y \in x \to y = j(y) \in j(x)$, thus $x \subset j(x)$ but $rank(j(x)) = rank(x)$ and $N \subset M$, so $y \in j(x) \to y = j(y) \to y \in x$ which means that $x = j(x)$, contradiction. Hence $\alpha < j(\alpha)$.

□

Definition 3.18. The least ordinal α such that $\alpha < j(\alpha)$ will be called the *critical point* of j.

We will now focus our interest in generalizing the technique of ultrapowers to classes, and specifically to V. The difficulty here is in defining the classes of functions so that they are sets.

Definition 3.19. Let U be an ultrafilter on S, M an inner model and $f, g, h : S \to M$, then:

(i) $f =_U g \leftrightarrow \{x \in S : f(x) = g(x)\} \in U$;

(ii) $f \in_U g \to \{x \in S : f(x) \in g(x)\} \in U$;

(iii) $[f] = \{g : f =_U g \wedge \forall h\ (h =_U f \to rank(g) \leq rank(h))\}$;

(iv) $Ult(M, U) = (\{[f] : f : S \to M\}, \in_U)$.[2]

[2] We use the same symbol for inclusion in classes, i.e. $[f] \in_U [g] \leftrightarrow f \in_U g$.

The following theorems also hold for classes and they are proved the same way as for sets.

Theorem 3.20.

(i) For any fomula ϕ and functions $f_1, \ldots, f_n : S \to M$,

$$Ult(M,U) \vDash \phi[[f_1], \ldots, [f_n]] \leftrightarrow \{x \in S : \phi[f_1(x), \ldots, f_n(x)]\} \in U$$

(Thus $Ult(M,U) \vDash ZFC$).

(ii) $j : M \to Ult(M,U)$, $j(x) = [c_x]$, where $c_x : S \to M$ is the identity function, is elementary. We call this embedding the canonical embedding.

Elementary embeddings of V

Now that we have generalized the notion of elementary embeddings and ultrapowers, we shall introduce our first example of a nontrivial elementary embedding $j : V \prec Ult(V,U)$ and study its properties. Before we do this though, we must make sure that $Ult(V,U)$ is an inner model, or at least that it is isomorphic to one. This will be carried out through the following lemma.

Lemma 3.21.

(i) The model $Ult(V,U)$ is extensional.

(ii) For every $[f] \in Ult(V,U)$, $ext_{\in_U}([f])$ is a set.

(iii) If U is σ-complete, then $Ult(V,U)$ is also well-founded.

Proof.

(i) $Ult(V,U) \vDash ZFC$ thus $Ult(V,U) \vDash$"extensionality axiom".

(ii) Let $[g] \in_U [f]$. By setting

$$h(x) = \begin{cases} g(x) & g(x) \in f(x) \\ \emptyset & else \end{cases}$$

we get a function h such that $h =_U g$, and $rank(h) \leq rank(f)$. Hence $ext_{\in_U}([f]) \in V_{rank(f)+1}$, which means that it is a set.

(iii) Suppose U is σ-complete and $Ult(V,U)$ is not well-founded. In this case there is an ω-sequence $\{[f_n]\}_{n<\omega}$ such that $\forall n < \omega\ [f_{n+1}] \in_U [f_n]$. Hence the sets $X_n = \{x \in S : f_{n+1}(x) \in f_n(x)\}$ are all members of U, and by σ-completeness $\bigcap_{n<\omega} X_n \in U \to \bigcap_{n<\omega} X_n \neq \emptyset$. If $x \in \bigcap_{n<\omega} X_n$, then there exists an infinite decreasing sequence $f_0(x) \ni f_1(x) \ni \ldots$, which is a contradiction, thus $Ult(V,U)$ is well-founded.

□

The above lemma combined with Mostowski's collapsing lemma produces an inner model M_U, isomorphic to $Ult(V, U)$. For our convenience, we will denote the elements of M_U by their isomorphic images in $Ult(V, U)$.

Theorem 3.22. *Suppose there is a measurable cardinal κ and U is a nonpriciple κ-complete ultrafilter on κ. Then the canonical embedding $j : V \prec M_U$ is a nontrivial elementary embedding.*

Proof. By theorem 3.20 we know j is elementary so we only need to show that it is nontrivial. We will first show that $\forall\alpha < \kappa\ j(\alpha) = \alpha$ and then, using this, we will prove that $\kappa < j(\kappa)$.

- $j(0) = 0$ and $\forall\alpha \in On\ j(\alpha + 1) = j(\alpha) + 1$ by the absoluteness of the corresponding terms. Suppose that $\alpha < \kappa \wedge limord(\alpha)$ and for every ordinal β less than α, $j(\beta) = \beta$. If $[f] < j(\alpha) = [c_\alpha]$ then for almost all $x \in \kappa\ f(x) < \alpha$ and since $\alpha < \kappa$, by the κ-completeness of U, there is a $\beta < \alpha$ such that $f =_U c_\beta$. Hence $j(\alpha) \leq \alpha$ so by lemma 3.17 $j(\alpha) = \alpha$.
- Let $id : \kappa \to V,\ id(\alpha) = \alpha$. Since U is κ-complete $|S \setminus x| < \kappa \to x \in U$, thus $\alpha = j(\alpha) = [c_\alpha] < [id],\ \alpha < \kappa$ which means that $\kappa \leq [id]$. On the other hand, $[id] < [c_\kappa] = j(\kappa)$, hence $\kappa < j(\kappa)$.

□

We will now prove the inverse:

Theorem 3.23. *(Keisler) If there is a nontrivial elementary embedding $j : V \prec M$, then its critical point is a measurable cardinal.*

Proof. Suppose κ is the critical point of j. κ is greater than ω so we only have to check that there is a κ-complete nontrivial ultrafilter U on κ. Let $U = \{X \subset \kappa : \kappa \in j(X)\}$, then:

- $j(\emptyset) = 0 \not\ni \kappa$, $\kappa \in j(\kappa)$, $j(X \cap Y) = j(X) \cap j(Y)$, $X \subset Y \to j(X) \subset j(Y)$ and $j(\kappa \setminus X) = j(\kappa) \setminus X$ by elementarity of j, therefore it is clear that U is an ultrafilter.
- If $\alpha < \kappa$, then $j(\{\alpha\}) = \{j(\alpha)\} = \{\alpha\}$, hence $\kappa \notin j(\alpha)$ which means that U is nonprincipal.
- Let $\alpha < \kappa$ and $S = \{X_\beta\}_{\beta<\alpha}$ be a sequence of subsets of κ such that for every β, $\kappa \in j(X_\beta)$. $j(S)$ will also be a sequence, of length $j(\alpha) = \alpha$, having the property $j(S)(\beta) = j(X_\beta),\ \beta < \alpha$. Hence

  $$j(\bigcap S''\alpha) = \bigcap j(S)''\alpha = \bigcap_{\beta<\alpha} j(X_\beta)$$

  which means that $\kappa \in j(\bigcap S''\alpha)$, so U is κ-complete.

□

Corollary 3.24. *The existence of a nontrivial elementary embedding $j : V \prec M$ is equivalent to the existence of a measurable cardinal.*

Proof. It has been carried out by the two preceding theorems. The only difference here is that, since we want to talk about the existence of an embedding j, we must work in the extended language $\mathcal{L}^* = \{\in, j\}$ or think of this result as a metatheorem. □

Corollary 3.25. *(Scott) If there exists a measurable cardinal then $V \neq L$.*

Proof. Suppose κ is the least measurable cardinal, U a κ-complete nonprincipal ultrafilter over κ and $j : V \prec M_U$ the elementary embedding described above. If $V = L$ then, since L is contained in every inner model, $M_U = L = V$. j is elementary so $V = M_U \vDash$"$j(\kappa)$ is the least measurable cardinal" which is a contradiction because $\kappa < j(\kappa)$. □

The last result, due to Scott, yields that the existence of large cardinals can drastically change the structure of V. It was also the first result which introduced a connection between elementary embeddings of V and large cardinals. We will now give some more information on the elementary embeddings of the form $j : V \prec M_U$.

Lemma 3.26. *If U is a nonprincipal κ-complete ultrafilter on κ and $j : V \prec M_U$ is the canonical elementary embedding, then:*

(i)
- $\forall x \in V_\kappa\ j(x) = x$ *so* $V_\kappa^{M_U} = V_\kappa$;
- $\forall X \subset V_\kappa\ j(X) \cap V_\kappa = X$, *thus* $V_{\kappa+1}^{M_U} = V_{\kappa+1}$ *and* $(\kappa^+)^{M_U} = \kappa^+$.

(ii) $2^\kappa \leq (2^\kappa)^{M_U} < j(\kappa) < (2^\kappa)^+$.

(iii) For every limit ordinal α:
- $cf(\alpha) = \kappa \to \lim_{\beta\to\alpha} j(\beta) < j(\alpha)$;
- $cf(\alpha) \neq \kappa \to \lim_{\beta\to\alpha} j(\beta) = j(\alpha)$.

(iv) If λ is a strong limit cardinal then $cf\lambda \neq \kappa \to j(\lambda) = \lambda$.

(v)
- $M_U^\kappa \subset M_U$ *(M_U is closed for κ-sequences);*
- $M_U^{\kappa+1} \not\subset M_U$ *(M_U is not closed for $\kappa+1$-sequences).*

(vi) $U \notin M_U$.

Proof.

(i)
- Suppose $x \in V_\kappa$ then, since κ is limit, $rank(x) < \kappa$. If $j(x) \neq x$ then we would have $j(rank(x)) \neq rank(x)$, just as in lemma 3.17, which is a contradiction.

  V_α is Π_1^{ZF} thus $(V_\kappa)^{M_U} = V_\kappa \cap M_U$, so from what we mentioned before $(V_\kappa)^{M_U} = V_\kappa$.

- If $X \subset V_\kappa$ then $x \in X \to x = j(x) \in j(X)$ thus $X \subset j(X) \cap V_\kappa$. On the other hand, $j(x) = x \in j(X) \cap V_\kappa \to x \in X$, hence $j(X) \cap V_\kappa = X$. Again $(V_{\kappa+1})^{M_U} = V_{\kappa+1} \cap M_U$ and for all $X \subset V_\kappa$ $X = j(X) \cap (V_\kappa)^{M_U} \to X \in M_U$ thus $(V_{\kappa+1})^{M_U} = V_{\kappa+1}$.
  The term $w.o.(x) = \{<: <$ well-orders $x\}$ is Π_1^{ZF} so $(w.o.(\kappa))^{M_U} = w.o.(\kappa) \cap M_U$, but $w.o.(\kappa) \subset V_{\kappa+1} \in M_U \to (w.o.(\kappa))^{M_U} = w.o.(\kappa)$. This means that κ^+ is $\Sigma_1^{M_U}$, hence $\kappa^+ \leq (\kappa^+)^{M_U}$. In order to show that $\kappa^+ = (\kappa^+)^{M_U}$ we must find an, absolute for M_U, function f, in M_U, which shows that $w.o.(\kappa)$ and κ^+ have the same cardinality. We define by transfinite recursion $F(\alpha, x, <) =$"the α-th element of $\langle x, <\rangle$ or else $\emptyset$", and then $G(<) = \min\{\alpha < \kappa^+ : F(\alpha, \kappa, <) = \emptyset\}$. $f = G''(w.o.(\kappa))$.

(ii) $2^\kappa = |P(\kappa)|$ and, since $P(\kappa)^{M_U} = P(\kappa)$ (it follows from the fact that $(V_{\kappa+1})^{M_U} = V_{\kappa+1}$) and $|x|$ is Π_1^{ZF}, $2^\kappa = (2^\kappa)^{M_U}$.
$\kappa < j(\kappa)$ and $j(\kappa)$ is measurable, therefore inaccessible in M_U, so

$$(\forall \alpha\ 2^\alpha < j(\kappa))^{M_U} \to (2^\kappa)^{M_U} < j(\kappa).$$

$|j(\kappa)| = |[c_\kappa]| = \kappa \cdot [\kappa]^\kappa = 2^\kappa < (2^\kappa)^+$.

(iii)
- $\beta < \alpha \to j(\beta) < j(\alpha)$ hence $\lim_{\beta\to\alpha} j(\beta) \leq j(\alpha)$.
  If $cf(\alpha) = \kappa$, then there is a cofinal function $f : \kappa \to \alpha$. $[f]$ acts similarly to $[id]$, i.e. $\forall \beta < \alpha\ j(\beta) < [f]$ but $[f] < j(\alpha)$ so $\lim_{\beta\to\alpha} j(\beta) < j(\alpha)$.
- If $cf(\alpha) > \kappa$, then $[f] < j(\alpha)$ implies that for almost every x $f(x) < \alpha$ therefore $\exists \beta < \alpha$ $[f] < j(\beta)$. For this reason $\lim_{\beta\to\alpha} j(\beta) = j(\alpha)$.
  If $cf(\alpha) < \kappa$, then for every $[f] < j(\alpha)$ there is again, by κ-completeness, a $\beta < \alpha$ such that $[f] < j(\beta)$.

(iv) If $\alpha < \lambda$, then $|[\alpha]| \leq |\alpha^\kappa| < \lambda$ thus $j(\lambda) = \lim_{\alpha\to\lambda} j(\alpha) \leq \lambda$. Since $\lambda \leq j(\lambda)$, $j(\lambda) = \lambda$.

(v)
- Let $s = \langle [f_\alpha] : \alpha < \kappa\rangle$ be a κ-sequence, $[g] = \kappa$ and $[h]$, $h(\alpha) = \langle [f_\beta] : \beta < g(\alpha)\rangle$. g is a function which approximates the sequence s and is such that $h(\alpha)$ is an α-sequence hence $[h]$ is an $[h] = \kappa$-sequence, and $[h]([c_\alpha]) = [f_\alpha]$, therefore $s = [h] \in M$.
- Suppose $s = \{j(\alpha) : \alpha < \kappa\}$ is in M_U. s is cofinal in $j(\kappa^+)$ because if $[f] < j(\kappa^+)$, then, since κ^+ is regular, there is an $\alpha < \kappa^+$ such that $[f] < [c_\alpha] = j(\alpha)$. We also have that $|s| = \kappa^+ \leq 2^\kappa < j(\kappa^+)$, so $j(\kappa^+)$ is not regular, thus, by the fact that $\neq regular(\mu)$ is Σ_1^{ZF}, $M_U \vDash$"$j(\kappa^+)$ is not regular" which is a contradiction.

(vi) ${}^\kappa\kappa$ is Π_1^{ZF} hence, by (v), $({}^\kappa\kappa) = {}^\kappa\kappa^{M_U}$ which belongs to M_U. If U also belongs in M_U then there is an absolute term $F(f) = [f]$, so $F''\ {}^\kappa\kappa \in M_U$ which is a function onto $j(\kappa)$. This means that $M_U \vDash |j(\kappa)| \leq 2^\kappa$ which is a contradiction since $M_U \vDash$"strong limit $j(\kappa)$" and $M_U \vDash \kappa < j(\kappa)$.

□

From (vi) we see that $M_U \neq V$, thus we cannot define a nontrivial embedding $j : V \prec V$ using the ultrafilter construction we have described. In the last chapter we will show that under reasonable assumptions no such embedding can exist.

Finally, we state the following application of this lemma, which shows that the existence of measurable cardinals has effects on cardinal arithmetic.

Corollary 3.27. *(Scott) Suppose κ is measurable. Then,*

$$\forall \alpha < \kappa \; 2^\alpha = \alpha^+ \to 2^\kappa = \kappa^+.$$

Proof. Let $j : V \prec M_U$ be the corresponding canonical embedding. By elementarity,

$$V \vDash \forall \alpha < \kappa \; 2^\alpha = \alpha^+ \to M_U \vDash \forall \alpha < j(\kappa) \; 2^\alpha = \alpha^+ \to (2^\kappa)^{M_U} = (\kappa^+)^{M_U}.$$

We already know that $2^\kappa \leq (2^\kappa)^{M_U}$ and that $\kappa^+ = (\kappa^+)^{M_U}$ so $2^\kappa = \kappa^+$ □

3.3 Normal measures, indescribability

We are now going to study normal measures and the connection of measurable cardinals with indescribability and Ramsey cardinals.

Definition 3.28. Let κ be a measurable cardinal. A normal filter D on κ is called a *normal measure* on κ.

As we have already mentioned, every normal filter D contains the club filter on κ, thus all of its sets are stationary. We have also seen that a filter is normal if and only if every regressive function on a set $X \in D$ is constant on a $Y \in D$, such that $Y \subset X$. We will show that there is a normal measure D for every measurable cardinal and then use this to prove that every measurable cardinal is Ramsey.

Theorem 3.29. *For every measurable cardinal κ there is a normal measure D on κ.*

Proof. Let U be a nonprincipal κ-complete ultrafilter on κ and $A = \{[g] \in M_U : \forall \alpha < \kappa \; [c_\alpha] < [g]\}$. A is a nonempty subset of M_U, since $[id] \in A$, thus it has a minimal element $[h]$. Suppose $f \in [h] \wedge rng(f) \subset \kappa$ (this is possible because $[h] \leq [id] \wedge rng(id) = \kappa$) and let $D = \{X \subset \kappa : f^{-1\prime\prime} X \in U\}$.

It is straightforward to see that D is a κ-complete ultrafilter. For every $\alpha < \kappa$, $f^{-1\prime\prime}\{\alpha\} \notin U$ because $[\alpha] < [f]$, thus D is nonprincipal.

Suppose $g : X \to \kappa$, $X \in U$ is a regressive function. Then $h = g \circ f$ has the property $\forall \alpha \in f^{-1\prime\prime} X \; h(\alpha) < f(\alpha)$, therefore by the minimality of f there is an α such that $\{\beta : h(\beta) < \alpha\} \in U$. Provided that U is κ-complete, h is constant on a $Y \in U$, hence g is constant on $f(Y) \in U$. For this reason, D is normal. □

Theorem 3.30. *Every measurable cardinal is a Ramsey cardinal.*

Proof. Let κ be measurable and D a normal measure on κ. We will show that every partition $f : [\kappa]^{<\omega} \to \alpha$ for every $\alpha < \kappa$ has a homogeneous set $X \in D$. In order to do this we will inductively define the sets $X_n,\ 0 < n < \omega$, so that X_n is homogeneous for $f \restriction [\kappa]^n$, hence $X = \bigcap_{n<\omega} X_n$ will have the desired properties.

By κ-completeness, at least one of the sets of any partition $f : [\kappa]^1 \to \alpha,\ \alpha < \kappa$ belongs to D, thus we take one of them as X_1. Suppose that every partition $f : [\kappa]^n \to \alpha,\ \alpha < \kappa$ has a homogenous set of size κ. If $f : [\kappa]^{n+1} \to \alpha,\ \alpha < \kappa$ then we define the functions $f_\beta : [\kappa \setminus \{\beta\}]^n \to \alpha,\ f_\beta(x) = f(\{\alpha\} \cup x)$ for every $\beta < \kappa$. For each f_β there is a set $X_\beta \in D$ homogeneous for f_β, i.e. there is an ordinal γ_β such that $f''_\beta[X_\beta]^n = \{\gamma_\beta\}$. If $X = \triangle_{\beta<\kappa} X_\beta$ then, since D is a normal measure, $X \in D$. It is straightforward that $\alpha_0 < \alpha_1 < \ldots < \alpha_n \in X \to \{\alpha_1, \ldots, \alpha_n\} \in [X_{\alpha_0}]^n$, hence $f(\alpha_0, \ldots, \alpha_n) = f_{\alpha_0}(\alpha_1, \ldots, \alpha_n) = \gamma_{\alpha_0}$. Since $\alpha < \kappa$, there is a subset of X, $H \in D$ so that $\forall \delta \in H\ \gamma_\delta = \gamma$, therefore $[H]^{n+1}$ is homogeneous in f. □

Remark 3.31. We have to notice here that in order to establish the above theorem we have used the axiom of choice. Under other circumstances measurable cardinals could be quite small, for example ZF+AD[3]$\vdash$"ω_1 is the least measurable cardinal".

Another theorem concerning normal measures is the following, which gives equivalent characterizations of normal measures, through the ultraproduct models generated by measurable cardinals and elementary embeddings.

Theorem 3.32. *If D is a nonprincipal κ-complete ultrafilter over κ, then the following are equivalent:*

(i) D is normal;

(ii) $\kappa = [id]$ (in M_U);

(iii) for every $X \subset \kappa$, $X \in D \leftrightarrow \kappa \in j(X)$.

Proof.

(i)→(ii) We already know that $\kappa \le [id]$. If $[f] \le [id]$ then f is regressive for almost all $\alpha < \kappa$ thus, by normality, it is constant for almost all $\alpha < \kappa$, therefore $[f] = [c_\alpha],\ \alpha < \kappa$. Hence, $\kappa < [id]$ cannot be the case, which means that $\kappa = [id]$.

(ii)→(iii) $X \in D \leftrightarrow \{\alpha : id(\alpha) \in X\} \in D$ which is equivalent to $[id] \in [X] = j(X)$. Thus, if $\kappa = [id]$, $X \in D \leftrightarrow \kappa \in j(X)$.

(iii)→(i) Suppose f is a regressive function on a $X \in D$. Then, by elementarity of j, $\alpha = j(f)(\kappa) < \kappa$. This means that $\kappa \in j(\{\beta < \kappa : f(\beta) = \alpha\})$ so f is equal to α for almost all $\beta < \kappa$.

[3]The axiom of determinacy, see [10] or [14].

□

Finally, we show that every measurable cardinal is Π^2_1-indescribable and we use this to prove that the least measurable cardinal is greater than the least Ramsey cardinal. We also see some more connections of measurable cardinals with indescribability.

Theorem 3.33. *Every measurable cardinal is Π^2_1-indescribable.*

Proof. Let $\phi = \forall X\ \psi(X)$, where $\psi(X)$ is a second order formula, $R \subset V_\kappa$ and $(V_\kappa, \in, R) \models \phi$. By the truth definition, we have that $\forall X \subset V_{\kappa+1}\ (V_{\kappa+1}, \in, X, V_\kappa, R) \models \psi^{V_\kappa}$. Since $(V_{\kappa+1})^{M_D} = V_{\kappa+1}$ and $(V_\kappa)^{M_D} = V_\kappa$, $((V_\kappa, \in, U) \models \phi)^{M_D}$ and furthermore $((V_\kappa, \in, j(R) \cap V_\kappa) \models \phi)^{M_D}$ because $R = j(R) \cap V_\kappa$. If $A = \{\alpha < \kappa : (V_\alpha, \in, R \cap V_\alpha) \models \phi\}$ then, by elementarity of j, $\kappa \in j(A)$, hence by normality of D, $A \in D \to A \neq \emptyset$. □

The above result is in some way the best possible since:

Theorem 3.34. *The least measurable cardinal is not Σ^2_1-indescribable.*

Proof. There is a sentence $\phi \in \Sigma^2_1$ such that $(V_\kappa, \in) \models \phi \leftrightarrow measurable(\kappa)$:

$$
\begin{aligned}
\exists U \in PP(V_\kappa)\ (\forall X \in P(V_\kappa)\ (X \in U \to X \subset On \wedge \emptyset \in U \wedge On \in U \wedge \\
\wedge \forall X, Y \in P(V_\kappa)\ X \cap Y \in U \wedge \\
\forall X, Y \in P(V_\kappa)\ (X \in U \wedge X \subset Y \to Y \in U) \wedge \\
\wedge \forall X \in P(V_\kappa)\ (X \in U \vee On \setminus X \in U) \wedge \\
\wedge \forall F : V_\kappa \to V_\kappa \forall \alpha \in On\ \bigcap F''\alpha \in U).
\end{aligned}
$$

Hence the least measurable cardinal κ is such that $(V_\kappa, \in) \models \phi$ but for all $\alpha < \kappa$, $(V_\alpha, \in) \not\models \phi$. Thus, it is not Σ^2_1-indescribable. □

Though the indescribability of measurable cardinals is up to Π^2_1, there are many totally indescribable cardinals below every measurable cardinal. In particular:

Theorem 3.35. *If κ is measurable and D is a normal measure on κ, then $\{\lambda < \kappa : \lambda$ is totally indescribable$\} \in D$.*

Proof. As we have seen in the previous proofs, we just have to show that (κ is totally indescribable)^{M_D} for a normal measure D. Suppose $(R \subset V_\kappa \wedge (V_\kappa, \in, R) \models \phi)^{M_D}$, $\phi \in \bigcup_{n,m<\omega} \Pi^m_n$. Then,

$$
\begin{aligned}
&(\exists \alpha < j(\kappa)\ (V_\alpha, \in, j(R) \cap V_\alpha) \models \phi)^{M_D} \to \\
&\exists \alpha < \kappa\ (V_\alpha, \in, R \cap V_\alpha) \models \phi \to \\
&(\exists \alpha < \kappa\ (V_\alpha, \in, R \cap V_\alpha) \models \phi)^{M_D}.
\end{aligned}
$$

□

Remark 3.36. The latter reveals the fact that indescribability is not enough to support the existence of measurable or greater large cardinals (provided that AC holds). Thus the elementary embeddings of V take its place as a stronger reflection principle, used to produce large cardinals. In the fifth chapter we will encounter more examples of such embeddings and large cardinals related to them.

Using Π^2_1-indescribability we prove the next corollary:

Corollary 3.37. *If D is a normal measure over the cardinal κ, then $\{\lambda < \kappa : Ramsey(\lambda)\} \in D$.*

Proof. As in 3.34, we can find a Π^2_1 sentence ϕ such that $Ramsey(\lambda) \leftrightarrow (V_\lambda, \in) \models \phi$. Hence the assertion follows from the proof of 3.30. $\square$

We close this section having a quick view at the models of the form $L[D]$, where D is a normal measure.

Theorem 3.38. *(Silver) If D is a normal measure on a cardinal κ, then $L[D] \models GCH$.*

Theorem 3.39. *(Kunen) If D is a normal measure on a cardinal κ and $V = L[D]$, then κ is the only measurable cardinal and D the only measure on κ.*

Chapter 4

$0^\sharp$ and elementary embeddings of L

4.1 Indiscernibles

Before we proceed further in this chapter, we first need to introduce one more model theoretical notion, indiscernibility. A set I is called a set of indiscernibles, if we cannot distinguish its elements using first order formulas. Thus, if one formula is satisfied by a n-tuple of I^n, then it is satisfied by all the other n-tuples of I^n. Since the sets of indiscernibles we will consider shall contain ordinals, which come along with a natural well-ordering, we will require that the n-tuples are entered in an increasing order in every formula we will consider. Finally, after introducing indiscernibility, we will see how new models can be created by extending sets of indiscernibles, using Skolem functions, and state some of their properties.

Definition 4.1.

(i) Let M be a structure and $(I, <)$ a linearly ordered set. I is a set of *indiscernibles* for M, if for every formula $\phi(v_1, \ldots, v_n)$ and all $x_1 < \ldots < x_n, y_1 < \ldots < y_n \in I$,

$$M \vDash \phi[x_1, \ldots, x_n] \leftrightarrow M \vDash \phi[y_1, \ldots, y_n].$$

(ii) If M is a structure and I a set of indiscernibles for M, then we denote by $\Sigma(M, I)$ the set of formulas true in M for increasing sequences of I.

A very useful theorem, concerning models which contain sets of indiscernibles, is the following:

Theorem 4.2. *(Ehrenfeucht-Mostowski) If T is a theory with infinite models, and $(I, <)$ is a linearly ordered set, then there exists a model M of T such that $(I, <)$ is a set of indiscernibles in M.*

Proof. It can be found in [2], [10] or [7] and it is an interesting application, of Ramsey's theorem, which gives us a connection of infinitary combinatorics with model theory. □

The following theorem provides us with the basic properties of models constructed from indiscernibles, and its proof can be found in [2].

Theorem 4.3. *Let M be a model, of a theory T with built in functions, and $(I, <)$ a set of indiscernibles of M. Then,*

(i) If $J \subset I$, then J with the inherited order is a set of indiscernibles in $\mathcal{H}(J)$ and $\mathcal{H}(J) \prec \mathcal{H}(I)$.

(ii) If I, J are infinite linearly ordered sets, then there is a model N of T, such that J is a set of indiscernibles for N and $\Sigma(M, I) = \Sigma(N, J)$.

(iii) Let N be a model of T and J a set of indiscernibles for N such that $\Sigma(M, I) = \Sigma(N, J)$. Every increasing function $j : I \to J$ can be uniquely extended to an elementary embedding $\bar{j} : \mathcal{H}(I) \to \mathcal{H}(J)$. Moreover, $\bar{j}''H(I) = H(j''I)$.

The theorems above are the starting point of an attempt of producing models with many automorphisms, initiated by Ehrenfeucht and Mostowski. By theorem 4.3 (iii), if $j : I \to I$ is an increasing onto I function then it can be extended to an automorphism of $\mathcal{H}(I)$. Thus, if a model is produced by a set of indiscernibles, with quite a flexible order[1], then it has a lot of automorphisms. The way we are going to use those theorems is much alike, i.e. we wish to verify the existence of a non-trivial elementary embedding $j : L \to L$. Of course, this comes with a cost; we must make stronger combinatorial assumptions, not provable in ZFC, which will guarantee the existence of appropriate sets of indiscernibles. This was first done using the existence of Ramsey cardinals and through this procedure it became clear that a weaker assumption, the existence of $0^\sharp$, is enough. $0^\sharp$ is a $\Sigma(M, I)$ set of formulas, having some specific properties.

4.2 $0^\sharp$

Near the sixties, it had already been noticed, by set theorists, that the existence of several large cardinals has a great impact on the structure of L relative to V. We have already seen in the previous chapter that the existence of a measurable cardinal implies that $V \neq L$. This impact could be described as a compression of L and it is connected to results such as $|L \cap \mathbb{R}| = \aleph_0$, $L \models inaccessible(\aleph_\alpha)$ for every $\alpha > 0$ and the existence of elementary embeddings of L. $0^\sharp$ emerged exactly from the study of those phenomena.

[1] One which has many $<$-automorphisms.

E-M sets

$0^\sharp$'s definition is not straightforward. In order to understand its nature, one must go through the procedure of its creation. This is why we first introduce the E-M sets, which are related to the theory of indiscernibles. Roughly speaking, in order to shrink L we must make sure that its structure is simple, which can be done by assuming that many of its elements have the same properties or, in other words, that it contains an appropriately large set of indiscernibles. From now on, the sets of indiscernibles for the models M we will consider, will contain only ordinals of M (elements of On^M), ordered by $\in^M$.

Definition 4.4. Let $<_L$ be the canonical well-ordering of L. We define, for every formula ϕ, the *canonical Skolem function* h_ϕ as follows:

$$h_\phi(\bar{x}) = \begin{cases} \text{the } <_L \text{-least } y \text{ such that } \phi(y,\bar{x}) \\ \emptyset \text{ otherwise} \end{cases}$$

Lemma 4.5. *Let α, β be limit ordinals, $j : L_\alpha \to L_\beta$ an elementary embedding and ϕ a formula. If $x_1, \ldots, x_n \in L_\alpha$, then*

$$h_\phi^{L_\beta}(j(x_1), \ldots, j(x_n)) = j(h_\phi^{L_\alpha}(x_1, \ldots, x_n)).$$

Proof. We may notice that the formula $x <_L y$ is absolute for all L_γ, where γ is a limit ordinal. Hence

$$\begin{aligned} h_\phi^{L_\beta}(j(x_1), \ldots, j(x_n)) &= \begin{cases} \text{the } <_L \text{-least } y \text{ such that } \phi^{L_\beta}(y, j(x_1), \ldots, j(x_n)) \\ \emptyset \end{cases} \\ &= j\left(\begin{cases} \text{the } <_L \text{-least } y \text{ such that } \phi^{L_\alpha}(y, x_1, \ldots, x_n) \\ \emptyset \end{cases}\right) \\ &= j(h_\phi^{L_\alpha}(x_1, \ldots, x_n)) \end{aligned}$$

□

In order to define $0^\sharp$ we introduce the following sets:

Definition 4.6. We will call a set of formulas Σ an *E-M set* (Ehrenfeucht-Mostowski), if there is a model M, an infinite set I of indiscernibles for M and a limit ordinal α such that:

(i) $M \equiv L_\alpha$

(ii) $\Sigma = \Sigma(M, I)$.

The following theorem is a special case of theorem 4.2:

Theorem 4.7. *If Σ is an E-M set and α an infinite ordinal, then there is a unique up to isomorphism pair (M, I), where M is a model of Σ and I a set of indiscernibles for M having the following properties:*

(i) $\Sigma = \Sigma(M, I)$;

(ii) the order type of I is α;

(iii) $M = \mathcal{H}(I)$.

Proof.

($\exists$) Since Σ is an E-M set, there is a model N and an infinite set of indiscernibles J for N such that $\Sigma = \Sigma(N, J)$. Let $\{j_n : n < \omega\}$ be a countable subset of J. By theorem 4.3 (ii), there exists a model M' and a set of indiscernibles I for it with order type α, such that $\Sigma = \Sigma(N, J) = \Sigma(M', I)$. The submodel $M = \mathcal{H}(I)$ of M', is exactly what we need, since it is an elementary submodel of M' thus I is a set of indiscernibles for it, and $\Sigma(M, I) = \Sigma(M', I) = \Sigma$.

($\exists!$) The uniqueness of the pair (M, I) is immediate by theorem 4.3 (iii). If (N, J) is another pair, then there is an $<$-isomorphism, $j : I \to J$ which can be extended to an isomorphism between M and N.

□

Definition 4.8. We will call the unique pair (M, I), of theorem 4.7, the (Σ, α)-model.

It is immediate from theorem 4.3 (iii), that

$$\alpha \leq \beta \to (\Sigma, \alpha) \prec (\Sigma, \beta).$$

We are now going to analyze three properties related to (Σ, α) models. Specifically we are going to consider the cases where a (Σ, α) model is well-founded, unbounded, or remarkable.

These properties are used in order to identify some of the (Σ, κ) models, $cardinal(\kappa)$, $\kappa > \omega$, with the models L_κ. This way the models L_κ and L will be built up from sets of indiscernibles, thus we will have the opportunity of getting the results we have already described.

Definition 4.9. Let (M, I) be a (Σ, α)-model and $I = \{i_\beta : \beta \in \alpha\}$, $\alpha < \beta \leftrightarrow i_\alpha < i_\beta$. Then,

(i) (Σ, α) is unbounded if whenever $M \vDash x \in On$, there exists an $i \in I$ such that $M \vDash x < i$.

(ii) (Σ, α) is remarkable if it is unbounded and $M \vDash (x \in On \wedge x < i_\omega)$ implies that $x \in \mathcal{H}^M(\{i_n : n < \omega\})$.

Definition 4.10. An E-M set Σ is called well-founded, unbounded or remarkable if every (Σ, α)-model is respectively well-founded, unbounded or remarkable.

The following lemma provides us with expressions equivalent to the properties described above:

Lemma 4.11. *Let Σ be an E-M set. Then,*

(a) The following are equivalent:

(i) For every ordinal α, (Σ, α) is well-founded.

(ii) For some ordinal $\alpha \geq \omega_1$, (Σ, α) is well-founded.

(iii) For every ordinal $\alpha < \omega_1$, (Σ, α) is well-founded.

(b) The following are equivalent:

(i) For every ordinal α, (Σ, α) is unbounded.

(ii) For some ordinal α, (Σ, α) is unbounded.

(iii) For every Skolem term $t(\bar{v})$,

$$[t(\bar{v}) \in On \rightarrow t(\bar{v}) < u] \in \Sigma.$$

(c) The following are equivalent if Σ is unbounded:

(i) For every ordinal $\alpha > \omega$, (Σ, α) is remarkable.

(ii) For some $\alpha > \omega$, (Σ, α) is remarkable.

(iii) For every Skolem term $t(\bar{v}, \bar{u})$,

$$[t(\bar{v}, \bar{u}) < u_1 \rightarrow t(\bar{v}, \bar{u}) = t(\bar{v}, \bar{w})] \in \Sigma.$$

(d) If (M, I) is a remarkable (Σ, α) set, then for every limit ordinal $\beta < \alpha$, $M \models (x \in On \wedge x < i_\beta)$ implies that $x \in \mathcal{H}^M(\{i_\gamma : \gamma < \beta\})$.

(e) If (M, I) is remarkable, then club(I) in On^M.

Proof. It is straightforward and can be found in [8], [10] or [5]. □

Definitions of $0^\sharp$

We are nearly ready to define $0^\sharp$. It will be the unique well-founded, remarkable E-M set, but we have not yet verified this uniqueness. In order to do this, we will first prove that the existence of E-M sets of this kind have two important consequences stated in 4.12. Later on, those properties will lead us to a second definition of $0^\sharp$.

Theorem 4.12. *(Silver) The existence of a well-founded, remarkable E-M set has the following consequences:*

(i) For all cardinals $\omega < \kappa < \lambda$, $(L_\kappa, \in) \prec (L_\lambda, \in)$.

(ii) There is a unique club class I in On, containing all cardinals and such that for every $\omega < \kappa$:

- $|I \cap \kappa| = \kappa$

– *$I \cap \kappa$ is a set of indiscernibles for L_κ*

– *$\forall \alpha \in L_\kappa$, α is definable in L_κ from $I \cap \kappa$.*

Proof. We will carry out the proof using the following lemmas. In every case we will assume that Σ is a well-founded, remarkable E-M set. □

Lemma 4.13. *If $\kappa > \omega$ is a cardinal, then the (Σ, κ) model is isomorphic to L_κ.*

Proof. By checking the proof of Gödel's condensation lemma, we can notice that since Σ is an E-M set, it satisfies ϕ_L thus the (Σ, κ)-model, being well-founded, is isomorphic to (L_α, I) for some ordinal α. $|I| = \kappa$ so $\kappa \leq \alpha$. Suppose $\kappa < \alpha$, then since (Σ, κ) is unbounded there is a $\beta < \kappa$ such that $L_\alpha \vDash \kappa < i_\beta$ and since (Σ, κ) is remarkable, for every $\gamma \leq i_\beta$,

$$\gamma \in \mathcal{H}^{L_\alpha}(\{i_\delta : \delta < \beta\}) \rightarrow \kappa \subset \mathcal{H}^{L_\alpha}(\{i_\delta : \delta < \beta\}).$$

The last inclusion is false, because $|\mathcal{H}^{L_\alpha}(\{i_\delta : \delta < \beta\})| = \beta < \kappa$, and as a result of that $\beta = \kappa$. □

Relying on the above lemma, if Σ is well-founded and remarkable we can identify the (Σ, κ)-model with (L_κ, I_κ). Thus if $\omega < \kappa < \lambda$, we have that $(\Sigma, \kappa) \prec (\Sigma, \lambda)$, which means that $L_\kappa \prec L_\lambda$ and this settles (i) of theorem 4.12.

Lemma 4.14. *If $\omega < \kappa < \lambda$, then*

(i) $I_\lambda \cap \kappa = I_\kappa$

(ii) $\mathcal{H}^{L_\lambda}(I_\kappa) = L_\kappa$.

Proof.

(i) Let $J = \{i_\alpha : \alpha < \kappa\} \subset I_\lambda$. Then, $(\mathcal{H}^{L_\lambda}(J), J)$ is a (Σ, κ)-model hence ,by the above lemma it, is isomorphic to (L_κ, I_κ). By lemma 4.11 (d) and (e), we can see that the ordinals of $\mathcal{H}^{L_\lambda}(J)$ are exactly i_κ. This means that $i_\kappa = \kappa$ and $J = I_\kappa$, thus

$$I_\lambda \cap \kappa = J = I_\kappa.$$

(ii) We already know that $(\mathcal{H}^{L_\lambda}(I_\kappa), I_\kappa) \cong (L_\kappa, I_\kappa)$. Thus the isomorpism of $\mathcal{H}^{L_\lambda}(I_\kappa)$ and L_κ is the identity function on I_κ. Provided that every Skolem function is uniquely defined, by $<_L$, $\mathcal{H}^{L_\lambda}(I_\kappa) = L_\kappa$.

□

This lemma proves the second part of theorem 4.12, except from the uniqueness of I. In particular, if we set $I = \bigcup_{\omega < \kappa} I_\kappa$, then:

- $|I \cap \kappa| = |I_\kappa| = \kappa$
- $I \cap \kappa = I_\kappa$ which is a set of indiscernibles for L_κ

- $L_\kappa = \mathcal{H}(I_\kappa)$ so every element of L_κ is definable in L_κ from $I_\kappa = I \cap \kappa$.
- $I \cap \kappa$ is unbounded for any cardinal κ so, since it is closed, $\kappa \in I$.

The next two lemmas are used to prove the uniqueness of I.

Lemma 4.15. *There is at most one well-founded remarkable E-M set.*

Proof. We already know that $\{\aleph_n : n \in \omega\} \subset I$. If Σ is well-founded and remarkable, then $L_{\aleph_\omega}$ is the $(\Sigma, \aleph_\omega)$-model hence for every formula ϕ

$$\phi(v_1, \ldots, v_n) \in \Sigma \leftrightarrow L_{\aleph_\omega} \models \phi[\aleph_1, \ldots, \aleph_n].$$

Thus Σ is unique. □

Lemma 4.16. *For every regular $\kappa > \omega$ there is at most one club set of indiscernibles I_κ for L_κ having the property $L_\kappa = \mathcal{H}(I_\kappa)$.*

Proof. Let J be another set of indiscernibles having the above properties. Since both J and I_κ are club their cut is also club thus infinite. Provided that they are both sets of indiscernibles, this means that

$$\Sigma(L_\kappa, J) = \Sigma(L_\kappa, J \cap I_\kappa) = \Sigma(L_\kappa, I_\kappa)$$

hence (L_κ, J) is the (Σ, κ)-model so $J = I_\kappa$. □

It follows from the lemma above, that I is unique because each of the I_κ is unique. Thus we have completed the proof of theorem 4.12.

By lemma 4.15 we can define $0^\sharp$:

Definition 4.17. $0^\sharp$ is the unique well-ordered remarkable E-M set.

The equivalent definition below is immediate from the proof of 4.15:

Corollary 4.18. *If $0^\sharp$ exists then,*

$$0^\sharp = \{\phi \in Form : L_{\aleph_\omega} \models \phi[\aleph_1, \ldots, \aleph_n]\}.$$

Another definition is the following:

Corollary 4.19. *If $0^\sharp$ exists then,*

$$0^\sharp = \{\phi \in Form : L \models \phi[\aleph_1, \ldots, \aleph_n]\}.$$

Proof. It follows from the reflection principle that for every formula ϕ there is a cardinal $\kappa > \aleph_\omega$ such that for all $x_1, \ldots, x_n \in L_\kappa$

$$L \models \phi[x_1, \ldots, x_n] \leftrightarrow L_\kappa \models \phi[x_1, \ldots, x_n].$$

By letting $x_1, \ldots, x_n$ be $\aleph_1, \ldots, \aleph_n$ and considering that $L_{\aleph_\omega} \prec L_\kappa$, we have that

$$L_\kappa \models \phi[\aleph_1, \ldots, \aleph_n] \leftrightarrow L_{\aleph_\omega} \models \phi[\aleph_1, \ldots, \aleph_n]$$

thus if we replace L_ω by L in corollary 4.18, we get the same set, i.e. $0^\sharp$. □

If we replace the formulas by their Gödel numbers, we can view $0^\sharp$ as a set of integers. It can also be considered as a real number and there are many more equivalent definitions. The one in the corollary above though, is probably the most common one.

Finally, we have to state that $0^\sharp$ may not be a large cardinal, but its existence implies the existence of η_α, for every $\alpha < \omega_1$, and if $\eta_{\omega+1}$ exists, then $0^\sharp$ also exists. Hence, its existence can be regarded to be a large cardinal axiom.

4.3 The connection of $0^\sharp$ with L

As we have already mentioned, the existence of $0^\sharp$ is related to the difference between the structure of V and L. Here we will analyze this relation in two ways. First we will see that the existence of $0^\sharp$ implies that $V \neq L$. It also has some more consequences which show that V is much larger than L assuming this existence. Afterwards, we shall examine what happens in the case $0^\sharp$ does not exist. This time we do not have a result of the form $V = L$[2], but Jensen's covering theorem will guarantee that L is really close to V. Most of the proofs here are omitted and can be found in [8], [10] and [5].

Assuming that $0^\sharp$ exists,

The next theorems are all corollaries of Silver's theorem.

Corollary 4.20.

(i) There exists a subset of $Form$ containing exactly $Th(L)$.

(ii) $L_\kappa \prec L$.

(iii) I is a set of indiscernibles for L.

(iv) $def_L(I) = L$.

Proof.

(i) By corollary 4.19 this set is $\{\phi \in 0^\sharp : sentence(\phi)\}$.

(ii) By the reflection principle, for every formula ϕ there is an uncountable λ such that ϕ is absolute for L_λ, L. We know from Silver's theorem that either $L_\kappa \prec L_\lambda$ or $L_\lambda \prec L_\kappa$, depending on which of κ, λ is bigger. Thus, ϕ is also absolute for L_κ, L_λ which means that it is absolute for L_κ, L too. Since this is true for every ϕ, $L_\kappa \prec L$.

(iii) This is immediate from the fact that $I \cap \kappa$ is a set of indiscernibles for L_κ, for every cardinal κ, and from (ii).

[2]It would be impossible for a large cardinal axiom to be equivalent with $V \neq L$, because then its consistency relative to ZFC would be provable in ZFC

(iv) If $x \in L$ then there is an cardinal κ such that $x \in L_\kappa$, so x is definable in L_κ by $I \cap \kappa$. Hence, by (ii), x is also definable in L by $I \cap \kappa$, thus also from I.

□

Corollary 4.21.

$$\forall x \in L\ (x \in def_L(\emptyset) \to |x| \leq \omega).$$

Proof. Suppose x is defined in L by a formula $\phi(v)$. In this case, the formula $\psi(u) = \forall v\ (v \in u \leftrightarrow \phi(v))$ is such that only x satisfies it in L. Since $L_{\aleph_1} \prec L$,

$$L \vDash \exists! u\ \psi(u) \to L_{\aleph_1} \vDash \exists! u\ \psi(u)$$

therefore there must be a $y \in L_{\aleph_1}$ satisfying ψ which ought to be equal to x. Thus, $x \in L_{\aleph_1}$ which means that x is countable. □

By the lemma above we have that no uncountable cardinals can be defined in L. As we will see below, those cardinals apart from being undefinable are also inaccessible in L.

Corollary 4.22. *Every uncountable cardinal is inaccessible in L.*

Proof. We will use the indiscernibility of uncountable cardinals here, in order to transfer properties from specific cardinals to all uncountable cardinals. $L \vDash regular(\aleph_1)$ and also $L \vDash limcard(\aleph_\omega)$, thus every uncountable cardinal has those properties in L, i.e. every uncountable cardinal is inaccessible in L. □

Remark 4.23. Since the existence of $0^\sharp$ implies the existence of η_ω, we know from the second chapter that $\eta_\omega \in L$. Therefore, using the technique of the lemma above, we can see that every uncountable cardinal in L is H_ω and thus it is also Mahlo and weakly compact.

Corollary 4.24. $|\mathbb{R} \cap L| \leq \omega_1$. *Furthermore,* $|V_\alpha \cap L| \leq |\alpha|$.

Proof. $V_\alpha \cap L \in def_L(\{\alpha\})$ so, by indiscernibility, $V_\alpha \cap L \in def_{L_\kappa}(\{\alpha\})$ for $\kappa = |\alpha|^+$. As we did before, we can find a formula $\psi(v, \alpha)$ such that $V_\alpha \cap L$ is the only set satisfying ψ and so $V_\alpha \cap L \in L_\kappa$. Hence, $|V_\alpha \cap L| < |L_\kappa| = |\alpha|^+$. □

The above corollary was first proved by Rowbottom for the case where a Ramsey cardinal exists. This is a stronger result which shows that $0^\sharp$ is closer to the collapse of L caused by the existence of specific large cardinals.

Finally, using any of the corollaries 4.21 to 4.24, one can prove that $V \neq L$.

Theorem 4.25. *If $0^\sharp$ exists then $V \neq L$.*

Assuming that $0^\sharp$ does not exist,

The most important result towards this direction is Jensen's covering lemma. Everything else we mention afterwards follows from it.

Theorem 4.26. *(Jensen's covering theorem) $0^\sharp$ does not exist iff for every uncountable set X of ordinals there is a set $Y \in L$ such that $X \subset Y$ and $|X| = |Y|$.*

Proof. See [8]. □

An example of a set we cannot cover if $0^\sharp$ exists is $X = \{\aleph_n : n \in \omega\}$. This is because $\aleph_\omega$ is uncountable, thus regular in L and $\sup(X) = \aleph_\omega$ so X cannot be covered by a set $Y \in L$ of cardinality less than $\aleph_\omega$. This way, if we also take under consideration that the existence of $0^\sharp$ implies that $\aleph_\omega$ is regular in L, we see that the existence of $0^\sharp$ is equivalent to the proposition $L \vDash regular(\aleph_\omega)$.

The following three corollaries demonstrate some of the consequences of the non-existence of $0^\sharp$.

Corollary 4.27. *Suppose $0^\sharp$ does not exist. Then, for every ordinal $\alpha \geq \aleph_2$, such that $L \vDash regular(\alpha)$, $cf(\alpha) = |\alpha|$.*

Proof. Let X be a cofinal subset of λ and Y a constructible set covering X (or an uncountable set X', $X \subset X' \subset \lambda$, if X is countable). Since both Y and λ belong to L, $Z = Y \cap \lambda \in L$ and it also covers X, i.e. $X \subset Z \subset \lambda$ and $|Z| = |X| + \aleph_1 = cf(\lambda) + \aleph_1$. Z is unbounded in λ and, provided that $L \vDash regular(\lambda)$, $|Z| = |\lambda|$. Hence $|\lambda| = cf(\lambda) + \aleph_1 = cf(\lambda)$ because $\lambda \geq \aleph_2$. □

Corollary 4.28. *If $0^\sharp$ does not exist, then $singular(\kappa) \to L \vDash singular(\kappa)$.*

Proof. If κ is singular and $L \vDash regular(\kappa)$ then $\kappa \geq \aleph_2$ so by corollary 4.27 $cf(\kappa) = \kappa$, which is a contradiction. □

Corollary 4.29. *If $0^\sharp$ does not exist then $singular(\kappa) \to (\kappa^+)^L = \kappa^+$.*

Proof. Suppose $\alpha = (\kappa^+)^L$. $\kappa \leq \alpha \leq \kappa^+$ thus if $\alpha \neq \kappa^+$, then $|\alpha| = \kappa$. This means that $cf(\alpha) < |\alpha|$ since κ is singular but this is a contradiction since $L \vDash regular((\kappa^+)^L)$ and by corollary 4.27 $cf(\alpha) = |\alpha|$. □

4.4 Elementary embeddings of L

The main purpose of this section is to present Kunen's theorem, which states that the existence of $0^\sharp$ is equivalent to the existence of a nontrivial elementary embedding $j : L \prec L$. This looks like the case of the nontrivial elementary embeddings of V, which exist exactly when a measurable cardinal exists. In fact the last one is a stronger kind of elementary embedding, because if $j : V \prec M$ is nontrivial then $j \restriction L : L \prec L$ is also nontrivial. Since L is contained in every inner model M, N, if there is a nontrivial embedding $j : M \prec N$ then there is also a nontrivial embedding $j' : L \prec L$. Therefore, the weakest assumption one can make on the existence of nontrivial elementary embeddings of inner models, is the existence of a nontrivial elementary embedding of L.

Theorem 4.30. *(Kunen) $0^\sharp$ exists iff there is a nontrivial elementary embedding $j : L \prec L$.*

Proof.

($\rightarrow$) We have proved the existence of a set I of indiscernibles in L, such that $L = \mathcal{H}(I)$. By theorem 4.3, every order-preserving function $j : I \rightarrow I$ can be extended to an elementary embedding $\bar{j} : L \prec L$ which is nontrivial if j is nontrivial. There are many such functions j, therefore there also many nontrivial elementary embeddings of L (for example take $j(i_\alpha) = j(i_{\alpha+1})$).

($\leftarrow$) This direction needs more work which will be carried out into three steps. In the first step, we will introduce a variant of the ultraproduct construction for nontrivial embeddings of L. This way using a nontrivial embedding $L \prec L$ we will define an inner model $M_D \cong L$ and a new nontrivial elementary embedding $j_D : L \prec L$. In the second step, we study the properties of j_D and of some of its reductions. Using those reductions we come up with a set of indiscernibles $\{\gamma_\alpha : \alpha < \omega_1\}$ for L_κ, where κ is an appropriate cardinal. In the third step, we see that the existence of an uncountable set of indiscernibles for a model L_κ implies the existence of $0^\sharp$, which completes the proof.

□

Step 1

We define below the notions of ultrafilters and ultrapowers, relativized to a model M.

Definition 4.31. Suppose M is a transitive model and $\kappa \in M$. A *M-ultrafilter* on κ is a subset D of $P^M(\kappa)$ having the following properties:

(i) $\emptyset \notin D \wedge \kappa \in D$;

(ii) $X, Y \in D \rightarrow X \cap Y \in D$;

(iii) $X \in D \wedge Y \in M \wedge X \subset Y \rightarrow Y \in D$;

(iv) $\forall X \in P^M(\kappa)\ (X \in D \vee \kappa \setminus X \in D)$.

D is called *nonprincipal* if:

(v) $\forall \alpha \in \kappa\ \{\alpha\} \notin D$.

D is called *κ-complete* if:

(vi) If $\alpha < \kappa$, $\{X_\beta : \beta < \alpha\} \in M$ and $\forall \beta < \alpha\ X_\beta \in D$, then $\bigcup_{\beta<\alpha} X_\beta \in D$.

D is called *normal* if:

(vii) Every regressive function $f \in M$ on a $X \in D$ is constant on a $Y \in D$.

The ultrapower M_D of M by D is remains the same as the one defined for ultrafilters and we should only have in mind that D is a M-ultrafilter. We also note by j_D the canonical embedding of M in M_D. The following theorems we have provided for ultrapowers and elementary embeddings, still hold for M-ultrapowers and they are proved the same way:

Theorem 4.32. *Suppose D is an M-ultrafilter on κ. Then for every formula ϕ:*

$$M_U \vDash \phi([f_1], \ldots, [f_n]) \leftrightarrow \{\alpha < \kappa : M \vDash \phi(f_1(\alpha), \ldots, f_n(\alpha))\} \in D.$$

Theorem 4.33. *Let $j : M \to N$ be an elementary embedding having κ as its critical point. Then, $M \vDash (regular(\kappa) \wedge \kappa > (\omega)^M)$ and $D = \{X \in P^M(\kappa) : \kappa \in j(X)\}$ is a normal nonprincipal κ-complete M-ultrafilter on κ.*

Suppose $j : M \prec N$ is an elementary embedding, κ is a critical point of j and D is is the M-ultrafilter described above. It is not hard to check that the function $k : M_D \to N$, $k([f]) = j(f)(\kappa)$, is an elementary embedding such that $k \circ j_D = j$. This way, we can see that M_D is well-founded, since an infinite decreasing sequence $\{[f_n]\}_{n<\omega}$ of M_D would be mapped to an infinite decreasing sequence $\{k([f_n])\}$ of N. In this case, as in chapter 3, we will identify M_D with its transitive collapse which is an inner model.

Now we will focus our interest on elementary embeddings $j : L \prec L$. $L_D = L$ because using theorem 4.32 we can show that $L_D \vDash V = L$. From this point on, we will only work with $j_D : L \prec L$ and symbolize its critical point by γ for convenience (κ will be used as an appropriate cardinal for which L_κ has an uncountable set of indiscernibles) .

Step 2

Lemma 4.34. *If κ is a limit cardinal such that $cf(\kappa) > \gamma$, then $j_D(\kappa) = \kappa$.*

Proof. As in the previous chapter, we can check that $j_D(\kappa) = \lim_{\alpha\to\kappa} j_D(\alpha)$. Additionally, for every $\alpha < \kappa$, $|j_D(\alpha)| = |[c_\alpha]| \leq |(\alpha^\gamma)^L|$ and $|(\alpha^\gamma)^L| < \kappa$ (because κ is limit so there is a cardinal $\alpha < \lambda < \kappa$ for which $\alpha^\gamma \leq \lambda^\gamma = \lambda < \kappa$). Thus, $j_D(\kappa) \leq \kappa \to j_D(\kappa) = \kappa$. □

We define by transfinite induction the following decreasing sequence of sets:

$$\begin{aligned} U_0 =& \{\kappa : cardinal(\kappa) \wedge cf(\kappa) > \gamma\} \\ U_{\alpha+1} =& \{\kappa \in U_\alpha : |U_\alpha \cap \kappa| = \kappa\} \\ U_\alpha =& \bigcap_{\beta<\alpha} U_\beta \text{ if limord}(\alpha) \end{aligned}$$

It is easy to prove by induction that every class U_α is unbounded and closed for sequences of length β such that $cf(\beta) > \kappa$, thus U_{ω_1} is nonempty. We are going to use an element κ of U_{ω_1} in order to establish our results.

We reduce the mapping $j_D : L \prec L$ we have described before, to $i = j \restriction L_\kappa$, which is an elementary embedding of L_κ. We define the sets $X_\alpha = U_\alpha \cap \kappa$, $\alpha < \omega_1$, which have cardinality κ, and afterwards the following elementary submodels of L_κ:

$$M_\alpha = \mathcal{H}(\gamma \cup X_\alpha).$$

Suppose F_α is the transitive collapse of M_α. Then, provided that $|\gamma \cup X_\alpha| = \kappa$, $F''_\alpha M_\alpha = L_\kappa$ by Gödel's condensation lemma. Let $i : L_\kappa \prec L_\kappa$, $i_\alpha = F_\alpha^{-1}$, and $\gamma_\alpha = i_\alpha(\gamma)$. Our goal is to prove that the ordinals γ_α, $\alpha < \omega_1$, form a set of indiscernibles for L_κ. In order to do this, we are going to create a family of elementary embeddings $i_{\alpha,\beta}$ of L_κ, which map γ_α to γ_β. In the next two lemmas we give some properties of the ordinals γ_α and define the embeddings $i_{\alpha,\beta}$.

Lemma 4.35.

(i) *γ_α is the least ordinal of M_α greater than γ.*

(ii) *If $\alpha < \beta$ then $i_\alpha \restriction M_\beta$ is the identity function, thus $i_\alpha(\gamma_\beta) = \gamma_\beta$.*

(iii) $\alpha < \beta \to \gamma_\alpha < \gamma_\beta$.

Proof.

(i) Since $\gamma \subset M$, $\gamma_\alpha = i_\alpha(\gamma)$ is the smallest ordinal of M containing all the elements of γ, thus $\sup_{M_\alpha}(\gamma)$. We will show that $\gamma \notin M_\alpha$, therefore $i_\alpha \neq \gamma$. The elements of M_α are of the form $t[\delta_1, \ldots, \delta_n]$ (t is a Skolem term), where the ordinals $\delta_1, \ldots, \delta_n$ belong either to γ or X_α, hence they are not changed by the embedding i. This means that

$$i(t[\delta_1, \ldots, \delta_n]) = t[i(\delta_1), \ldots, i(\delta_n)] = t[\delta_1, \ldots, \delta_n]$$

so $\gamma \notin M_\alpha$ because $i(\gamma) \neq \gamma$.

(ii) Since the elements of M_β are of the form $t[\delta_1, \ldots, \delta_n]$, we have to show that the elements of γ and X_β are preserved by i_α. For γ, it is straightforward because $\gamma \subset M_\alpha$. As for X_β, we can see by transfinite induction that $i_\alpha(\lambda) = \lambda$, since

$$\alpha < \beta \to |X_\alpha \cap \lambda| = \lambda \to F_\alpha(\lambda) = \sup_{M_\alpha}\{F_\alpha(\mu) : \mu < \lambda\}.$$

(iii) It is immediate that $\gamma_\alpha \leq \gamma_\beta$, so we only need to chck that $\gamma_\alpha \neq \gamma_\beta$. This is also true though, since

$$\gamma < \gamma_\alpha \to \gamma_\alpha = i_\alpha(\gamma) < i_\alpha(\gamma_\alpha) \leq i_\alpha(\gamma_\beta) = \gamma_\beta.$$

□

Lemma 4.36. *For all $\alpha < \beta < \omega_1$ there are elementary embeddings $i_{\alpha,\beta} L_\kappa \prec L_\kappa$, which preserve all ordinals of ω_1 that are not in the interval $[\alpha, \beta]$ and map γ_α to γ_β.*

Proof. The embeddings $i_{\alpha,\beta}$ are defined in a similar way to i_α. Specifically, we let $M_{\alpha,\beta} = \mathcal{H}(\gamma_\alpha, X_\beta)$ and $i_{\alpha,\beta} = F^{-1}_{\alpha,\beta}$, where $F_{\alpha,\beta}$ is the transitive collapse of $M_{\alpha,\beta}$.

$\gamma_\alpha \subset M_{\alpha,\beta}$, therefore $i_{\alpha,\beta}(\gamma_\xi) = \gamma_\xi$ for all $\xi < \alpha$. By repeating the procedure in lemma 4.35 for X_β we get that $i_{\alpha,\beta}(\gamma_\xi) = \gamma_\xi$ for all $\xi > \beta$.

What is left now is to prove that $i_{\alpha,\beta}(\gamma_\alpha) = \gamma_\beta$. Similarly to lemma 4.35, $\gamma_\alpha \leq i_{\alpha,\beta}(\gamma_\alpha) \leq \gamma_\beta$. Suppose there is an ordinal $\zeta \in M_{\alpha,\beta}$, such that $\gamma_\alpha \leq \zeta < \gamma_\beta$ and $\zeta = t[\delta_1, \ldots, \delta_n, \epsilon_1, \ldots, \epsilon_m]$, where $\delta_1 < \ldots < \delta_n < \gamma_\alpha,\ \gamma_\beta < \epsilon_1, \ldots, \epsilon_m$. This means that

$$L_\kappa \models \exists \delta_1, \ldots, \delta_n < \gamma_\alpha\ \gamma_\alpha \leq t[\delta_1, \ldots, \delta_n, \epsilon_1, \ldots, \epsilon_m] < \gamma_\beta.$$

Using the information we have for $F_\alpha = i_\alpha^{-1}$ from lemma 4.35, we get that

$$L_\kappa \models \exists \delta_1, \ldots, \delta_n < \gamma\ \gamma \leq t[\delta_1, \ldots, \delta_n, \epsilon_1, \ldots, \epsilon_m] < \gamma_\beta$$

which is a contradiction since $t[\delta_1, \ldots, \delta_n, \epsilon_1, \ldots, \epsilon_m] \in M_\beta$, thus there should be an ordinal $\zeta \in M_\beta$ such that $\gamma \leq \zeta < \gamma_\beta$. □

Lemma 4.37. $\Gamma = \{\gamma_\alpha : \alpha < \omega_1\}$ *is a set of indiscernibles for* L_κ.

Proof. Suppose that $\alpha_1 < \ldots < \alpha_n$, $\beta_1 < \ldots < \beta_n$ and $\max\{\alpha_n, \beta_n\} < \delta_1 < \ldots < \delta_n$ are increasing sequences of elements of Γ. Using the above lemma $2n$ times, we get that for every formula ϕ:

$$L_\kappa \models \phi[\alpha_1, \ldots, \alpha_n] \leftrightarrow L_\kappa \models \phi[\delta_1, \ldots, \delta_n]$$

and

$$L_\kappa \models \phi[\beta_1, \ldots, \beta_n] \leftrightarrow L_\kappa \models \phi[\delta_1, \ldots, \delta_n].$$

This means that Γ is a set of indiscernibles for L_κ. □

Step 3

The proof is completed using the following lemma:

Lemma 4.38. *If there is a model* L_λ *which has a set of indiscernibles of order* $\kappa > \omega$*, then there is a limit ordinal* γ *and a set* $I \subset \gamma$*, of order type* κ*, such that* (L_γ, I) *is an E-M remarkable set.*

Proof. See [8].

By the preceding lemma we see that there is a well-founded remarkable E-M set, hence $0^\sharp$ exists. □

In the previous chapter we saw that the existence of an nontrivial elementary embedding of V is equivalent to the existence of a measurable cardinal. Theorem 4.30 is important because it implies that the existence of elementary embeddings of inner models of V can yield the truth of weaker large cardinal axioms. Using the methods described in this section one could prove relative results for elementary embeddings of the models $L(X)$, $X \subset \omega$ and $X^\sharp = \{\phi : (L_{\aleph_\omega}(X), \in, X) \vDash \phi[\aleph_1, \ldots, \aleph_n]\}$. Those methods can be extended further more using the technique of ultrapowers and lead to a finer analysis of the structure of large cardinals. What we wish to study though, is how strong large cardinals we can get by extending the embeddings of V, thus we will not proceed further more to this subject. As we will see in the next chapter, assuming the existence of a nontrivial elementary embedding $j : V \prec M$, the closer M is to V the stronger large cardinals we get.

Chapter 5

Stronger embeddings of V

The strongest large cardinals we have met up to now are the measurable cardinals. In terms of elementary embeddings, the existence of such cardinals is equivalent to the existence of an elementary embedding $j : V \prec M$, where M is an inner model. In the case $M = M_D$ -D is a normal κ-complete ultrafilter on κ- we have seen that $M \neq V$, because $D \notin M$, and even more M is not closed under sequences of length κ^+. This means that the inner model M can be quite thin in comparison with V. In this chapter we are going to require that M is closer to V and introduce this way stronger and stronger large cardinals. Finally, we will investigate the case where $M = V$.

5.1 Strong, Woodin and superstrong cardinals

Our first step in strengthening the the elementary embeddings $j : V \prec M$, is done by introducing strong cardinals, Woodin cardinals and superstrong cardinals. We will define even stronger large cardinals in the next sections, but those described here have the advantage that they yield important results with relatively weaker hypotheses. They can also be described, by the notion of extenders, which are roughly sets of measures which approximates the corresponding embedding $j : V \prec M$.

Strong cardinals

Definition 5.1.

(i) A cardinal κ is called *α-strong*, if there is an elementary embedding $j : V \prec M$, with critical point κ, such that $\alpha < j(\kappa)$ and $V_{\kappa+\alpha} \subset M$.

(ii) A cardinal κ is called *strong*, if it is α-strong for every $\alpha \in On$.

The following theorem contains some of the interesting consequences of the existence of α-strong or strong cardinals:

Theorem 5.2.

(i) If there is a strong cardinal, then for every set A $V \neq L(A)$.

(ii) For every α*-strong cardinal* κ*, if* GCH *holds for every* $\beta < \kappa$*, then it also holds for every* $\beta < \kappa + \alpha$.

(iii) If κ *is strong, then* $V_\kappa \prec_2 V$.

Theorem 5.3. *Suppose* κ *is 2-strong. Then, there is a normal ultrafilter* D *over* κ *such that* $\{\lambda < \kappa : measurable(\lambda)\} \in D$.

Therefore strong cardinals are stronger than measurable cardinals. Actually, the measurability of a cardinal is equivalent to its 1-strongness.

Another interesting result, is the fact that the existence of a strong cardinal does not imply the existence of greater large cardinals.

Theorem 5.4.

$cons(ZFC{+}\exists\kappa\ strong(\kappa)) \rightarrow cons(ZFC{+}\exists\kappa\ strong(\kappa){+}\ \nexists\lambda > \kappa\ inaccessible(\lambda))$.

Woodin cardinals

Definition 5.5. A cardinal κ is called a *Woodin cardinal*, if for every $f \in^\kappa \kappa$, there is an $\alpha < \kappa$ and a $j : V \prec M$, such that:

(i) $f''\alpha \subset \alpha$;

(ii) the critical point of j is α;

(iii) $V_{j(f)(\alpha)} \subset M$.

The great importance of Woodin's cardinals springs from their consequences in descriptive set theory. We mention the axiom of determinacy (AD), without giving a definition because we did so, we should also state several results which reveal its importance in descriptive set theory.

Theorem 5.6. *The following theories are equiconsistent:*

(i) ZFC + "$there\ are\ infinitely\ many\ Woodin\ cardinals$".

(ii) $ZF + AD$.

Theorem 5.7. *The existence of infinitely many Woodin cardinals and of a measurable cardinal above them, yields* $AD^{L(\mathbb{R})}$.

The following theorem, provides us with a characterization of Wooding cardinals. On the same time, it proves that

$$cons(ZFC + \exists\kappa\ Woodin(\kappa) \rightarrow cons(ZFC + \exists\kappa\ strong(\kappa)$$

and that the existence of a Woodin cardinal, implies the existence of all α-strong cardinals, $\alpha < \kappa$. Thus Woodin cardinals have greater consistency strength than strong cardinals.

Theorem 5.8. *Let κ be a Woodin cardinal. Then, the set $\{\alpha < \kappa : \forall \beta < \kappa\ \beta - strong(\kappa)\}$ is stationary in κ.*

One additional information we get from theorem 5.8, is that every Woodin cardinal κ is κ-Mahlo. On the contrary, it can not be weakly compact, as we will see that the property of being a Woodin cardinal is Π^1_1.

Superstrong cardinals

Definition 5.9. A cardinal κ is called *superstrong*, if there is a $j : V \prec M$ with critical point κ, and such that $V_{j(\kappa)} \subset M$.

If a cardinal is superstrong, then it is Woodin and there are many Woodin cardinals below it. In particular:

Theorem 5.10. *Let κ be superstrong. Then κ is Woodin and there is a normal ultrafilter D on κ such that*

$$\{\lambda < \kappa : Woodin(\lambda)\} \in D.$$

Extenders

An extender is a set of measures, defined in a similar way we defined a measure from a nontrivial embedding $j : V \prec M$.

Definition 5.11. Suppose that the critical point of $j : V \prec M$ is κ and $\kappa \leq \gamma \leq j(\kappa)$.

(i) For every $\alpha \in [\gamma]^{<\omega}$, E_α is a measure on $[\kappa]^{<\omega}$ such that

$$X \in E_\alpha \leftrightarrow \alpha \in j(X).$$

(ii) The (κ, γ)-*extender derived from* j is the set $E = \{E_\alpha : \alpha \in [\beta]^{<\omega}\}$.

Every element $E_[\alpha]$ of an extender (κ, γ), defines an ultrapower M_{E_α} of V. It is not hard to find embeddings $i_{\alpha,\beta} : M_{E_\alpha} \prec M_{E_\beta}$, for all $\alpha, \beta \in [\gamma]^{<\omega}$, such that$\alpha \subset \beta$. The collection $\{M_{E_\alpha} : \alpha \in [\gamma]^{<\omega}\}$ along with the embeddings $i_{\alpha,\beta}$, forms a *directed system* of models. Thus there is a model M_E, such that every model E_α is elementary embedded in M_E and M_E is the union of the images of E_α through those elementary embeddings. This idea is similar to that of iterated ultrapowers. By studying the properties of extenders we can conclude to the fact that they can be defined in ZFC without the assumption of the existence of the embedding j.

The following characterizations of strong, Woodin and superstrong cardinals, using extenders, imply that the correlated large cardinals can be defined in ZFC.

Theorem 5.12.

(i) *κ is α-strong iff there is a $(\kappa, |V_{\kappa+\alpha}|^+)$-extender E such that $V_{\kappa+\alpha} \subset M_E$ and $\alpha < j_E(\kappa)$. Therefore, κ is strong iff there are such extenders for every α.*

(ii) *κ is Woodin iff for all $f \in^\kappa \kappa$ there is an $\alpha < \kappa$ and an extender $E \in V_\kappa$ such that:*

- *$f''\alpha \subset \alpha$;*
- *the critical point of j_E is α;*
- *$j_E(f)(\alpha) = f(\alpha)$;*
- *$V_{j_E(f)(\alpha)} \subset M_E$.*

(iii) *κ is superstrong iff there is a (κ, β)-extender E, $\beta > \kappa$, such that $V_{j_E(\kappa)} \subset M_E$.*

5.2 Strongly compact cardinals

Strongly compact cardinals have greater consistency strength than Woodin cardinals, but there is not a known connection between them and superstrong cardinals. We define them below using elementary embeddings.

Definition 5.13. Suppose $\kappa \leq \lambda$.

(i) κ is λ-compact, if there is an elementary embedding $j : V \to M$ with critical point κ and such that:

$$X \subset M \wedge |X| \leq \lambda \to \exists Y \in M \ (X \subset Y \wedge M \vDash |Y| < j(\kappa)).$$

(ii) κ is strongly compact if it is λ-compact for every $\lambda \geq \kappa$.

Definition 5.14. For every $x \in P_\kappa(X)$, $\hat{x} = \{y \in P_\kappa(X) : x \subset y\}$. A κ-complete ultrafilter U on $P_\kappa(X)$ is called a *fine measure* on $P_\kappa(X)$, if $\forall x \in P_\kappa(X)\ \hat{x} \in U$.

Using complete filters, fine filters and infinitary languages, we can give the following characterizations of strongly compact cardinals:

Lemma 5.15. *For every regular cardinal κ the following are equivalent:*

(i) *κ is strongly compact.*

(ii) *For every set S, every κ-complete filter on S can be extended to a κ-complete ultrafilter on S.*

(iii) *For every X, $|X| \geq \kappa$, there is a fine measure on $P_\kappa(X)$.*

(iv) *$\mathcal{L}$ satisfies the compactness theorem.*

Each of those definitions yields that strong cardinals can be defined in ZFC.

As we have already mentioned, the consistency strength of strong cardinals is greater than that of Woodin cardinals, therefore greater than that of measurable cardinals. We will state a stronger theorem towards this direction. In order to do this we are going to define Mitchell's order. Mitchell's order assists us in constructing measurable cardinals with greater consistency strength and such that many measurable cardinals lie below them. For example, if D is a normal measure on κ of order greater than 0, then $\{\lambda < \kappa : measurable(\lambda)\} \in D$.

Definition 5.16.

(i) Suppose κ is a measurable cardinal and $\mathcal{D}$ is the collection of all the normal measures on κ. *Mitchell's order*, is an order on $\mathcal{D}$ defined the following way:

$$D_1 < D_2 \leftrightarrow D_1 \in M_{D_2}.$$

(ii) It is easy to check that the above order is well-founded, thus we may define $o(D) = rank_<(D)$, for every $D \in \mathcal{D}$. By $o(\kappa)$, we denote the supremum of $\{o(D) : D \in \mathcal{D}\}$.

Theorem 5.17. *(Mitchell) If there is a strongly compact cardinal, there exists an inner model with a measurable cardinal κ, such that $o(\kappa) = \kappa^{++}$.*

The following yields that if there is a strongly compact cardinal, then the universe is not even relatively constructible.

Theorem 5.18. *(Vopěnka-Hrbáček) If there is a strongly compact cardinal, then for every set A, $V \neq L[A]$.*

We finally give the following result, which relates strongly compact cardinals with the computation of the continuum function.

Theorem 5.19. *(Solovay) If κ is strongly compact then the singular hypothesis holds above κ.*

5.3 Supercompact cardinals

Supercompact cardinals form a very important class of large cardinals since they have strong consequences on the structure of V above them. It is hard though, to approach the structure of the universe below them, because no procedure, such as approximation with extenders, is known to work for them neither has a canonical inner model been found for them yet. The consistency strength of a supercompact cardinal is greater or equal to that of a strongly compact cardinal, but it is not known if it is strictly above.

Definition 5.20.

(i) Suppose $\kappa \leq \lambda$. κ is *λ-supercompact* if there is an embedding $j : V \prec M$ such that:

– the critical point of j is κ;

– $j(\kappa) > \lambda$;

– ${}^{\lambda}M \subset M$.

(ii) κ is *supercompact* if it is λ-supercompact for all $\lambda \geq \kappa$.

The following characterization of λ-supercompactness implies that it can be defined in ZFC. The same of course holds for supercompactness. The characterization uses normal filters on $P_\kappa(\lambda)$ and it works in a similar way as ultrafilters are used in the characterization of measurable cardinals.

Definition 5.21. A fine filter on $P_\kappa(\lambda)$ is *normal*, if it is closed under diagonal intersections of the form

$$\triangle_{\alpha<\lambda} X_\alpha = \{x \in P_\kappa(\lambda) : x \in \bigcap_{\alpha \in x} X_\alpha\}.$$

The same way as we did for normal measures, we can prove the following equivalence:

Lemma 5.22. *A filter D over $P_\kappa(\lambda)$ is normal iff every choice function*[1] *on a D-stationary set*[2] *X is constant on a D-stationary set.*

Theorem 5.23. *If $\kappa \leq \lambda$, then κ is λ-supercompact iff there is a normal ultrafilter over $P_\kappa(\lambda)$.*

We give one more characterization of supercompactness, which introduces it as a reflection principle.

Theorem 5.24. *(Magidor) κ is supercompact iff for every $\alpha > \kappa$ there is a $\beta < \kappa$ and an embedding $i : V_\beta \prec V_\alpha$ with critical point δ, $i(\delta) = \kappa$.*

We present some results which -as we have already seen- are also true for strong cardinals.

Theorem 5.25.

(i) If κ is 2^κ-supercompact, then there is a normal ultrafilter D over κ such that

$$\{\lambda < \kappa : measurable(\lambda)\} \in D.$$

(ii) If κ is γ-supercompact, then $\forall \alpha < \kappa\ 2^\alpha = \alpha^+$ implies that $\forall \beta \leq \gamma\ 2^\beta = \beta^+$.

(iii) If κ is supercompact, then $V_\kappa \prec_2 V$.

Finally, we see that supercompact cardinals have many measures and their Mitchell order is high.

[1] $\forall x \in X \setminus \emptyset\ f(x) \in x$.

[2] If F is a filter on S then $X \subset S$ is F-stationary if $\forall Y \in F\ X \cap Y \neq \emptyset$.

Theorem 5.26. *Let κ be supercompact, then*

(i) there are 2^{2^κ} normal measures on κ;

(ii) for every $\lambda \geq \kappa$ there are $2^{2^{<\lambda}}$ normal measures on $P_\kappa(\lambda)$;

(iii) the Mitchell order of κ is $(2^\kappa)^+$.

5.4 Extendible cardinals and Vopěnka's principle

Here we see some large cardinal of even greater consistency strength.

Extendible cardinals

Definition 5.27. ($\alpha \neq 0$)

(i) A cardinal κ is *α-extendible* if there is a β and an embedding $i : V_{\kappa+\alpha} \to V_\beta$ with critical point κ and such that $\alpha < i(\kappa)$.

(ii) κ is *extendible* if it is α-extendible for every α.

The following theorem shows that the existence even of a 1-extendible cardinal implies the existence of many measurable cardinals.

Theorem 5.28. *Suppose κ is 1-extendible. Then,*

(i) κ is measurable;

(ii) there is a normal measure D on κ such that $\{\alpha < \kappa : measurable(\alpha)\} \in D$.

There is a deep connection between extendible and supercompact cardinals. Although extendible cardinals are stronger, supercompact cardinal are above every α-extendible.

Theorem 5.29. *Suppose κ is extendible. Then,*

(i) κ is supercompact;

(ii) there is a normal measure D on κ such that $\{\alpha < \kappa : supercompact(\alpha)\} \in D$.

Theorem 5.30.

(ii) Suppose κ is $|V_{\kappa+\alpha}|$-supercompact. Then, there is a normal measure D on κ such that $\{\beta < \kappa : \alpha - extendible(\beta)\} \in D$.

(ii) If κ is α-extendible and $\beta + 2 < \alpha$, then κ is $|V_\kappa|$-supercompact.

Finally, we give one important reflection property of extendible cardinals, out of which, many consequences of the existence of extendible cardinals can be derived.

Theorem 5.31. *If κ is extendible, then $V_\kappa \prec_3 V$.*

Vopěnka's principle

In contrast with the large cardinal axioms we have considered up to now, Vopěnka's principle does not imply the existence of a specific large cardinal. This is because it requests the existence of several elementary embeddings.

Definition 5.32. *Vopěnka's principle* (VP): Let C be a class of structures of the language $\mathcal{L}$. There are $A, B \in C$ such that there is an embedding $i : A \prec B$.

Vopěnka's principle is stronger than extendible cardinals:

Theorem 5.33. *VP implies that the class of extendible cardinals κ, carrying a normal measure D such that $\{\lambda < \kappa : extendible(\lambda)\} \in D$, is stationary on On.*

We can restate VP, in terms of A-supercompact and A-extendible cardinals, which are relativized versions of supercompact and extendible cardinals respectively.

Definition 5.34. Let A be a class and κ a cardinal.

(i) κ is *A-supercompact* if for all $\alpha > \kappa$ there is a $\beta < \kappa$ and an embedding

$$i : (V_\beta, \in, V_\beta \cap A) \prec (V_\alpha, \in, V_\alpha \cap A)$$

with critical point γ such that $i(\gamma) = \kappa$.

(ii) κ is *A-extendible* if for all $\alpha > \kappa$ there is an ordinal β and an embedding

$$i : (V_\alpha, \in, V_\alpha \cap A) \prec (V_\beta, \in, V_\beta \cap A)$$

with critical point κ and such that $\alpha < i(\kappa) < \beta$.

Theorem 5.35. *The following are equivalent:*

(i) VP;

(ii) for every class A there is an A-extendible cardinal;

(iii) for every class A there is an A-supercompact cardinal.

Finally, we present the strongest large cardinal axioms known.

5.5 Huge cardinals and I0-I3

Huge cardinals

Definition 5.36.

(i) κ is *n-huge* if there is an embedding $j : V \prec M$ with critical point κ and such that ${}^{j^n(\kappa)}M \subset M$.

(ii) κ is *huge* if it is 1-huge.

The following equivalent form of n-hugeness, yields that it can be defined in ZFC.

Theorem 5.37. *A cardinal κ is n-huge iff there are cardinals $\kappa = \lambda_0 < \lambda_1 < \ldots < \lambda_n = \lambda$ and a κ-complete normal ultrafilter D over $P(\lambda)$ such that for every $i < n$, $\{x : ordtype(x \cap \lambda_{i+1}) = \lambda_i\} \in D$.*

Although huge cardinals have greater consistency strength, the existence of a huge cardinal does not imply the existence of a supecompact cardinal. Even more:

Theorem 5.38. *If there are both a huge and a supercompact cardinal, then the least huge cardinal is below the least supercompact cardinal.*

Theorem 5.39. *$cons(ZF + \exists\kappa\ huge(\kappa)) \to cons(ZF + VP)$. In particular, if κ is huge, then $V_\kappa \models VP$.*

It is known that n-huge cardinals form an increasing in strength hierarchy.

Theorem 5.40. *If κ is $n+1$-huge, then there is a normal measure D on κ such that $\{\lambda < \kappa : n - huge(\lambda)\}$.*

I0-I3

The principles I0-I3 are very close to inconsistency since they require the existence of embeddings close to $j : V \prec V$. We define them below and place them in the hierarchy of large cardinals.

Definition 5.41.

- I0: There is an α and an $i : L(V_{\alpha+1}) \prec V_{\alpha+1}$ with critical point strictly less than α.
- I1: There is an α and an embedding $i : V_{\alpha+1} \prec V_{\alpha+1}$.
- I2: There is an α and an embedding $j : V \prec M$ such that $V_\alpha \subset M$, the critical point of j is strictly less than α and $j(\alpha) = \alpha$.
- I3: There is an α and an embedding $i : V_\alpha \prec V_\alpha$.

Theorem 5.42.

$$I0 \to I1 \to I2 \to I3.$$

Theorem 5.43. *If $I3$ holds and the critical point of the corresponding embedding j is κ, then there is a normal measure D on κ such that $\{\lambda < \kappa : \forall n < \omega\ n - huge(\kappa)\} \in D$.*

5.6 Kunen's theorem

Up to now we have used elementary embeddings in order to define new large cardinal axioms. We may notice that as we try to increase the consistency strength of those axioms, we defined each time an elementary embedding closer to $j : V \prec V$. In the following two we investigate whether such an embedding exists and in the case it exist we state its consequences. By a theorem due to Kunen, we see that if we extend the language of set with a function symbol j such that j is an elementary embedding and it satisfies all instances of replacement for the formulas of the extended language. On the contrary, if we do not require that j satisfies the replacement axiom we cannot get to a contradiction as in Kunen's theorem. This situation is described by the wholeness axiom, introduced by Corazza, which is a large cardinal axiom with consistency strength between n-huge[3] cardinals and I_3.

We will show that there is no extension $(V, \in, j)$ of V such that j is a non-trivial embedding $j : V \prec V$ which satisfies all replacement for every formula ϕ of the language $(\in, j)$. In order to achieve this, we will first define ω-Jónsson functions that will help us conclude to a contradiction.

Definition 5.44. A function $f :^{\omega} x \to x$ is called *ω-Jónsson* on x if $y \subset x \wedge |y| = |x| \to f''^{\omega}y = x$.

Theorem 5.45. *(Erdös-Hajnal) For every $\lambda \geq \omega$, there is an ω-Jónsson function over λ.*

Theorem 5.46. *If $j : V \prec M$, then $M \neq V$.*

Proof. Suppose $M = V$, the critical point of j is κ and $\lambda = \sup\{j^n(\kappa) : n < \omega\}$. We have that $j(\lambda) = \sup\{j^{n+1}(\kappa) : n < \omega\} = \lambda$.

On the other hand, there is an ω-Jónsson function f over λ. $j(f)$ is also ω-Jónsson. Let $X = j''\lambda$. $|X| = \lambda$, thus there is a $x \in^{\omega} X$ such that $j(f)(x) = \kappa$. If $x(n) = j(\alpha_n)$ for every $n < \omega$, then $x = j(y)$, where $\forall n < \omega\ y(n) = \alpha_n$. The latter means that $\kappa = j(f)(j(y)) = j(f(y))$ which is a contradiction, as κ is the critical point of j. □

The following theorem is similar to the above, and it emerges from some other proofs of 5.46.

Theorem 5.47.

(i) For any α there is no $j : V_{\alpha+2} \prec V_{\alpha+2}$.

(ii) If α is the least ordinal above the critical point of j, such that $j(\alpha) = \alpha$, then $j''\alpha \notin M$.

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[3]In fact super-n-huge which are stronger than n-huge.

5.7 The wholeness axiom

As we have already seen, the axiom of replacement was used in order to construct the sequence $\{j^n(\kappa)\}_{n<\omega}$. Hence, in the case we do not require that j satisfies the replacement axiom the proof is no longer valid. This situation is captured by the notion of weak definability.

Definition 5.48. Let M be a model of ZF and $X \subset M$. X is *weakly definable* in M if $(M, \in, X)$ satisfies the axiom of replacement for the extended language.

Definition 5.49. *Wholeness Axiom* (WA): There is a nontrivial embedding $j; V \prec V$ which is not weakly definable in V and such that for every set A, $j \restriction A$ is a set.

We call j a *WA-embedding*.

The request for $j \restriction A$ to be a set is necessary in order not to lose all the strength obtained by the replacement axiom. This specific part of WA places it high in the hierarchy of large cardinals. In particular:

Theorem 5.50. *Let $j : V \prec V$ be WA-embedding with critical point κ. Then, κ is n-huge for every $n < \omega$.*

Theorem 5.51. *If α is the least limit ordinal for which $V_\alpha \prec V_\alpha$, then $V_\alpha \models WA + I_3$. Therefore, $cons(ZFC + I_3) \rightarrow cons(ZFC + WA)$ while $cons(ZFC + WA) \not\rightarrow cons(ZFC + I_3)$.*

Following the proof of Kunen's theorem we can disprove the existence of some sets if WA holds. For example:

Theorem 5.52. *If $j : V \prec V$ is an elementary embedding with critical point κ, then the sequence $\kappa, j(\kappa), j^n(\kappa), \ldots$ is not weakly definable in V and it is unbounded in On.*

We finish this section with an important reflection principle emerging from WA.

Theorem 5.53. *If $j : V \rightarrow V$ is a WA-embedding, then $\forall n < \omega V_{j^n(\kappa)} \prec V$.*

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