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Opinion Dynamics in the presence of Social Choice Rules

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Last but not least, I reminisce all friends and members of the Computation and Reasoning Lab for the everyday correspondence we had, so valuable for me in order to enjoy the countless hours in the lab. I hope the best for all of you, as a commited member of Corelab.

Abstract

Networks rise in several aspects of modern life, motivating us to model and understand them. Several types of mathematical and algorithmic problems arise in networks. Opinion dynamics is a process in networks modelling the effect of local interactions on agents' beliefs. The beliefs can be thought as a belief for some common question of interest, for instance the probability of some event. In this thesis, I will present mathematical models which capture such processes in networks and state mathematical problems on them. I try to evaluate these processes in terms of economic behavior and convergence time of dynamic processes. We would ideally like to connect these quantities with parameters of the network structure. Finally, motivated by polls present on political elections, we study the effect of partial global information on agents' local interactions and behavior.

Contents

Chapter 1

Introduction - Game Theory

Through the years Game Theory has been studied rigorously, examining mathematical models of situations that arise in several aspects of life, starting from economic situations. In the long run it became valuable in many different sciences such as economics, computer science, political science and biology. Game Theory makes a step further from Optimization Theory, studying models where several different parties interact in a common environment, acting selfishly. Optimization Theory (individual decision making) sets the groundwork for the study of how the simultaneous behavior of selfish individuals generates outcomes. Each agent has available actions and applies these actions in order to optimize an objective function, using information for the other agents' behavior. A game can be described in two forms, the extensive form game and the normal form, but in this thesis I will deal with normal form games. During the years, there have been studied several solution concepts, which present an action profile which agents tend to follow. The most well known is the concept of the Nash Equilibrium, but not the only one. There is a big debate involving a variety of solution concepts, such as dominant strategy profiles, subgame perfect equilibria, trembling hand equilibria, mixed Nash and correlated equilibria, bayesian equilibria, sequential rationality. In this thesis I will mainly use the idea of a pure Nash Equilibrium. Philosophical issues arise about the plausibility of such a concept if it is not reached by the agents. Based on a specific solution concept, one can describe several important values of the game, depending in its applications. The most basic ideas of Game Theory can be found in [20] and [17].

In the introduction of my thesis, I would like to introduce some basic tools used in the study of games. These tools can only apply to a special class of games.

·	H	T
H	1/-1	-1/1
T	-1/1	1/-1

Table 1.1: Normal Form Game (Matching Pennies)

However, when applied they facilitate the analysis of the game, implying nice properties for the Nash equilibria and economic behavior of the game. Moreover, they seem as a starting point towards the understanding of a game and the discovery of other more general analytic ideas, arising from general fixed point theorems such as Brower fixed point theorem and Banach fixed point theorem [6].

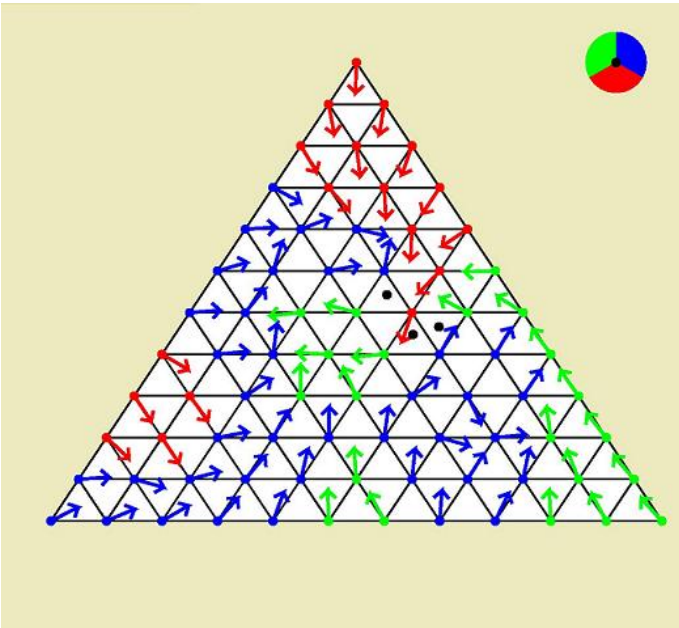

Figure 1.1: Fixed Point (Sperner's Lemma)

1.1 Potential Games

In this subsection, I will introduce a specific class of games with nice properties. Potential games can be seen as an optimization problem, where the agents optimize a unified function. I will mention several classes of potential games and point out some guides on how to examine if a game admits a potential.

Let's consider a game with n agents. Each agent has a strategy space X_i and a utility function $u_i : X \to \mathcal{R}$, where $X = X_1 \times \ldots \times X_n$. We say that a game has

1. an exact potential function $\Phi : X \to \mathcal{R}$, if for every agent i, every two strategies $x_i, x_i' \in X_i$ and every strategy vector $x_{-i} \in X_{-i}$

$$u_i(x_i, x_{-i}) - u_i(x_i', x_{-i}) = \Phi(x_i, x_{-i}) - \Phi(x_i', x_{-i})$$

2. a weighted w-potential function $\Phi : X \to \mathcal{R}$, where $w = (w_i)$, if for every agent i, every two strategies $x_i, x_i' \in X_i$ and every strategy vector $x_{-i} \in X_{-i}$

$$u_i(x_i, x_{-i}) - u_i(x_i', x_{-i}) = w_i(\Phi(x_i, x_{-i}) - \Phi(x_i', x_{-i}))$$

3. an ordinal potential function $\Phi : X \to \mathcal{R}$, if for every agent i, every two strategies $x_i, x_i' \in X_i$ and every strategy vector $x_{-i} \in X_{-i}$

$$u_i(x_i, x_{-i}) - u_i(x_i', x_{-i}) > 0 \text{ iff } \Phi(x_i, x_{-i}) - \Phi(x_i', x_{-i}) > 0$$

Of course an exact potential function for a game can also be called a weighted potential function or an ordinal potential function. A game with an ordinal potential function is called a potential game. Consider a game with an ordinal potential function. If the potential function admits a maximum value in $x \in X$, then the game admits a Nash equilibrium in X, since for every agent i and strategy x_i'

$$\Phi(x) \geq \Phi(x_i', x_{-i}) \Rightarrow u_i(x) \geq u_i(x_i', x_{-i})$$

When considering finite games (finite strategy sets), the potential function surely admits a maximum value. So, every finite potential game possesses a pure Nash equilibrium. An analogous result can be proven for infinite potential games, considering Weierstrass's theorem. Every game with a continuous potential function and compact strategy sets possesses a pure Nash equilibrium.

From now on I will present an important theorem which points out the existence of an exact potential function. When seen as a tool it can be extended to point the existence of a weighted potential function. Before stating the theorem, I must define some notation. Let's consider a finite path of strategy profiles $P = (x^0, x_1, \ldots, x^m)$, where consequent strategy profiles differ in a unique agent's strategy. We define the function I given by

$$I(P, u) = \sum_{k=1}^{m} (u_{i_k}(x^k) - u_{i_k}(x^{k-1}))$$

where i_k is the agent who deviates at step k. A path is closed if $x^0 = x^m$ and simple if a strategy profile does not appear twice in the path.

Theorem 1.1.1 *Consider a game G. The following claims are equivalent:*

1. *G has an exact potential function.*
2. *$I(P,u) = 0$ for every finite closed path P.*
3. *$I(P,u) = 0$ for every finite simple closed path P.*
4. *$I(P,u) = 0$ for every simple closed path of length 4.*

Proof (1) $\Rightarrow$ (2): $I(P,u) = \sum_{k=1}^{m}(u_{i_k}(x^k) - u_{i_k}(x^{k-1})) = \sum_{k=1}^{m}(\Phi(x^k) - \Phi(x^{k-1})) = \Phi(x^m) - \Phi(x^0) = 0$

(2) $\Rightarrow$ (1): For every two strategy profiles x and y and every two paths P_1 and P_2 from x to y

$$I(P_1,u) = I(P_2,u)$$

That is because path $(P_1, -P_2)$, where $-P_2$ is the path P_2 when reversed, is closed, i.e.

$$0 = I((P_1,-P_2),u) = I(P_1,u) + I(-P_2,u) = I(P_1,u) - I(P_2,u)$$

Choose an initial strategy profile x^0 and consider a path $P(x)$ from x^0 to x. We claim that the function $\Phi(x) = I(P(x),u)$ is an exact potential function. Consider a strategy profile x. Then $\Phi(x) - \Phi(x'_i, x_{-i}) = I(P(x),u) - I((P(x),(x'_i,x_{-i})),u) = u_i(x_i,x_{-i}) - u_i(x'_i,x_{-i})$, which proves that $\Phi(\cdot)$ is an exact potential function.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4): The proof is trivial.

(4) $\Rightarrow$ (2): We proceed by induction. Assume that for every closed path P of length $m-1$, $I(P,u) = 0$. Consider a closed path P of length m. Assume that $i_1 = 1$, then there exists a step $2 \leq k \leq m$, such that $i_k = 1$, since agent 1 should deviate back to his initial strategy. If $i_2 = 1$ or $i_m = 1$, then we can a smaller path P', where $x^0 \to x^2$ or $x^{n-1} \to x^1$ respectively. Of course $I(P,u) = I(P',u) = 0$. Assume now that $3 \leq k \leq m-1$. Then there exists another path $P' = (x^0, \ldots, x^{k-2}, x', x^k, \ldots, x^m)$, where $x' = (x_1^k, x_{i_{k-2}}^{k-2}, x_{-\{i(k-2),1\}}^{k-2})$. Since $I((x^{k-2}, x^{k-1}, x^k, x'),u) = 0$ by (4), we conclude that $I(P,u) = I(P',u)$. Step by step, one can find a closed path P'' with $i_2 = 1$. So, $I(P,u) = I(P'',u) = 0$.

This completes the proof.

Finally, I will present another theorem which applies on infinite games.

Theorem 1.1.2 *Consider a game G, where the strategy sets are intervals of $\mathcal{R}$. Suppose the utility functions are twice continuously differentiable. Then G has an exact potential function if and only if*

$$\frac{\partial^2 u_i}{\partial x_i \partial x_j} = \frac{\partial^2 u_j}{\partial x_i \partial x_j}$$

for every two agents i and j.

The above theorem is completely analogous to the theorem stated before and points out a way to compute the exact potential function

$$\Phi(x) = \sum_i \int_0^1 \frac{\partial u_i(x(t))}{\partial x_i} \frac{dx_i(t)}{dt} dt$$

where $x : [0, 1] \to X$ is a piecewise continuously differentiable path in X, such that $x(0) = x^0$ and $x(1) = x$.

Potential games have nice properties, since we can apply tools and theorems from Optimization Theory in a straightforward manner, exploiting the potential function. It is direct to seek for a concave potential function in order to guarantee uniqueness of a local maximum or a Nash Equilibrium. Convergence analysis can be thoroughly applied in the potential function.

1.2 Concave Games

The class of potential games is a well studied class of games with nice properties and results. However, this is not the case most times. We would like to introduce a broader class of games with nice properties, maybe begining with the existence of a Nash Equilibrium. The analysis of such a class would be a generalization of the Optimization Theory Mathematics. The quest begins by studying Concave Games, using Fixed Point theorems.

We consider games with n agents. Each agent has a strategy x_i in the Euclidean strategy space $\mathcal{R}^{m_i}$. So, there is a vector of all players strategies

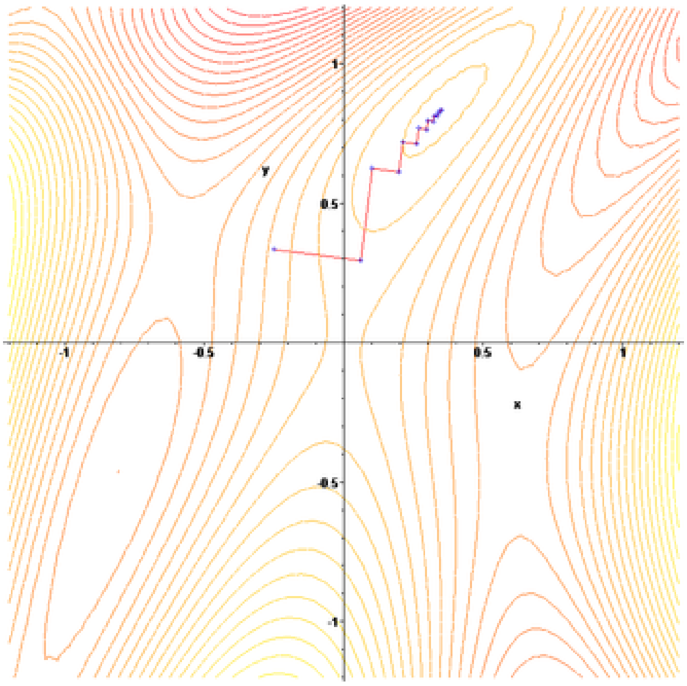

Figure 1.2: Convergence (Gradient Descent)

$x \in \mathcal{R}^{m_1} \times \ldots \times \mathcal{R}^{m_n}$. Of course the strategy set is restricted to a compact (closed and bounded) and convex set $X = X_1 \times \ldots \times X_n \subset \mathcal{R}^{m_1} \times \ldots \times \mathcal{R}^{m_n}$. Each agent has a utility function $u_i : X \to \mathcal{R}$ and tries to maximize it. The utility function is continuous in x and concave in x_i. In order to prove the result, we will need a function $\rho : X \times X \to \mathcal{R}$ given by the following formula

$$\rho(x, y) = \sum_i u_i(x_1, \ldots, y_i, \ldots, x_n)$$

The above function can be proven trivially to be continuous in x and y and is concave in y for every fixed x. The class of games satisfying these properties are called concave games. There is an equivalent formulation for minimization problems and convexity, which we will call convex games. The same results occur. The following result is fundamental

Theorem 1.2.1 *An equilibrium point exists for every concave game.*

Proof We consider a mapping $U : X \to \mathcal{P}(X)$ given by

$$U(x) = \{y | \rho(x, y) = \max_{z \in X} \rho(x, z)\}$$

Since $\rho(x, y)$ is continuous in x and y and concave in y for every fixed x, it can be proven trivially that $U(\cdot)$ is upper semicontinuous mapping, which maps every point x in a closed and convex subset of X. Using the Kakatuni fixed point theorem, there exists a point $x \in X$ such that $x \in U(x)$, which means that

$$\rho(x, x) = \max_{z \in X} \rho(x, z)$$

By using the above equality, we conclude that for each agent i and strategy vector $x'_i \in X_i$

$$\rho(x, x) \geq \rho(x, (x'_i, x_{-i})) \Rightarrow$$
$$u_i(x) \geq u_i(x'_i, x_{-i})$$

This completes the proof.

From a point of view, $\rho(x, \cdot)$ functions as a potential function on each point x. This is a generalization of the potential function as one cannot find a potential function. It remained to prove the existence of a point which is a solution of maximizing the potential function of that point.

In the rest of [21], Rosen studies concave games using methods of optimization theory, with the key point of introducing the pseudogradient $(\bigtriangledown_{x_1} u_1(x), \ldots, \bigtriangledown_{x_n} u_n(x))^T$, as a generalization of the gradient of a function. It seems that the only information needed is the change of $u_i(x)$ when changing x_i. It follows a result on the uniqueness of a Nash equilibrium, when making assumptions about diagonally strict concavity similar to the strict concavity of a function. The convergence to a single point and convergence time can be studied directly in well structured concave games by exploiting the pseudogradient stated above.

1.3 Smooth Games

In this section I will examine a specific class of games called "smooth". This class of games satisfy a given property concerning the cost function. By using an argument one can prove PoA bounds for the game. I will introduce the framework, present the argument and the extent of it's results and mention an example concerning congestion games. I will shortly discuss an analogous class of games called local smooth games.

In [22] Roughgarden introduced the smoothness framework for a cost minimization game. A game is called (λ, μ)- smooth if for every two strategy profiles x and x^*,

$$\sum_i c_i(x_i^*, x_{-i}) \leq \lambda \sum_i c_i(x^*) + \mu \sum_i c_i(x)$$

It seems that a game which is (λ, μ)-smooth, with $\lambda > 0$ and $\mu < 1$, guarantees good economic efficiency. Specifically, at a Nash equilibrium x the game satisfies the following inequalities

$$\sum_i c_i(x) \leq c_i(x_i^*, x_{-i}) \leq \lambda \sum_i c_i(x^*) + \mu \sum_i c_i(x)$$

By considering x^* as the optimal strategy profile, with respect to the social cost $SC(x) = \sum\limits_i c_i(x)$, the above inequality guarantees that

$$SC(x) \leq \frac{\lambda}{1-\mu} SC(x^*)$$

This means that the price of anarchy for pure Nash equilibria is upper bounded

$$PoA_{pure} \leq \frac{\lambda}{1-\mu}$$

It seems that the arguments used for upper bounding the PoA can be also be used for mixed, correlated and coarse correlated equilibria of a game. The value of (λ, μ) do not change since a smooth game satisfies the smoothness inequality for every two outcomes x and x^*. This means that the PoA remains the same for more general classes of equilibria, specifically mixed N.E., correlated N.E. and coarse correlated N.E.. For this reason the PoA proved with the smoothness framework is called the robust Price of Anarchy. When the smoothness framework was introduced it became clear that many bounds of the PoA in several games were proved by using this specific argument. I will present an example for congestion games with affine cost functions to illustrate the functionality of the smoothness framework.

Example A congestion game with n agents is defined by a set E of resources and strategy sets $X_1, \ldots, X_n \subseteq \{0,1\}^E$. Each resource $e \in E$ has an affine cost function $c_e(k) = a_e x_e + b_e$ where x_e is the number of agents using the resource at the strategy profile $x \in X_1 \times \ldots \times X_n$. The cost of agent i is defined as $c_i(x) = \sum\limits_{e \in x_i} c_e(x_e)$. The social cost of the congestion game is defined by $SC(x) = \sum\limits_i c_i(x) = \sum\limits_{e \in E} x_e c_e(x_e)$.

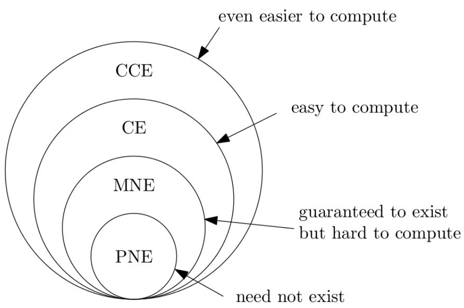

Figure 1.3: Nash Equilibrium Concepts

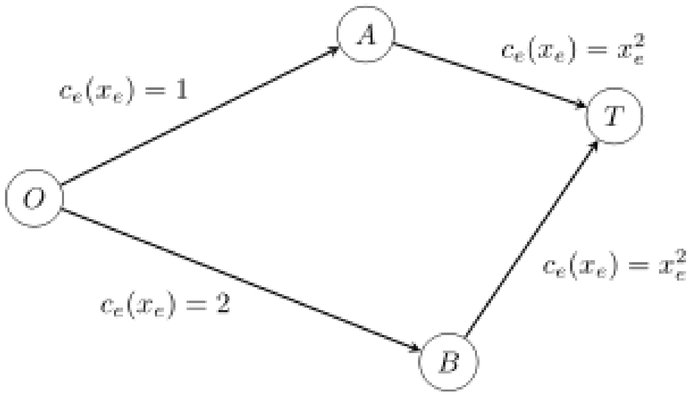

Figure 1.4: Congestion Games

Let's try to prove that a game is (λ, μ)-smooth using every two strategy profiles x and x^*

$$\sum_i c_i(x_i^*, x_{-i}) \leq \sum_{e \in E} (a_e(x_e + 1) + b_e)x_e^* \leq$$

$$\sum_{e \in E} \frac{5}{3}(a_e x_e^* + b_e)x_e^* + \sum_{e \in E} \frac{1}{3}(a_e x_e + b_e)x_e =$$

$$\frac{5}{3}\sum_i c_i(x^*) + \frac{1}{3}\sum_i c_i(x)$$

where the last inequality results from a basic algebraic inequality

$$y(z+1) \leq \frac{5}{3}y^2 + \frac{1}{3}z^2$$

The above computation implies an upper bound of $\frac{5}{2}$ on the PoA of pure Nash equilibria in every congestion game with affine cost functions, as proved in [9].

Consider a class of games $\mathcal{G}$ which admit a pure Nash equilibrium. For every game $G \in \mathcal{G}$ we get the PoA_{pure} which is the ratio of the **worst** Nash equilibrium to the optimal strategy profile. When proving that a game is "smooth" we need the **best** choice of (λ, μ) to derive an upper bound $\dfrac{\lambda}{1-\mu}$. Suppose that $\mathcal{A}(\mathcal{G})$ is the set of (λ, μ), for which all games in $\mathcal{G}$ are (λ, μ)-smooth. It is straightforward that

$$\sup_{G \in \mathcal{G}} PoA_{pure}(G) = \inf_{(\lambda,\mu) \in \mathcal{A}(\mathcal{G})} \frac{\lambda}{1-\mu}$$

which can be thought as a weak duality condition. If we derive an equality for a specific class of games then one can conclude that the upper bound provided by smooth arguments is tight. It seems a challenge to characterize the class of games that such a max-min condition is true.

The above results are true for every possible strategy set. However, there is an idea which brings better results when the strategy sets are continuous. In [23] Roughgarden et al. introduced the local smoothness framework, which takes effect into games where X_i is a convex subset of $\mathcal{R}_{m_i}$ and each cost function c_i is continuously differentiable with a bounded derivative.

A game is called locally (λ, μ)-smooth with respect to strategy profile x^* if for every strategy profile x,

$$\sum_i [c_i(x) + \bigtriangledown_{x_i} c_i(x)^T (x_i^* - x_i)] \leq \lambda \sum_i c_i(x^*) + \mu \sum_i c_i(x)$$

A locally (λ, μ)-smooth game with respect to the optimal outcome x^* has good economic efficiency. At a Nash equilibrium x

$$\sum_i c_i(x) \leq \sum_i [c_i(x) + \bigtriangledown_{x_i} c_i(x)^T (x_i^* - x_i)] \leq \lambda \sum_i c_i(x^*) + \mu \sum_i c_i(x)$$

The above inequality guarantees that

$$SC(x) \leq \frac{\lambda}{1-\mu} SC(x^*)$$

This means that the price of anarchy for pure Nash equilibria is upper bounded

$$PoA_{pure} \leq \frac{\lambda}{1-\mu}$$

The above result can be extended for mixed and correlated equilibria. However, that is not the case for coarse correlated equilibria. The intuition behind the local smoothness framework derives from the need for a “small” deviation which is effectively bounded. We would like to move away from the Nash equilibrium in order to use the Nash inequalities, and then bound the resulted increase of the cost functions. In the smoothness framework we tried to compute (λ, μ) in order to bound the change for every possible strategy profile. In the local smoothness framework we assume a slight (local) move, which restricts the increase that one has to bound. If the cost functions are convex, this “small” deviation would be more effectively bounded. To conclude the above results seem the only tools in hand to derive upper bounds on the PoA.

1.4 Best Responce Matrix

The games which studied mainly in this thesis are continuous games, escaping from the classical algorithmic framework. That means that we should recall convergence results from “Optimization Theory” to examine such games, especially since a potential function exists. In many games the best response of each agent can be given by a combination of the other agents’ strategies. If agents reply simultaneously a system of equations is implied

$$x(t+1) = Ax(t) + Bu(t)$$

There is a rich literature ([7]) analyzing such systems with nice properties depending on the structure of matrix A and B. Using properties of matrix A one examine if $x(t)$ converges and subsequently the convergence time.

Chapter 2

Several Models

This section could be the introduction. I preferred to introduce my thesis by introducing the main techniques discovered writing this thesis. A reader may stop in the introduction, but I will be confident that he would have read the best points and learned the most. From now and on I present applications of the theory. So this chapter will not have any results but a few, but mainly a presentation of the wide range of the Opinion Dynamics Theory. It is a very important chapter of the Social Networks Theory, as it can be seen in [16].

Social Networks arise in several aspects of everyday life since they depict every kind of interaction between people. Nowadays, the Internet can been seen as a huge social network, since the last years there is a great rise on the platforms accomodating social interaction such as Facebook and Twitter. Ofcourse the information and ideas spread through the "social media" seem to be a gear of the modern civilized world. Moreover, learning procedures are accomodated in a social network since agents share information about several beliefs. So, there is a need to model and understand the interaction in a social network and how the structure of the network influences learning and the spread of information.

I will mainly present models and the framework that they are studied. I will shortly present the kind of results researched. Before beginning stating the models I would like to define an opinion game between n agents. In order to keep track with the "ingredients" needed, one can suppose that it is a "different" kind of optimization problem. An optimization problem is defined by it's parameters (or exogenous variables or input), it's endogenous variables and the optimization procedure (objective function). The optimization procedure is the dynamics of the game and would be discussed later. The input of the opinion game would be

1. The agents' internal beliefs s_i (in some models).
2. The Social Network given by functions on each couple of agents. For example, it can be a graph $G = (V, E)$ (SN) with weights w_{ij} on edges (constant function).

Of course the endogenous variables would be a vector of the agents' opinions x.

Special talk can be done for the optimization procedure, meaning the set of solutions. Now it is not decentralized for the purpose of optimizing an objective function, but it can be thought as the result of a dynamical procedure driven by the agents. This procedure can be described step-by-step or can be vague as in the Nash Dynamics. The step-by-step procedure is mostly described by the agent's response function and an ordering of the agents' responses. The Nash Dynamics are described by the agents' cost function, knowing that a solution of the system is the one where each agent has no incentive to change his opinion.

After defining the game by stating it's 3 ingredients, it would be ideal to clarify the goals of the research done. I will state briefly three of these goals:

1. The existence of a solution.
2. The economic behavior of the game.
3. The convergence rate of the dynamics, when applicable.

2.1 The Degroot Model

Firstly, I would like to present the first model considered in [10] and define the notion of consensus. It is a starting point to examine how the structure of a social network can influence opinion formation. From now each agent's internal belief could serve as a starting point of his opinion formed, namely $x_i(0) = s_i$. This will mainly be used when describing a kind of dynamics by the agent's response formula. DeGroot used a matrix W to represent the social network, where w_{ij} represents the weight of trust that each agent i places on agent j, i.e.

$$x(t+1) = Wx(t)$$

So $x(t) = W^t s$ and the final opinion vector will be given by

$$x = \lim_{t \to \infty} x(t) = \lim_{t \to \infty} W^t s$$

We would assume that $\sum_j w_{ij} = 1$ for each agent i. The following figure illustrates

$$W = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 0 & 2/3 \\ 3/4 & 1/4 & 0 \end{pmatrix}$$

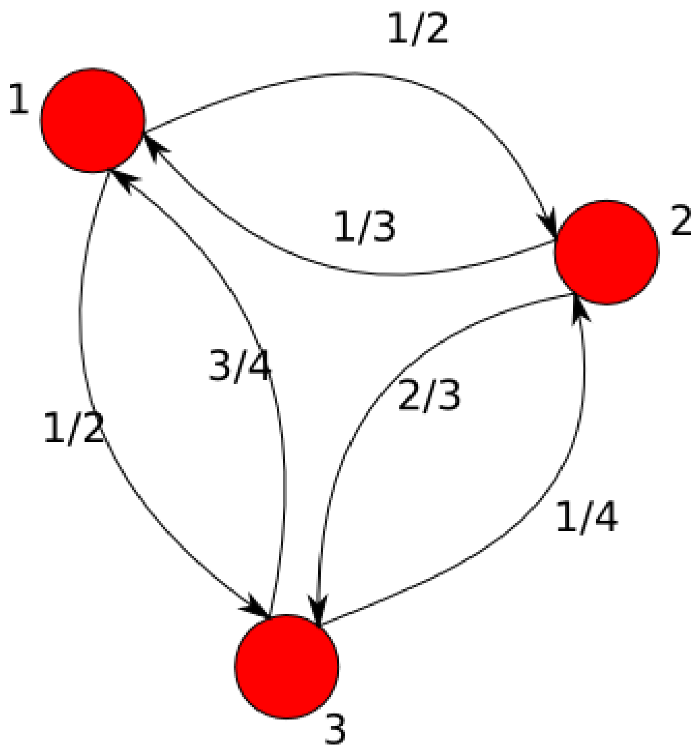

Figure 2.1: Degroot Model (3 agents)

There are two major technical questions that arise for this model. The first one concerns if this process converge to a specific strategy vector, namely does $\lim_{t \to \infty} W^t s$ exist? Of course the process should converge for each possible choice of s, so we are interested on the convergence of W^t. The nodes form strongly connected components such that the agents in a strongly connected component influence each other. A strongly connected component is closed when it has not a directed edge to some other strongly connected component. So it is intuitive to think that the closed strongly connected components should converge to some value by their own and the other sets will assimilate their opinions. It is clear that a strongly connected component should not be periodic in order to converge to a specific value, which is a standard result from Markov Chain Theory. A matrix W is aperiodic if the greatest common divisor of all the directed cycle lengths is one, where the directed cycles are defined relative to a directed network where a directed link exists from i to j if $w_{ij} > 0$. We conclude that,

Theorem 2.1.1 *W is convergent if and only if every set of nodes that is strongly connected and closed is aperiodic.*

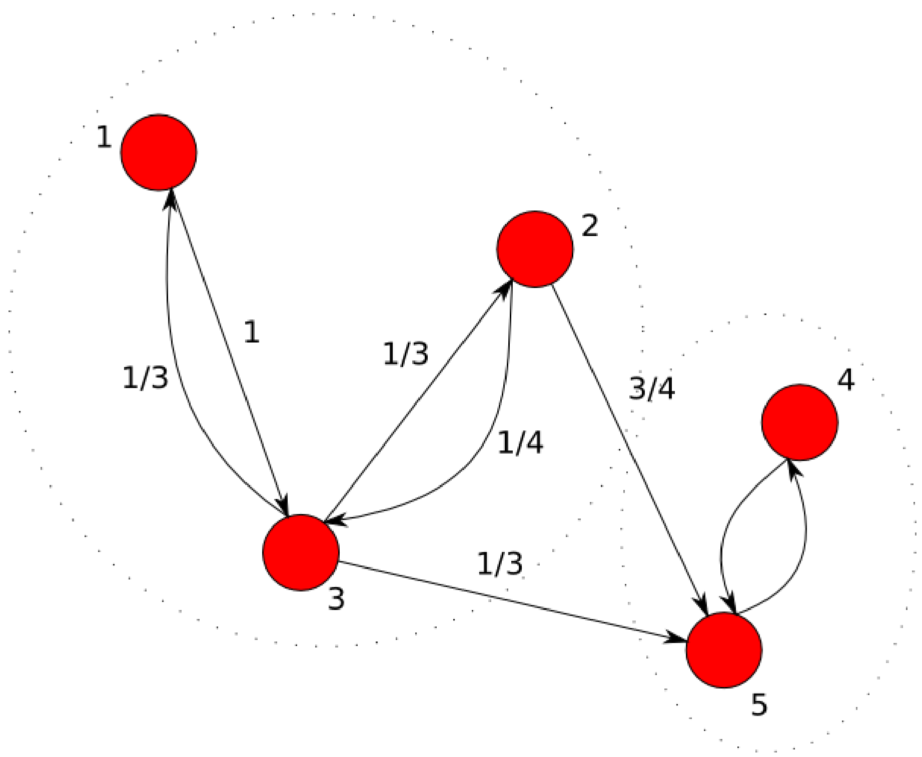

Figure 2.2: Strongly Connected Components ($\{4,5\}$ is closed)

As stated above this model can be thought as a learning process. It is important for the agents to exchange information and converge to the same opinion. This would be the extracted information from the network and we say that the set of agents have reached a consensus. In one of the following chapters we will see what can be done to extract information from the network, when the agents do not reach a consensus. So the second major technical question concerns if the process reaches a concensus, trying to characterize the well behaved social networks. The following theorem characterizes the convergent value for any strongly connected and closed component of the graph.

Theorem 2.1.2 *Any strongly connected and closed group of individuals reaches a consensus for every initial vector of beliefs if and only if it is aperiodic.*

It is clear from above that each strongly connected component will be independent of s on other nodes. So if there are two closed strongly connected components, for different values of s they cannot possibly extract the same value. I state the theorem

Theorem 2.1.3 *A consensus is reached if and only if there is exactly one strongly connected and closed group of agents and W is aperiodic on that group.*

The following theorem seems a more welcomed characterization of social networks which reach a consensus.

Theorem 2.1.4 *A consensus is reached if and only if there exists t such that some column of W^t has all positive entries.*

Now it is time to proceed with generalizations and variations of the above model.

2.2 Symmetric Coevolution Games

The main model studied by the Algorithmic Game Theory line of research is the one introduced in [4].We assume n agents, with internal beliefs (input) s_i. Each agent's strategy set is the line and his strategy is represented by x_i. The selfish agent tries to reduce his cost

$$c_i(x_i, x_{-i}) = w_i(x_i - s_i)^2 + \sum_{j \neq i} w_{ij}(x_i - x_j)^2$$

Of course in this model they assume quadratic costs. The model can be generalized further by assuming general functions:

$$c_i(x_i, x_{-i}) = w_i g_i(x_i - s_i) + \sum_{j \neq i} w_{ij} f_{ij}(x_i - x_j)$$

It remains to picture the game as a system and set an objective function, which ideally is optimized. I will point out the main one used, which is the social cost:

$$SC(x) = \sum_i w_i(x_i - s_i)^2 + \sum_i \sum_{j \neq i} w_{ij}(x_i - x_j)^2$$

or, more generally

$$SC(x) = \sum_i w_i g_i(x_i - s_i) + \sum_i \sum_{j \neq i} w_{ij} f_{ij}(x_i - x_j)$$

Due to the impossiblity results proved in this line of research, as will be evident afterwards, we assume a symmetric model:

$$w_{ij} = w_{ji}$$

and

$$f_{ij} = f_{ji}$$

The nice property about symmetric coevolution games is the existence of a potential function. The potential function for the general case stated above would be

$$\Phi(x) = \sum_i w_i g_i(x_i - s_i) + \sum_i \sum_{j<i} w_{ij} f_{ij}(x_i - x_j)$$

This is easy to prove:

$$c_i(x) - c_i(x'_i, x_{-i}) = w_i g_i(x_i - s_i) - w_i g_i(x'_i - s_i) + \sum_j w_{ij} f_{ij}(x_i - x_j) - \sum_j w_{ij} f_{ij}(x'_i - x_j)$$

$$= \Phi(x) - \Phi(x'_i, x_{-i})$$

It seems that if the functions $g_i(\cdot)$ and $f_{ij}(\cdot)$ are strictly convex, then the potential function is stricty convex, which means that the Nash Equilibrium is unique. That is the case for quadratic cost functions.

2.3 Assymetric Coevolution Games

There are several models which differentiate concerning the "friendship" relationships and their effect on the agents' cost. Of course the relationships would be illustrated by a graph and weights on it's edges. The only assumptions could be that a friendship is powered up by the proximity on the agents' opinions. So a well formed model would assume agents' costs

$$c_i(x_i, x_{-i}) = w_i(x_i - s_i)^2 + \sum_{ij} q_{ij}(x)(x_i - x_j)^2$$

We assume quadratic costs. One can generalize but the quadratic costs have not been explored yet. The friendship function would be $q_{ij}(x) = F_i(d^i_j, d^i_{-i,-j})$, where $d^i_j = |x_j - s_i|$. We make three assumptions:

1. F_i is a continuous funtion.
2. F_i is decreasing in d^i_j.
3. F_i is increasing in d^i_{-j}.

An agent j with an opinion close to agent's i internal belief, influences him more and agent j influences him more if agent i is far from other agents. The continuity of F_i is assumed for simplicity, in order to classify assymetric coevolution games in a well celebrated class of games.

It seems difficult to find a potential function for the above game and by applying the machinery to test the existence of a potential function it may be impossible. However, the cost function is continuous in x and convex in x_i, since $q_{ij}(x)$ does not depend on x_i. This means that the assymetric coevolution game is a convex game and admits a Nash equilibrium.

2.4 Discrete Strategy Sets

Another direction recently studied is restrict the strategy set. Goldberg et al. [12] introduced a model by restricting the strategy set to $X_i = \{0, 1\}$. They followed the symmetric model with constant weights and quadratic costs, stated above. It seems they assume that each agent's distance from his internal belief has the same weight for all agents, which is important for the results given. Each agent has a cost

$$c_i(x_i, x_{-i}) = (x_i - s_i)^2 + D_i(x)$$

where $D_i(x) = \sum\limits_{j:x_i \neq x_j} w_{ij}$ is the sum of weights between agent i and each agent having the opposite opinion.

It seems that the specific model admits a potential function

$$\Phi(x) = \sum_i (x_i - s_i)^2 + \frac{1}{2} \sum_i D_i(x)$$

where $D(x) = \frac{1}{2} \sum\limits_i D_i(x)$. This occurs since

$$c_i(x) - c_i(x_i') = (x_i - s_i)^2 - (x_i' - s_i)^2 + D_i(x) - D_i(x_i', x_{-i})$$

$$\begin{aligned}\Phi(x) - \Phi(x_i', x_{-i}) = \sum_j (x_j - s_j)^2 + D(x) - \sum_{j \neq i} (x_j - s_j)^2 - (x_i' - s_i)^2 - D(x_i', x_{-i}) = \\ (x_i - s_i)^2 - (x_i' - s_i)^2 + D_i(x) - D_i(x_i', x_{-i}) = \\ c_i(x) - c_i(x_i', x_{-i})\end{aligned}$$

Since there exists a potential function, we can justify the existence of a Nash equilibrium.

The objective function, when studying the model, could be the social cost

$$SC(x) = \sum_i (x_i - s_i)^2 + 2D(x)$$

2.5 Discontinuous Social Network functions

A well studied model with discontinuous social network is the Hegselmann-Krause model. Most literature assumes that there are no internal beliefs. The dynamics are given in the form of each agent's response

$$x_i(t+1) = \frac{\sum\limits_{j:|x_j - x_i| \leq 1} x_j(t)}{|\{j : |x_j - x_i| \leq 1\}|}$$

Each agent forms an opinion as the average of his very close friends. The above formula gives an idea of the order followed on the agents responses. All agents act simultaneously. So the existense of a solution is a convergence concept.

Another model would be the K-NN Model. Each agent's cost is given by

$$c_i(x_i, x_{-i}) = \rho K(x_i - s_i)^2 + \sum_{j \in S(i)} (x_i - x_j)^2$$

where K is a constant and $S(i)$ would be the set of the K closest agents to i, with respect to the distance $|x_j - s_i|$. Each agent forms an opinion as the average of his K closest friends. This model is studied, with respect to it's Nash Dynamics.

Chapter 3

Economic Behavior

In this chapter, I examine the economic behavior of some opinion dynamics models. I will revisit two of the above models, where a solution is guaranteed. The above models are defined in terms of a cost function and the solution is a Nash Equilibrium. The measure of economic behavior will be the well celebrated, in the algorithmic game theory research line, Price of Anarchy. This measure is the ratio between the social cost of the worst Nash equilibrium and the social cost of an optimal solution where the agents' beliefs could be derived from a centralized authority

$$PoA = \sup_{x:N.E.} \frac{SC(x)}{SC(x^*)}$$

The first to examine will be the symmetric coevolution game by using smoothness arguments.

3.1 Symmetric Coevolution Games

This result is important for two reasons. Firstly, it is the model that is heavily studied by the reseach conducted and secondly, it cathces the case of the best-response dynamics, since the N.E. is unique and is identical to the vector the dynamics converge. Before studying the symmetric model, it would be ideal to state an economic result for the assymetric case (directed graph) with quadratic costs.

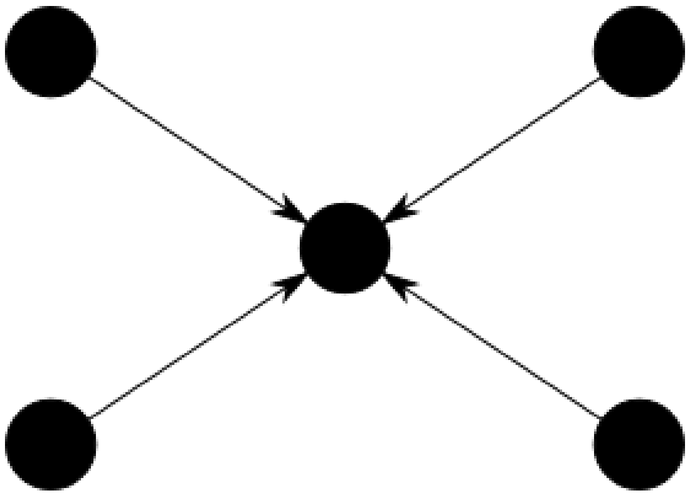

Figure 3.1: Star Network

Example Consider a star with n agents, where agent 1 is the center of the star and $w_{i1} = 1$. Agent 1 has internal belief $s_1 = 1$ and all other agents have internal belief $s_i = 0$. At the Nash equilibrium agent 1 would choose his internal belief $x_1 = s_1 = 1$ and all other agents would choose $x_i = \frac{1}{2}$. In the optimal profile x^* (for large n), agent 1 changes his opinion to 0 and all other agents declare their internal belief. So, the example has a ratio

$$\frac{SC(x)}{SC(x^*)} = \frac{\frac{n-1}{4} + \frac{n-1}{4}}{1} = \Omega(n)$$

As the ratio is lower bounded by n, one could try to find a special graph class with better economic efficiency. It seems that even for graphs with constant degree the ratio of economic efficiency remains high.

Example Let G be a complete 2^k-ary tree with n nodes and each edge pointing to a layer above has weight 1. The tree's depth is $\log_{2^k} n$. The root has internal belief 1 and all other agents have internal belief 0. At a Nash equilibrium the agents at layer i declare opinion 2^{-i}. Each agent's cost in layer i would be $(2^{-i})^2 + (2^{-(i-1)} - 2^{-i})^2 = 2^{-2i}$. There are 2^{ki} agents in layer i. So, the total social cost at the Nash equilibrium is

$$SC(x) = 2 \sum_{i=1}^{\log_{2^k} n} 2^{(k-2)i}$$

For $k > 2$ this cost is equal to

$$SC(x) = 2 \cdot 2^{k-2} \frac{(2^{k-2})^{\log_{2^k} n} - 1}{2^{k-2} - 1} = 2^{k-1} \frac{n^{\frac{k-2}{k}} - 1}{2^{k-2} - 1}$$

The cost of the optimal solution is at most 1 since the root agent can declare opinion 0. It seems that the PoA gets higher as one increases k. The PoA for $k = 2$ would be $\Theta(\log n)$.

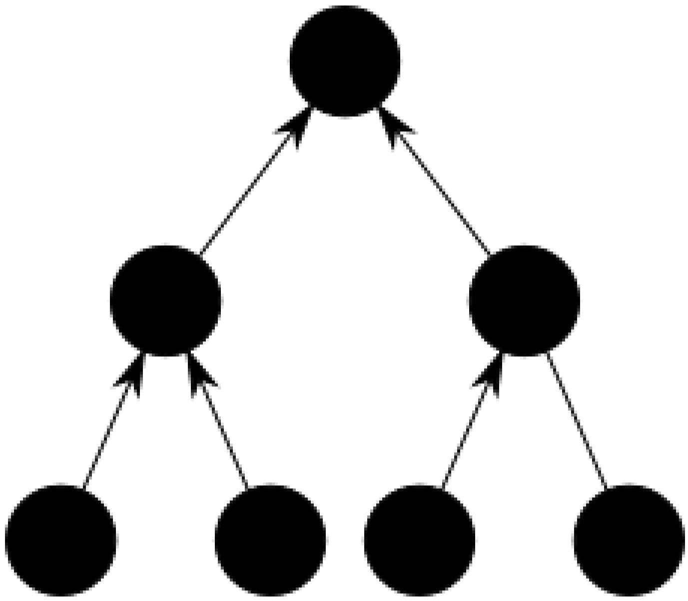

Figure 3.2: Tree Network

Now that it seems difficult to bound the PoA in the assymetric model, it would be straightforward to consider the symmetric case. In the symmetric model with quadratic costs, the social cost is given by the following form:

$$SC(x) = \sum_i (x_i - s_i)^2 + 2 \sum_{(i,j) \in E, i > j} w_{ij} (x_i - x_j)^2$$

The potential function is given by the following form:

$$\phi(x) = \sum_i (x_i - s_i)^2 + \sum_{(i,j) \in E, i > j} w_{ij} (x_i - x_j)^2$$

Let x denote the Nash Equilibrium and x^* the social cost minimizer. By using the potential function (it's form resembles the social cost), one can derive a PoA bound:

$$PoA = \frac{c(x)}{c(x^*)} = \frac{\sum_i (x_i - s_i)^2 + 2 \sum_{(i,j) \in E, i > j} w_{ij} (x_i - x_j)^2}{c(x^*)} \leq$$

$$\frac{2\sum_i (x_i - s_i)^2 + 2 \sum_{(i,j)\in E, i>j} w_{ij}(x_i - x_j)^2}{c(x^*)} = \frac{2\phi(x)}{c(x)} \leq \frac{2\phi(x^*)}{c(x^*)} \leq$$

$$\frac{2c(x^*)}{c(x^*)} = 2$$

However, the same job can be done by using the local smoothness framework. Let x denote possible profile and x^* a fixed profile (I use the optimal sign because in order to prove any results x^* is chosen to be the social cost minimizer). A game is local smooth if for a fixed profile x^*, there are values $\mu < 1$, $\lambda > 0$, such that for every x:

$$\sum_i c_i(x_i, x_{-i}) + (x_i^* - x_i)\frac{\partial c_i(x_i, x_{-i})}{\partial x_i} \leq \lambda c(x^*) + \mu c(x)$$

Theorem 3.1.1 *Let σ denote a correlated equilibrium. If a game is local smooth (according to the above definition), with respect to the social cost minimizer x^* as the fixed profile, the ratio of $E_{x\sim\sigma}[c(x)]$ to $c(x^*)$ is at most $\frac{\lambda}{1-\mu}$. In particular, the correlated PoA is at most $\frac{\lambda}{1-\mu}$.*

Let's assume a more general model for each agent's cost:

$$c_i(x_i, x_{-i}) = \sum_{j\neq i} f_{ij}(x_i - x_j) + w_i g_i(x_i - s_i)$$

Let's try to prove that the above game is smooth.

$$\sum_i c_i(x_i, x_{-i}) + (x_i^* - x_i)\frac{\partial c_i(x_i, x_{-i})}{\partial x_i} =$$

$$\sum_i [\sum_{j\neq i} f_{ij}(x_i - x_j) + w_i g_i(x_i - s_i)] +$$

$$(x_i^* - x_i)[\sum_{j\neq i} f'_{ij}(x_i - x_j) + w_i g'_i(x_i - s_i)] =$$

$$\sum_{i\neq j} [2f_{ij}(x_i - x_j) + (x_i^* - x_i)f'_{ij}(x_i - x_j) + (x_j^* - x_j)f'_{ij}(x_j - x_i)] +$$

$$\sum_i w_i [g_i(x_i - s_i) + (x^* - x_i)g'_i(x_i - s_i)] =$$

$$\sum_{i \neq j} 2[f_{ij}(x_i - x_j) + \frac{1}{2}[(x_i^* - x_j^*) - (x_i - x_j)]f'_{ij}(x_i - x_j)] +$$
$$\sum_i w_i[g_i(x_i - s_i) + ((x_i^* - s_i) - (x_i - s_i))g'_i(x_i - s_i)]$$

In order to prove the game is smooth, one must bound the right part of the above equation by using the following terms:

- $SC(x^*) = \sum_i w_i g_i(x_i - s_i) + \sum_{i \neq j} 2 f_{ij}(x_i^* - x_j^*)$
- $SC(x) = \sum_i w_i g_i(x_i - s_i) + \sum_{i \neq j} 2 f_{ij}(x_i - x_j)$

So one can desire the best possible (λ, μ), in order to minimize $\frac{\lambda}{1-\mu}$, such that:

- $g_i(x_i - s_i) + ((x_i^* - s_i) - (x_i - s_i))g'_i(x_i - s_i) \leq \lambda g_i(x_i^* - s_i) + \mu g_i(x_i - s_i)$
- $f_{ij}(x_i - x_j) + \frac{1}{2}[(x_i^* - x_j^*) - (x_i - x_j)]f'_{ij}(x_i - x_j) \leq \lambda f_{ij}(x_i^* - x_j^*) + \mu f_{ij}(x_i - x_j)$

Convexity of the functions $g_i(\cdot)$ and $f_{ij}(\cdot)$ implies that:

- $g_i(x_i - s_i) + ((x_i^* - s_i) - (x_i - s_i))g'_i(x_i - s_i) \leq g_i(x_i^* - s_i)$
- $f_{ij}(x_i - x_j) + [(x_i^* - x_j^*) - (x_i - x_j)]f'_{ij}(x_i - x_j) \leq f_{ij}(x_i^* - x_j^*) \Rightarrow f_{ij}(x_i - x_j) + \frac{1}{2}[(x_i^* - x_j^*) - (x_i - x_j)]f'_{ij}(x_i - x_j) \leq \frac{1}{2} f_{ij}(x_i - x_j) + \frac{1}{2} f_{ij}(x_i^* - x_j^*)$

So it seems that the game is $(1, \frac{1}{2})$-smooth, since $(\lambda, \mu) = (1, \frac{1}{2})$ satisfies the smoothness inequality. The PoA is equal to $\frac{\lambda}{1-\mu} = 2$. When using more restricted cost functions the PoA could be lower. I will analyze the case where the cost function is a polynomial, i.e. $g_i(d) = |d|^a$ and $f_{ij}(d) = |d|^a$.

Again, we shall find (λ, μ) such that

$$(x_i - x_j)^a + \frac{1}{2}[(x_i^* - x_j^*) - (x_i - x_j)]a(x_i - x_j)^{a-1} \leq \lambda(x_i^* - x_j^*)^a + \mu(x_i - x_j)^a$$

Suppose that $x = x_i - x_j$ and $y = x_i^* - x_j^*$. So, we need

$$x^a + \frac{1}{2}(y - x)ax^{a-1} \leq \lambda y^a + \mu x^a$$

The same should be proven for functions $g_i(d) = |d|^a$, i.e.

$$(x_i - s_i)^a + ((x_i^* - s_i) - (x_i - s_i))a(x_i - s_i)^{a-1} \leq \lambda(x_i^* - s_i)^a + \mu(x_i - s_i)^a$$

Suppose that $x = x_i - s_i$ and $y = x_i^* - s_i$, so the above inequality is transformed to

$$x^a + (y - x)ax^{a-1} \leq \lambda x^a + \mu y^a$$

The inequalities above are true for

$$(\lambda, \mu) = ((\frac{a}{a-1})^{-(a-1)} \cdot \frac{(2^{\frac{a}{a-1}} - 1)^{a-1}}{2}, 1 - \frac{a}{2} + \frac{a}{2}(\frac{1}{2^{\frac{a}{a-1}} - 1}))$$

So, we can a derive equal to

$$PoA = \frac{(a-1)^{(a-1)}}{a^a} \cdot \frac{(2^{\frac{a}{a-1}} - 1)^a}{2^{\frac{a}{a-1}} - 2}$$

The following table presents the PoA for sereral degrees on the polynomial of the cost function

α	POA
1.001	1.986
1.008	1.919
1.062	1.647
1.5	1.188
2	9/8 =1.125
4	1.083
8	1.071
16	1.066
32	1.064

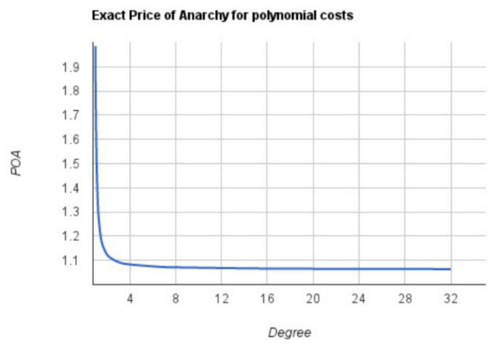

Figure 3.3: PoA bounds for polynomial cost functions

In order to end the discussion for polynomial cost function, it seems that one can construct a lower bound, for each choice of a, such that the PoA coincides with the value stated and proven above. I will present a generalized instance with 3 agents.

Example There are 3 agents with internal beliefs $s_1 = 0$, $s_2 = 1$, $s_3 = 2$. Each agent's cost is given by the formula

$$c_i(x) = wg(x_i - s_i) + \sum_{j \neq i} f(x_i - x_j)$$

For polynomial cost functions, each agent's cost function is given by

$$c_i(x) = w|x_i - s_i|^a + \sum_{j \neq i} |x_i - x_j|^a$$

I assume arbitrary values w_1, w_2, satisfying the constraint $(1 - w_1)(1 - w_2) \leq 0$. Of course the ordering between the agents in the example will remain the same. I will prove that, for some w, the Nash equilibrium of the above example is the following:

$$(x_1, x_2, x_3) = (\frac{(1 - w_1)w_2}{w_2 - w_1}, 1, 2 - \frac{(1 - w_1)w_2}{w_2 - w_1})$$

I will combine it with an arbitrary solution of the above example

$$(x_1^*, x_2^*, x_3^*) = (\frac{(1 - w_1)}{w_2 - w_1}, 1, 2 - \frac{(1 - w_1)}{w_2 - w_1})$$

with cost at least as much as the optimal. It is trivial to prove that the above value of x is a Nash Equilibrium for some x. Since $w_{ij} = 1$, the game is totally symmetric, meaning that $x_2 = 1$. So, imagine to reduce the value w. Agent's 1 equilibrium starts moving to the right and agent's 3 equilibrium starts moving to the left, with the same velocity. For some w the equilibrium reaches the above value, since $x_1 = 2 - x_3$. So, one can choose that w for our lower bound example. It is easy to check that for the value of (λ, μ) chosen above, polynomial cost functions the smoothness inequalities are tight, i.e.

$$g(x_i - s_i) + ((x_i^* - s_i) - (x_i - s_i))g'(x_i - s_i) = \lambda g(x_i^* - s_i) + \mu g(x_i - s_i)$$

$$f(x_i - x_j) + \frac{1}{2}((x_i^* - x_j^*) - (x_i - x_j))f'(x_i - x_j) = \lambda f(x_i^* - x_j^*) + \mu f(x_i - x_j)$$

Summing up one can conclude for the above example that

$$\frac{SC(x)}{SC(x^*)} = \frac{\lambda}{1 - \mu}$$

We have discussed the symmetric coevolution game, when the cost function is continuous. I presented a nice proof for the case when the cost function is convex, proving that the Price of Anarchy is constant. By deepening in the case of polynomial cost functions and producing a closed-form computation of the PoA, depending on the degree of the polynomials, one can see that as the degree of the polynomial rises the PoA gets closer to 1. The cost function can be written as

$$f_{ij}(x_i - x_j) = (x_i - x_j)^a = (x_i - x_j)^{a-2} \cdot (x_i - x_j)^2$$

This means that as the degree rises, agents care more about the agents' opinions which may be far. So, the social outcome is better if agents are more concerned about "different" opinions.

3.2 Discrete Strategy Sets

I will analyze the economic behavior of the model introduced in [12]. Unfortunately, the local smoothness framework cannot be used since the cost function is not continuous.

The case in this model is disheartening. The following example points out that that the PoA is unbounded.

Example There are n agents with internal beliefs $s_i = 0$, and a clique with edge weights $w_{ij} = 1$. The optimal profile is the one where $x_i^* = 0$, which admits $SC(x^*) = 0$. However, the profile $x_i = 1$ is a Nash Equilibrium, which admits $SC(x) = n$.

It seems that nothing can be done. However, in the above example it seems that the optimal outcome is also a Nash Equilibrium. So, it would be interesting to study the Price of Stability, where the price of stability is defined as the ratio between the social cost of the best Nash equilibrium and the social cost of the optimal solution

$$PoS = \frac{SC(x_{best})}{SC(x^*)}$$

We proceed immediately with the price of stability result

Theorem 3.2.1 *The Price of Stability is 2.*

Proof The proof for an upper bound for the price of anarchy is straightforward by using the potential function. The best Nash equilibrium x has lower social cost than the Nash equilibrium x' which minimizes the potential function. So,

$$SC(x) \leq SC(x') = \Phi(x') + D(x') \leq 2\Phi(x') \leq 2\Phi(x^*) \leq 2SC(x^*)$$

In order to prove that $PoS = 2$ it remains to construct a lower bound example. Consider a star (agent 1 is the center of the star) of $n+1$ agents, where $s_1 = s_2 = 1$ and each other agent i has internal belief $s_i = 0$. The star's edges have weight $w = \frac{1}{n}$. The optimal outcome would be x^*, where $x_1^* = 1 - s_1 = 0$ and, for $i \neq 1$, $x_i^* = s_i$. The optimal outcome admits social cost

$$SC(x^*) = 1 + \frac{2}{n}$$

The example has a unique Nash equilibrium x. Each agent declares his belief. It is clear for the agents but the center to prefer declaring their belief, since their cost would surely be lower than 1. But then, the center, prefers also declaring his belief since (not yet written). The social cost of the Nash equilibrium is

$$SC(x) = \frac{n-1}{n} + (n-1)\frac{1}{n} = 2\frac{n-1}{n}$$

The lower bound is $\frac{2\frac{n-1}{n}}{\frac{n+2}{n}} = 2\frac{n-1}{n+2}$ which approaches 2 as n approaches infinity.

Chapter 4

Convergence Results

There are two types of convergence results. Firstly, we are interested whether a specific process converges. It may be the case that when we write down the model in terms of a cost function, that there are some stable points (N.E.) but the system will not ever converge. It is important to guarantee that best-response dynamics converge for every possible initial beliefs. Secondly, if convergence is guaranteed the next step would be to derive the convergence time of the best-response dynamics, and ideally in terms of the social structure.

4.1 Symmetric Coevolution Games - Simultaneous Best-Response Dynamics

I will study the symmetric coevolution game with a more refined model. The cost function will be given by the following formula

$$c_i(x) = w_i(x_i - s_i)^2 + \sum_{j \in N(i)} (x_i - x_j)^2$$

Of course, we cannot study the convergence rate of the Nash Dynamics. So we study simultaneous response dynamics, where each agent's response is given by the following formula

$$x_i(t+1) = \frac{w_i}{w_i + |N(i)|} s_i + \frac{1}{w_i + |N(i)|} \sum_{j \in N(i)} x_j(t)$$

We can describe the above equations with the vectors $x = (x_i)$, $s = (s_i)$ and the matrices A and B, where $A_{ij} = \frac{1}{w_i + |N(i)|}$ when $j \in N(i)$ and 0 otherwise, and $B_{ii} = \frac{w_i}{w_i + |N(i)|}$ and 0 otherwise. So

$$x(t+1) = Ax(t) + Bs$$

The above equation implies the following

$$x(t) = A^t s + \sum_{s=0}^{t-1} A^s Bs$$

4.1.1 Convergence

In order to prove convergence, I will make a further assumption about the stubborness of the agents. I suppose that $w_i = 0$ for each agent i. The model without stubborn players resemble the DeGroot model [10]. This means that $B = \mathbb{O}$ and $x(t+1) = A^t s$.

The above matrix is nonnegative. I will assume that the matrix is primitive which means that there is a constant t such that all elements of A^t are strictly positive. This is equivalent with assuming that the social network is not bipartite. I will examine this case since the bipartite graph case is easy to characterise.

Since matrix A is nonnegative and primitive, by the Perron-Frobenius theorem the largest eigenvalue is real and nonnegative, i.e. $\rho(A) \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$. Since, A is stohastic, by the Perron-Frobenius theorem

$$1 = \min_i \sum_j a_{ij} \leq \rho(A) \leq \max_i \sum_j a_{ij} = 1$$

So, $\rho(A) = 1$. Since A is stochastic, the eigenvector corresponding to the $\rho(A)$ is the unit vector:

$$A\mathbf{1}_n = \mathbf{1}_n$$

Let's consider the eigenvector π of $\rho(A)$. Then A^t converges to $\mathbf{1}\pi^T$, by the above theorem. Since A is stochastic, A^t remains stochastic. It is easy to check that the left eigenvector of A is

$$\pi_i = \frac{N(i)}{2|E|}$$

since it satisfies the equation

$$(\pi^T A)_i = \sum_{j \in N(i)} \frac{N(j)}{2|E|} \cdot \frac{1}{N(j)} = \frac{N(i)}{2|E|} = \pi_i$$

This is a standard result of Markov chains theory, where A can be interpreted as transition probability matrix of a random walk and π is the stationary distribution.

We know that $\lim_{t \to \infty} A^t = \mathbf{1}\pi^T$. This means that $x = \mathbf{1}\pi^T s$. So, the best-response dynamics will converge to the unique equilibrium (concave potential function, as stated above)

$$x_i = \sum_{j=1}^{n} \pi_j s_j = \frac{1}{2|E|} \sum_{j=1}^{n} |N(j)| s_j$$

4.1.2 Convergence Time

I will prove a result for the convergence time with the assumption of non-stubborn agents. I define an error at step t

$$e(t) = x(t) - x_i$$

In order to bound the error at step t, I will introduce a norm, specially constructed for each graph, given by the following inner product, with respect to the vector π stated above

$$(z \cdot y)_\pi = \sum_{i=1}^{n} z_i y_i \pi_i$$

$$\|z\|_\pi = \sqrt{\sum_{i=1}^{n} z_i^2 \pi_i}$$

The best-response dynamics reduce the error significantly, i.e.

$$\|e(t)\|_\pi \leq \rho_2^t \|e(0)\|_\pi$$

where $\rho_2 = \max_{i \neq 1} |\lambda_i|$ is the second largest eigenvalue of A.

Proof The error can be computed recursively

$$\begin{gathered}
e(t+1) = x(t+1) - x_i = Ax(t) - x_i = Ax(t) - \mathbf{1}\pi^T s = \\
Ax(t) - \mathbf{1}\pi^T s - \mathbf{1}\pi^T x(t) + \mathbf{1}\pi^T x(t) = \\
Ax(t) - A\mathbf{1}\pi^T s - \mathbf{1}\pi^T x(t) + \mathbf{1}\pi^T \mathbf{1}\pi^T s = \\
A(x(t) - \mathbf{1}\pi^T s) - \mathbf{1}\pi^T (x(t) - \mathbf{1}\pi^T x) = \\
(A - \mathbf{1}\pi^T)(x(t) - \mathbf{1}\pi^T s) = \\
(A - \mathbf{1}\pi^T)e(t)
\end{gathered}$$

Let λ_i be A's eigenvalues and v_i the corresponding eigenvectors. So we get the following equations

- $(A - \mathbf{1}\pi^T)\mathbf{1} = \mathbf{1} - \mathbf{1} = 0$
- For $i \geq 2$, $(A - \mathbf{1}\pi^T)v_i = \lambda_i v_i - \mathbf{1}\pi^T v_i = \lambda_i v_i$

Furthermore the following are true for the error

- $e(t) = \sum_{i=2}^{n} \lambda_i (e(t) \cdot v_i)_\pi v_i$
- $(A - \mathbf{1}\pi^T)e(t) = \sum_{i=2}^{n} \lambda_i (e(t) \cdot v_i)_\pi v_i$

So the error's norm can be extracted

$$\begin{gathered}
\|e(t+1)\|_\pi^2 = \sum_{i=2}^{n} \lambda_i^2 (e(t) \cdot v_i)_\pi^2 \|v_i\|_\pi^2 = \\
\sum_{i=2}^{n} \lambda_i^2 (e(t) \cdot v_i)_\pi^2 \leq \\
\rho_2^2 \sum_{i=2}^{n} (e(t) \cdot v_i)_\pi^2 = \\
\rho_2^2 \|e(t)\|_\pi^2
\end{gathered}$$

This means that $\|e(t+1)\|_\pi \leq \rho_2 \|e(t)\|_\pi \leq \rho_2^{t+1} \|e(0)\|$.

The convergence time is defined as the time needed to reduce the error under a value v:

$$\tau(v) = \inf\{t \geq 0 : \|e(t)\|_\pi \leq v\}$$

By using the theorem proven we get that

$$(\frac{1}{1-\rho_2} - 1)\log(\frac{\|e(0)\|_\pi}{v}) \leq \tau(v) \leq \frac{1}{1-\rho_2}\log(\frac{\|e(0)\|_\pi}{v})$$

which means that the convergence time for each graph is $\Theta(\frac{1}{1-\rho_2})$.

4.1.3 Stubborn Agents

When agents are stubborn is a bit more difficult to analyze and it can be read through in [14]. The matrix A is irreducible sub-stohastic and some row has a sum less than 1. This means that the spectral radius is less than 1 and A^t converges to $\mathbb{O}$ as t converges to infinity. Moreover, since A^t converges to zero one can derive that the sum $\sum_{\sigma=0}^{\infty} A^\sigma$ converges to $(\mathbb{I} - A)^{-1}$. So the agents' opinions converge to

$$x = \sum_{\sigma=0}^{\infty} A^\sigma Bs = (\mathbb{I} - A)^{-1} Bs$$

Using an analogy with random walks in graphs, as opinions travel through the graph, one can prove that agents' opinions converge to a convex combination of their initial values. With a similar analysis as before it is proven that the convergence time can be bounded by

$$(\frac{1}{1-\lambda_A} - 1)\log(\frac{\|e(0)\|_\pi}{v}) \leq \tau(v) \leq \frac{1}{1-\lambda_A}\log(\frac{\|e(0)\|_\pi}{v})$$

where λ_A now is the largest eigenvalue of A. So, as above, the convergence time for each graph is $\Theta(\frac{1}{1-\lambda_A})$.

I would like to highlight the importance of these results, since they strengthen the notion of the Nash equilibrium when the network is undirected. When considering undirected networks it holds that the Nash equilibrium is unique, so the best response dynamics converge to the unique equilibrium. This means that studying the economic behavior at that equilibrium is actually the case we know the agents will converge to. It would be ideal to understand the economic behavior on the point that the system converges when the network is directed, since the results stated in the previous chapter for directed networks hold for the worst stable point. Unfortunately, there are cases, a subset of all bipartite graphs that the system does not converge. A simple example would be for the DeGroot model to have the following matrix which is aperiodic and hence non primitive

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

4.2 Discrete Strategy Sets - Best-Response Dynamics

In this section I will discuss the best-response dynamics of the model introduced in [12]. One cannot know the ordering that the agents respond, so we may assume the worst ordering. The existence of a potential function is important to prove convergence of the dynamics and bound the steps needed to converge. Of course, the dynamics converge, since there are 2^n profiles and each time an agent responds the potential function decreases and a new profile occurs. There cannot be a cycle on the profiles, since a potential function exists. I will demonstrate convergence time results by assuming unitary weights, i.e. $w_{ij} = 1$. Same results can be proven for general weights.

Let's prove a convergence time result in the class of opinion games $\mathcal{G}$ where the internal beliefs set is $S_i = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$.

Theorem 4.2.1 *The best-response dynamics of a game in $\mathcal{G}$ converge after a polynomial number of steps.*

Proof In the proof, I will use the potential function $\Phi(\cdot)$. It is clear that, $\sum_i (x_i - s_i)^2 \leq n$ and $D_i \leq n$. So $0 \leq \Phi(x) \leq n + \frac{1}{2}n^2$. It remains to show that at each step (an agents responds) the potential function decreases by a constant. Let's assume that an agent changes his profile from x_i to x_i', at a best-response move.

$$\Phi(x) - \Phi(x_i', x_{-i}) = (x_i - s_i)^2 - (x_i' - s_i)^2 + D_i(x) - D_i(x_i', x_{-i})$$

Of course $D_i(x) - D_i(x_i', x_{-i}) > -1$, since $c_i(x) > c_i(x_i', x_{-i})$

- If $D_i(x) - D_i(x'_i, x_{-i}) \geq 2$ then $\Phi(x) - \Phi(x'_i, x_{-i}) \geq 1$, since $(x_i - s_i)^2 - (x'_i - s_i)^2 \geq -1$

- Assume $D_i(x) - D_i(x'_i, x_{-i}) = \{0, 1\}$. If $x_i = 0$, since the cost decreases $s_i \geq \frac{1 - D_i(x) + D_i(x'_i, x_{-i}) + \frac{1}{2}}{2}$, and it follows that $\Phi(x) - \Phi(x'_i, x_{-i}) \geq \frac{1}{2}$.

So,

$$\Phi(x) - \Phi(x'_i, x_{-i}) \geq \frac{1}{2}$$

and the best-response dynamics converge in polynomial number of steps.

4.3 Hegselman-Krause Model

In this section, I will prove an upper bound on the convergence time of the HK model. I will discuss about the simultaneous best-response dynamics, where each agent's best response is given by

$$x_i(t+1) = \frac{\sum\limits_{j:|x_j - x_i| \leq 1} x_j(t)}{|\{j : |x_j - x_i| \leq 1\}|}$$

Before proving the convergence time result, it would be useful to state the order preserving property of the dynamics, i.e. if $s_i \leq s_j$ then $x_i(t) \leq x_j(t)$ for all t. Suppose that $x_i(t) \leq x_j(t)$ and denote $N_i(t)$ the agents connected to i and not to j, $N_{ij}(t)$ the agents connected to both i and j and $N_j(t)$ the agents connected to j and not to i. For any $k_1 \in N_i(t)$, $k_2 \in N_{ij}(t)$ and $k_3 \in N_j(t)$, $x_{k_1}(t) \leq x_{k_2}(t) \leq x_{k_3}(t)$. This means that $\frac{\sum\limits_{k_1 \in N_i(t)} x_{k_1}(t)}{|N_i(t)|} \leq \frac{\sum\limits_{k_2 \in N_{ij}(t)} x_{k_2}(t)}{N_{ij}(t)} \leq \frac{\sum\limits_{k_3 \in N_j(t)} x_{k_3}(t)}{N_j(t)}$. So,

$$x_i(t+1) = \frac{|N_i(t)| \sum\limits_{k_1 \in N_i(t)} x_{k_1}(t) + |N_{ij}(t)| \sum\limits_{k_2 \in N_{ij}(t)} x_{k_2}(t)}{|N_i(t)| + |N_{ij}(t)|} \leq \frac{\sum\limits_{k_2 \in N_{ij}(t)} x_{k_2}(t)}{N_{ij}(t)}$$

$$\leq \frac{|N_j(t)| \sum\limits_{k_3 \in N_j(t)} x_{k_3}(t) + |N_{ij}(t)| \sum\limits_{k_2 \in N_{ij}(t)} x_{k_2}(t)}{|N_j(t)| + |N_{ij}(t)|} = x_j(t+1)$$

Now I can prove an upper bound for the convergence time

Theorem 4.3.1 *The simultaneous best-response dynamics of the HK model converge within $O(n^3)$ steps.*

Proof Consider the leftmost non-frozen agent $l(t)$, at time step t. All the agents left from $l(t)$ have frozen meaning, that they do not have neighbors with different opinions. For every t, at time $t+2$, agent $l(t)$ "meets" another agent on his right, or freezes or moves to the right by at least $\frac{1}{2n^2}$.

Assume that $l'(t)$ is the leftmost agent in agent's $l(t)$ neighborhood. If $N_{l(t)}(t) = N_{l'(t)}(t)$ then the two agents meet. If not there must exist an agent $r(t)$ on the right in $N_{l'(t)}(t) \setminus N_{l(t)}(t)$. This means that $x_{r(t)} - x_{l(t)} > 1$ and agent $l'(t)$ moves to the left by at most

$$\frac{\delta \cdot (1-k) - (1-\delta) \cdot 1}{k} = \delta - \frac{1}{k} \leq \delta - \frac{1}{n}$$

where $\delta = x_{l'(t)}(t) - x_{l(t)}(t)$ and $k = |N_{l'(t)}(t)|$. If $x_{l(t)}(t+1) \geq x_{l(t)}(t) + \frac{1}{2n}$, then $l(t)$ has moved at least $\frac{1}{n^2}$. Otherwise, $x_{l'(t)}(t+1) - x_{l(t)}(t+1) \geq \frac{1}{2n}$. If $x_{l'(t)}(t+1) - x_{l(t)}(t+1) \geq 1$ then $l(t)$ freezes, otherwise, $\frac{1}{2n} \leq x_{l'(t)}(t+1) - x_{l(t)}(t+1) \leq 1$. Then $l(t)$ moves by $x_{l(t)}(t+2) - x_{l(t)}(t+1) \geq \frac{\frac{1}{2n}}{|N_{l(t)}(t+1)|} \geq \frac{1}{2n^2}$.

One can assume that $s_n - s_1 \leq n$, because otherwise two consecutive agents must have distance at least 1. Then, we could assume that the game is seperated to two other games, which evolve independently. An agent $l(t)$ can meet other agents at most n times and agents, being the leftmost non-frozen, can only happen to move to the right $2n^3$ times. So, after $2(n+2n^3)$ steps the above statement cannot apply.

Chapter 5

Social Choice Rules

In this chapter I will introduce a new model. Before doing so I will present a significant result of social choice rules. The following results will be presented in a general framework, where the strategy sets is a common one-dimensional strategy set X. This case may include choosing an alternative from an ordered set of alternatives or choosing a point of an interval in $\mathcal{R}$. The most standard strategy set is the interval $[0, 1]$. Generalizing the notion of a utility function, in order to model each agent's i choice behavior we use a preference relation $\succcurlyeq_i$ between alternatives. A preference relation is single-peaked if there exists a point $p_i \in X$, such that for all $x \in X \setminus \{p_i\}$ and $\lambda \in [0, 1)$, $(\lambda x + (1-\lambda)p_i \succ_i x$. The vector of the agents' preference relations is $\succcurlyeq \in \mathcal{R}$. A social choice rule is a function $f : \mathcal{R} \to X$, which produces an outcome from the agents' preference relations.

A social choice rule is strategy-proof if every agent has a dominant strategy to declare his preferences truthfully. A social choice rule is onto if for each $x \in X$, there exists a preference relations vector $\succcurlyeq$, such that $f(\succcurlyeq) = x$. A social choice rule is unanimous if $p_1 = \ldots = p_n = p$ implies $f(\succcurlyeq) = p$. A social choice rule is Pareto-optimal, there exists no $x \in X$ and $x \neq f(\succcurlyeq)$, such that $x \succcurlyeq_i f(\succcurlyeq)$ for each agent i. A social choice rule is anonymous if for each $\succcurlyeq$ and $\succcurlyeq'$, which results from a permuation on $\succcurlyeq$, $f(\succcurlyeq) = f(\succcurlyeq')$. This means that the rule is irrelevant of the agents' labels. The above properties should be satisfied for natural social choice rules.

Theorem 5.0.2 *A rule f is strategy-proof, onto and anonymous if and only if* *there exist* $x_1, \ldots, x_{n-1} \in X$ *such that for all* $\succcurlyeq \in \mathcal{R}$,

$$f(\succcurlyeq) = med\{p_1, \ldots, p_n, x_1, \ldots, x_{n-1}\}$$

The above theorem is a full characterization of all strategy-proof, onto and anonymous mechanisms for the above problem. It seems that the median is the right choice when dealing with social choice problems. This theorem applies even in similar settings to opinion dynamics, where the strategy set is the real line and the preference relations are expressed by cost functions depending on a distance. The cost function defines a single-peaked preference relation, which is completed for pairs which belong to other sides of the peak. Since the preference relations are defined by a cost function, we have to deal with rational, single-peaked preference relations. When dealing with rational, single-peaked relations and need a strategy-proof rule, the median scheme is the first option. When dealing with such preference relations, we should only examine if the median scheme remains strategy-proof.

Consider now the opinion dynamics framework, where the agents have an extra incentive to influence the global outcome and bring it near their internal belief. We would like to design a mechanism which defines the global outcome, as if it was the outcome of political elections.

5.1 Social Choice Rules with the Presence of a Social Network

It seems that searching for strategy-proof social choice rules is a significant restriction. We may consider the solution concept of the Nash equilibrium to deal with this problem. The Nash equilibrium concept may be problematic and not realistic since there is a convergence issue. I will present a simple example on opinion dynamics, where a social choice rule is applied in their opinions and an outcome is produced. This is the main purpose of this chapter. As seen before a nice property of opinion games is reaching a consensus. It seems that this do not happen for the models stated this far. So a social choice rule could be applied to gather the information of the social network. For simplicity, I will assume that there does not exist a social network yet.

Example Consider the standard opinion dynamics setting with n agents and no social network ($w_{ij} = 0$). Each agent i has an internal belief s_i and his cost function is given by

$$c_i(x) = (f(x) - s_i)^2 + (x_i - s_i)^2$$

where $f : \mathcal{R}^n \to \mathcal{R}$ is the social choice rule. It is trivial to infer that when the social choice rule is the median, each agent's i dominant strategy is $x_i = s_i$. The median scheme is strategy-proof. However, I will examine the average scheme $f(x) = av\{x_1, \dots, x_n\}$. It is trivial to infer that the game admits the following potential function

$$\Phi(x) = (av(\{x_1, \dots, x_n\})^2 - \sum_i \frac{s_i x_i}{n} + \sum_i (x_i - s_i)^2$$

The potential function is strictly concave which means that there exists a unique Nash equilibrium. In the Nash equilbrium, each agent's strategy would be

$$x_i = s_i + \frac{s_i - av\{s_1, \dots, s_n\}}{n}$$

The nice thing about the average scheme is that the outcome remains the same, i.e.

$$av\{x_1, \dots, x_n\} = av\{s_1, \dots, s_n\} + \frac{\sum_i s_i - n \cdot av\{s_1, \dots, s_n\}}{n} = av\{s_1, \dots, s_n\}$$

The social cost is given by

$$SC(x) = \sum_i (av\{x_1, \dots, x_n\} - s_i)^2 + \sum_i (x_i - s_i)^2$$

The social cost at the unique Nash equilibrium x is

$$SC(x) = \sum_i (av\{s_1, \dots, s_n\} - s_i)^2 + \sum_i (\frac{s_i - av\{s_1, \dots, s_n\}}{n})^2 =$$

$$(1 + \frac{1}{n^2}) \sum_i (av\{s_1, \dots, s_n\} - s_i)^2$$

The optimal solution x^* occurs when agents declare their internal beliefs, since the term $(x_i - s_i)^2$ becomes 0 and the term $\sum\limits_i (av\{x_1, \dots, x_n\} - s_i)^2$ gets minimized. So, $SC(x^*) = \sum\limits_i (av\{s_1, \dots, s_n\} - s_i)^2$ and the price of anarchy is given by

$$PoA = (1 + \frac{1}{n^2})$$

The average scheme behaves well as the number of agents increase, since each agent loses the power to change the outcome. That is the power of the average scheme. The median scheme could not have such a behavior since the median of the internal beliefs can be away from the average, independently of the number of agents. Consider the same game with n agents, where the social choice rule is the median scheme ($f(x) = med\{x_1, \ldots, x_n\}$). The agents have a dominant strategy to declare their internal belief. Consider w.l.o.g. that the internal beliefs of $\frac{n}{2}$ are 0 and the remaining 1. The median chooses $f(x) = 0$. The social cost of x is $SC(x) = \frac{n}{2}$. Consider, however, a strategy x', where an agent from the left moves to $\frac{1}{2}$. The social cost would be $SC(x') = \frac{1}{4}n + \frac{1}{4} = \frac{n+1}{4}$. The ratio between the two is

$$\frac{SC(x)}{SC(x')} = \frac{\frac{n}{2}}{\frac{n+1}{2}} = 2 - \frac{2}{n+1}$$

which becomes 2 as the number of agents increase. So, there is a good reason to consider the average scheme when studying opinion dynamics under the presence of a social choice rule. Apart from that when considering the median it seems that the the game may not have a Nash equilibrium.

By studying several models of opinion dynamics nothing has been said about consensus of the agents' opinions. It seems that when agents are stubborn, the effect of their internal belief cannot aveliate. This means that agents cannot agree to an 'overall' value to depict their information about a specific quantity. Ofcourse the overall quantity can be decided by applying a social choice rule on the agents' opinions. Before studying the outcome of such a social choice rule, we can assume that each agent has an incentive to influence the overall estimate and drive the overall behavior near his internal belief. This incentive directly affects a player's cost function:

$$c_i(x) = \alpha(avg(x) - s_i)^2 + w_i(x_i - s_i)^2 + \sum_{j \in N(i)} w_{ij}(x_i - x_j)^2$$

where α is an arbitrary constant measuring the effect on the agent's cost.

In this direction it remains to consider the economic behavior of the above model. It seems that the model has a potential function

$$\Phi(x) = (av(\{x_1, \ldots, x_n\})^2 - \sum_i \frac{s_i x_i}{n} + w_i \sum_i (x_i - s_i)^2 + \sum_i \sum_{j<i} w_{ij}(x_i - x_j)^2$$

The potential function is convex, which means that the Nash equilibrium is unique. So, if we will be able to find best-response dynamics converging to the aforementioned equilibrium, it would be important to evaluate the Price of Anarchy of the above model.

5.2 Convergence with Polls

On the other side of the coin we tried to examine opinion dynamics, when agents have an interest on the global outcome and have some kind of information on the global outcome. As stated before they prefer a social outcome near their internal belief. However, it would be superficial to assume that the agents have full information of the global outcome, as long as a mistreat on the subject of local interactions forming behavior. Nevertheless, a natural assumption would be the agents having some estimate on specific time periods, as suggested by the presence of polls during elections. There will be two treatments on this idea, the one with an initial estimate and another with estimates every time-period. As a first step towards this direction the estimates would be true values.

5.2.1 Single Estimate

I would assume that the agents have initial opinions given by $\mathbf{x}[0]$. The agents are informed about the average of their initial opinions and try to minimize the following cost function

$$c_i(\mathbf{x}) = \alpha(\frac{n-1}{n}avg(\mathbf{x}[0]) + \frac{x_i}{n} - s_i)^2 + w_i(x_i - s_i)^2 + \sum_{j \in N(i)} w_{ij}(x_i - x_j)^2 =$$

$$\alpha(\frac{1}{n}c_i'\mathbf{x}[0] + \frac{x_i}{n} - s_i)^2 + w_i(x_i - s_i)^2 + \sum_{j \in N(i)} w_{ij}(x_i - x_j)^2$$

where $c_i = \dfrac{n-1}{n}\mathbf{1}_n$ is a constant vector with all the entries equal to $\dfrac{n-1}{n}$.

By applying first order conditions we get a best response function given by

$$x_i = \sum_{j \in N(i)} \frac{w_{ij}}{W_i} x_j + \frac{w_i + \frac{\alpha}{n}}{W_i} s_i - \frac{\alpha}{n^2 W_i} c_i'\mathbf{x}[0]$$

where $W_i = w_i + \dfrac{\alpha}{n^2} + \sum_{j \in N(i)} w_{ij}$.

We can now define the matrix C with the i'th row equal to $\frac{1}{W_i} c_i'$ and define the following linear system

$$\mathbf{x}[k+1] = A\mathbf{x}[k] + B\mathbf{s} - \frac{\alpha}{n^2} C\mathbf{x}[0]$$

This means that

$$\mathbf{x}[k] = (A^k - \frac{\alpha}{n^2}\sum_{\sigma=0}^{k-1} A^\sigma C)\mathbf{x}[0] + \sum_{\sigma=0}^{k-1} A^\sigma B\mathbf{s}$$

It is clear that A is an irreducible sub-stochastic matrix with each row-sum less than one. So the spectral radius of A is less than one and we get that $\lim_{k\to\infty} A^k = 0$. Since the largest eigenvalue is less than one, we get that $\sum_{\sigma=0}^{\infty} A^\sigma$ converges to $(\mathbb{I} - A)^{-1}$. Hence,

$$\mathbf{x}[\infty] = \sum_{\sigma=0}^{\infty} A^\sigma (B\mathbf{s} - \frac{\alpha}{n^2} C\mathbf{x}[0]) = (\mathbb{I} - A)^{-1}(B\mathbf{s} - \frac{\alpha}{n^2} C\mathbf{x}[0])$$

We can assume that the initial opinions are the agents' internal beliefs, i.e. $\mathbf{x}[0] = \mathbf{s}$.

5.2.2 Estimate per time period

I would define a period T when the agents are informed about the average of their opinions. For simplicity, I will define a vector $\mathbf{x}'[k] = \mathbf{x}[kT]$. The above discussion points out the following linear system

$$\mathbf{x}'[k+1] = X\mathbf{x}'[k] + \sum_{\sigma=0}^{T-1} A^\sigma B\mathbf{s}$$

where $X = A^T - \frac{\alpha}{n^2} \sum_{\sigma=0}^{T-1} A^\sigma C$.

This means that

$$\mathbf{x}'[k] = X^k \mathbf{x}[0] + \sum_{\sigma_1=0}^{k-1} X^{\sigma_1} \sum_{\sigma_2=0}^{T-1} A^{\sigma_2} B\mathbf{s}$$

Consider the norm $||A||_\infty = \max_i \sum_j |a_{ij}|$. Since A is sub-stochastic we know that $||A||_\infty < 1$. This means that $||A^T||_\infty \leq ||A||_\infty^T < 1$. So we can say that for large enough n, we get that $||X||_\infty < 1$. Since, $\rho(X) \leq ||X||_\infty < 1$ we get that X^k converges to zero and and the sum $\sum_{\sigma=0}^{\infty} X^\sigma$ equals to $(\mathbb{I} - X)^{-1}$. Finally, we get that

$$x'[\infty] = (\mathbb{I} - X)^{-1} \sum_{\sigma_2=0}^{T-1} A^{\sigma_2} B\mathbf{s}$$

This is really interesting since the the system converges even with polls taking part every fixed time period. This result can be farther generalised by considering polls which take part after an arbitrary time period. Furthermore, it would be ideal to understand the relationship between the equilibrium without polls, with a single poll and when several polls are conducted.

Chapter 6

Conclusions

This thesis presented large part of the opinion dynamics literature. It presents several models introduced, their economic behavior and convergence results.

The networks research is a line of research heavily based on the models chosen. As seen in the context above even if we agree that the process will be described in terms of a cost function or a best-response function, it remains to define many parameters of the function. It would be ideal to use the data we poccess these days, extracted from the social networks platforms, in order to derive models and afterwards analyze them.

It seems that the economic behavior of the opinion dynamics models are studied using the notion of the Price of Anarchy present on the algorithmic game theory line of research. There are several results for undirected networks which are welcomed. On the other side the case for directed networks is disheartening. However, it seems that by considering the worst Nash equilibrium is not such a good choice. It was proven that the best-response dynamics converge to a specific stable point. What if that point has a good economic behavior?

With regards to convergence results, nothing is said for assymetric coevolution games. It would be interesting to study the best response dynamics when the friendship weight is a “continuous” function of the distance between the beliefs and internal beliefs of two agents.

Moreover, the presence of polls seem to influence the agents’ beliefs but even now they converge to a specific point (equilibrium). Is the equilibrium achieved when considering polls much different from the equilibrium achieved with a single poll is conducted or when they are totally absent. Furthermore, nothing is said about how polls influence the convergence time. Do the polls consistently influence the convergence time of the process?

Finally, the research on opinion dynamics considers questions about opinion leaders. Suppose that a firm has a limited budget to affect some agents. Which agents must be affected to achieve the best overall belief on the network? Furthermore, since reaching a consensus is an important feature for learning application, it would be ideal to know which agents must be influenced in order to achieve a better convergence time to consensus.

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