Opinion Dymanics with Local Interactions

by

© Stratis Skoulakis

A thesis submitted to the

Departement of Mathematics of National University of Athens

in partial fulfilment of the

requirements for the degree of

Graduate Program in Logic, Algorithms and Computation

Department of Mathematics

October 2016

Abstract

During the last century many researchers investigated the way individuals form their opinions. The rapid growth of social networks in the recent years (Facebook,Twitter e.t.c) has further intensified this interest. To this day, a lot of models, on how our opinions evolve, have been proposed. In the huge majority of these models, each agent has to learn a large amount of opinions of other agents in order to update her opinion. In this thesis, we investigate the well studied Hegelsmann-Krause and Freidkin-Johson Model, under the constraint that each agent can learn a small amount of opinions of other agents. We propose three vatiations of these models, namely Network Hegelsmann-Krause, Random Hegelsmann-Krause and Limited Information Friedkin-Johson Model and we investigate their convergence properties.

Acknowledgements

The submission of this thesis defines the end of my studies in the Graduate Program in Logic, Algorithms and Computation. I would like to thank all the professors of the Program for the knowledge that I have acquired through their teaching as well as the students of the Program for the really hospitable and cooperative envirement that we had these years. I also thank the professors Aris Pagourtzis, Stathis Zachos and all the members of corelab, in which I am very glad to be part of. Finally, I would like to thank specially my supervisor Dimitris Fotakis for the excellent cooperation that we had and for all the help and time that he provided me to complete this thesis.

Contents

Chapter 1

Introduction

The unterstanding of human behavior was always a major study held in various sciences. Psychologists, Sociologists and Political Scientists were always interested in how humans form their opinions and consequently their behaviour. Although biology has taught us that human characteristics such as height or colour are imprinted on us by our genes, opinions or beliefs have nothing to do with genes. So a major question arises: Where do the opinions come from?

Today we are quite confident that the way that we form our beliefs depends on the experiences that we get from our birth to our death. Apparently, different individuals have very different experiences, something that explains why there exist such vast differences in human's behavior around the world. The causes that lead a certain individual to adopt a certain opinion on a specific subject are various and very complicated. For example economic welfare, education, religion and cultural backround play a major role in someone's beliefs. All these factors are very heteregenous, but they all something in common: They are all trasmitted by the interaction of people with other people. Thus, society plays an very important role in the opinion formation process. Having the principle that $our\ opinions\ are\ similar\ to\ the\ opinion\ of\ our\ friends$ in mind many models concerning how we form opinion have been proposed. This is where Theoritical Computer Science comes in. All the algorithmic and mathematical toolbox developped all these years can be used to study all these models in a very productive and modern way. Studying the properties of a specific model can be a way to verify how realistic and meaningful a model is and at the same time a lot of novel sociological results can be produced.

More precisely, we may translate the above thoughts to the language of Game Theory. We may imagine that each individual participating in a social network is a selfish agent playing a game. Each agent i has an opinion $x_i \in [0, 1]$ and her cost is defined by a cost function $f_i : [0, 1]^n \mapsto R_{>0}$, where n is the number of agents. Inuitively, f_i denotes the cost that agent i has for disagreeing with the other individuals. The precise definition of f_i depends on the model. Now, each agent i tries to minimize her personal cost and thus updates her opinion to the opinion that minimizes $f_i(\vec{x})$, assuming that the opinions of all the other agents will stay the same. Since all the agents update simultaniously their opinions, each agent i won't have the cost that she expected to have at the end of the update step. As a result, another update step may take place and we can understand why opinions evolve in time. We can represent the opinion vector at time step t as $x(t)$ and studying the orbit of $x(t)$ in $[0, 1]^n$ is the major purpose of Opinion Dynamics.

$Previous\ Work$

In 1965 American statistician Morris H. DeGroot proposed a model according to which the opinions in a social network are formed [12]. This model is known as DeGroot model and it is one of the most influential and well studied models in Opinion Dynamics [19, 16]. DeGoot has represented the social network as a graph G(V, E) at which the nodes stand for the members of the social network and the edges stand for the trust between them. Initially, each agent i has an initial opinion $x_i(0) \in [0, 1]$ and at each time step each agent averages her opinion with the opinions of her neighbors. Using standard Markov Theory, it can be proven that all the nodes converge to a stable state, meaning that finally all the nodes adopt a specific opinion [19]. Observe that if this was not the case, it would mean that all the agents change opinion over time, something that apparently doesn't hold in human's societies. As a result, each model in opinions dynamics should answer positively to the following two questions, in order to be meaningful. Do equilibrium points exists? Does the system converges to an equilibria?

After DeGroot Model many other models have been proposed [3, 13, 14, 15, 18, 27], each of them has a different motivation and tries to capture different sociological phenomena that are observed. Here we have to mention the Friedkin-Johnson Model, which is generalization of DeGroot Model and tries to capture the fact that in social networks consesus is rarely acheived. In the most recent years, a series of more complex models were proposed, in which the cost function of each agent i changes over time. More formally for each agents i there exists a time series of cost functions $\{f_i^t\}_{t \in N}$ measuring the cost of disagreement of each agent a time step t. In this class of models belong the Hegelsmann-Krause Model, the K-NN Asymmetric and Generalized Asymmetric Games [3, 18]. The motivation behind these is the very common knowledge that the $bonds$ and the $trust$ between the individuals doesn't stay constant of over time, but evolves as the opinions change.

As we have already mentioned, for each one of these models we need a proof concerning the existence of equilibrium points and the convergence to equilibrium. In the last years, there exists a vast amount of both theoretical and experimental work answering the above questions for various models [4, 5, 7, 8, 14, 25]. Interestingly, the cases where the cost functions change over time ($\{f_i^t\}_{t \in N}$) are very difficult to be handeled by analytical methods such as Markov Chain Theory. Suprisingly, the algorithmic toolbox seems every promising in proving the above properties in many of the above cases. Many of the algorithmic based ideas can prove convergence in very complicated cases in which the analytical techniqeus fail, but they also provide bounds in the convergence time of the systems [8, 9, 17, 23]. As a result, many of the works inspired for the opinions dynamics provide a very refreshing point of view concerning the way we study all the dynamical systems [10, 11].

$Our\ Work$

We have already discussed that in order to consider a model suitable to simulate the natural opinion process, the convergence to equilibrium is necessary. In this thesis, we add another requirement that is motivated by the simple observation conserning the way we form opinions. Observe that at each time step, agent i has a disagreement cost $f_i^t(\vec{x}(t))$, at time step t. At the next time step, each agent will update her opinion to minimize her personal cost. As a result, each agent i must learn the opinions of the agents that she $trusts$ at time step t. Now, the problem is that this number can be very large in many of the already proposed models. For example in Hegelsmann-Krause Model, each agent i needs to learn the opinions of the all the other agents to update her opinion. Today's social network (e.g. Facebook,Twitter) tend to be huge, so how realistic a model can be if it requires for each agent to learn some hundrends of opinions at each time step? Instead, it is far more reasonable to assume that agents act within a social network and update their opinions by consulting the opinions of a small (possibly random) subset of their neighbors. Motivated by this natural observation and by experimental work on opinion dynamics with communication regions, we introduce variants of the Hegelsmann-Krause and the Freidkin-

Johson Model, where information exchange in each round is limited (for both models) and local (for the Hegelsmann-Krause Model). We thoroughly investigate to which extent the convergence properties of the above models are affected by such practical considerations in the opinion exchange between the agents.

Chapter 2

Local Interaction in Hegelsmann-Krause Model

2.1 Hegelsmann-Krause Model

A natural assumption about the way that opinions evolve in a social network is that individuals with similar opinions are more likely to interact. Through this assumption it is easy to reason why polarization exists in many subjects in society(e.g. Political Beliefs, Religion, etc). The Hegelsmann-Krause Model [18], tries to capture this phenomena with a very straight forward way. Up next we give the definition of HK-Model.

Let V a set of agent s.t. $|V| = n$, each agent $i \in V$ has an initial opinion $x_i(0) \in [0, 1]$. At each time step, agent i updates her opinion as follows:

$$x_i(t) = \frac{\sum_{j \in S_i(t)} x_j(t) + x_i(t)}{|S_i(t)| + 1}$$

, where $S_i(t) = \{j \in V : \ |x_i(t) - x_j(t)| \leq \epsilon\}$. The parameter $\epsilon > 0$ is fixed for each instance of the HK-Model and quantifies how *open minded* the agents are. Now, let $x_1^*, x_2^* \in [0,1]$ s.t. $|x_1^* - x_2^*| > \epsilon$ and assign the opinion x_1^* in some agents and x_2^* in the rest. It is easy to see that this is an equilibrium point for the HK-Model. Working in the same way, one may see that each instance of the HK-Model admits an infinite number of equilibrium points [5]. Each equilibrium point consists of some $opinion\ clusters$ in the $[0,1]$ line and any two clusters have distance greater than ϵ. As always, the existence of equilibrium point is not enough. We want the agents to converge to some equilibrium point using the update rule defined by the HK-Model. This question has be extensively studied [2, 5, 23] and it is answered positively. Any instance $(\vec{x}(0), \epsilon)$ of the HK-Model will certainly converge to an equilibrium point. We also know that HK-Model needs at most $O(n^3)$ time steps to converge to equilibrium and that there exist instances that need $\Omega(n^2)$ time steps to converge [2, 26]. Interestingly, closing the gap between $O(n^3)$ and $\Omega(n^2)$ is still an open problem.

Before procceding to the next session, we present some thoughts concerning the above model. As we have already mentioned HK-Model captures the fact of polarization in the society very efficiently. However, there are some things in the model that don't seem very "$natural''$. For example, it is implicitly assumed in the HK-Model, that any agent knows all the other agents, what is how she can learn whether agent j has an opinion similar to hers at time step t. Apparently, this is not the case in large social networks in which each agent knows a small fraction of the agents of the network. We can also understand that even each agent knows all the other agents, it is very time consuming to learn all of their opinions at each time step in order to update her opinion. Towards, this direction we propose two variations of the HK-Model that capture the above thoughts. We prove the convergence of these models to equilibrium points. Although we don't provide bounds on the convergence time, we believe that the proofs have mathematical interest.

2.2 Network HK-Model

In this section we present the Network Hegelsmann-Krause Model, which is a variation of the original Hegelsmann-Krause Model. As mentioned in the previous section, the purpose of this variation is to capture the fact that in large social networks each agent knows only a small fraction of the other agents. The agents that agent i knows are independent of the opinions at each time step and remain the same during the whole process. As in the HK-Model, at each time step each agents learns the opinions of the agents that she knows and then averages her opinion with the similar opinions. In a more formal way, there exists an undirected graph $G = (V, E)$ where V represents the agents and E the $friendship$ between the agents. Let $x_i(0) \in [0, 1]$ the initial opinion of agent i. At each time step, agent i update her opinion as follows:

$$x_i(t+1) = \frac{\sum_{j \in S_i(t)} x_j(t) + x_i(t)}{|S_i(t)| + 1}$$

where $S_i(t) = \{j \in V : \{i, j\} \in E$ and $|x_i(t) - x_j(t)| \leq \epsilon\}$ and $\epsilon > 0$ is a fixed constant capturing how open minded the agents are.

As in the previous section, we can easily prove the existence of equilibrium points. We just need to take x_1^*, x_2^* s.t. $|x_1^* - x_2^*| > \epsilon$ and assign arbitrarily at each agent either x_1^* or x_2^*. Again, the major question is whether the previous update rule leads the system to a stable state. In the rest of this session we prove that any instance $(G, \vec{x}(0), \epsilon)$ of the Network HK-Model converges to equilibria. Before procceding to the proof, observe that if we set G the complete graph K_n then we get the original HK-Model. Thus, the convergence of Network HK-Model directly implies the convergence of the HK-Model.

At first we may write the above proccess in more convenient form:

$$\vec{x}(t) = A_t \vec{x}(t-1) = A_t \cdots A_1 \vec{x}(0)$$

where $\vec{x}(0)$ is the initial opinion vector and A_t is the matrix produced by the update rule. A_t is a stochastic matrix, with positive diagonal elements which are greater than $1/n$. Matrix A_t can also be descrided as an multigraph with self loops and the some edges from E $activated$ at time step t. We say that an edge $\{i, j\} \in E$ is activated at time step t if and only if $|x_i(t) - x_j(t)| \leq \epsilon$. For simplicity, for now on we may refer to A_t as matrix or graph giving matrix or graph properties respectively.

Let $(S, V \backslash S)$ a cut of G and assume that there exists a $t_0 \in N$ s.t. for all $t \geq t_0$ $\delta_t(S, V \backslash S) = \varnothing$,where $\delta_t(S, V \backslash S) = \{\{i, j\} \in E : \ \{i, j\} \in A_t\}$. This means that after time step t_0 there is no interaction between any agent in S and $V \backslash S$, which means that the system $breaks$ into the independent subsystems S and $V \backslash S$. This is a simple but useful observation that leads us to the following definition.

Definition 1. *A set of agents $S \subseteq V$ is* weakly connected *if and only if for any non-empty $S' \subset S$ and any $t_0 \in \mathbb{N}$, there is a round $t \geq t_0$ so that A_t includes at least one edge connecting an agent in S' to some agent in $S \backslash S'$.*

Lemma 1. *Let $(G, \vec{x}(0), \epsilon)$ an instance of the Network HK-Model in which V is not weakly connected. Then there exists $(S, V \backslash S)$ and $t_0 \in \mathbb{N}$ s.t. for all $t \geq t_0 : \delta_t(S, V \backslash S) = \varnothing$.*

Proof. By definition of weakly connected. □

Using induction and the above Lemma we may reduce the question of convergence of Network HK-Model to the the convergence of Network HK-Model in cases where V is weakly connected. Up next, we present the main Theorem of this section that ensures the convergence of the Network HK-Model.

Theorem 1. *Let $(G(V, E), \varepsilon, \vec{x}(0))$ be an instance of network-HK, where the opinion dynamics keep V weakly connected. Then, all agents converge to a single opinion x^*.*

The intuition behind the previous Theorem is that all agents in a weakly connected set influence each other with their opinions. In the rest of this section we prove that this influence is enough to lead the agents to the same opinion. For simplicity, from now on we consider V as weakly connected without mentioning it.

As we have already seen the above process can be descrided as a the matrix product $\vec{x}(t+1) = A_t \cdot A_{t-1} \cdots A_0 \vec{x}(0)$. In order to prove that all agents adopt the same opinion, we just need to prove that $\lim_{t \to \infty} A_t \cdot A_{t-1} \cdots A_0 = A^*$ and $\text{rank}(A^*) = 1$. A very useful tool to study products of stochastic matrices is the coefficient of ergodicity [24].

Definition 2. *Let A be a stochastic matrix then the coefficient of ergodicity of matrix A, $\tau(A) = \frac{1}{2} \cdot max_{i,j} \sum_{k=1}^{n} |A_{ik} - A_{jk}|$ and has the following properties:*

- $\tau(A \cdot B) \leq \tau(A) \cdot \tau(B)$
- *if A has positive elements then $\tau(A) < 1$*
- $\tau(A) = 0$ *if and only if rank(A)=1*

To complete the proof, it suffices to show that there is a round t_0 so that the coefficient of ergodicity of the matrix $C = A_{t_0 - 1} \cdots A_0$ is $\tau(C) \leq \varepsilon/2$. Given this, we have that $\vec{x}(t_0) = C\vec{x}(0)$ and that for all agents i and j:

$$
\begin{aligned}
|x_i(t_0) - x_j(t_0)| &= |(C_i - C_j)\vec{x}(0)| && (2.1)\\
&\leq \|C_i - C_j\|_1 && (2.2)\\
&\leq 2\tau(C) \leq \varepsilon && (2.3)
\end{aligned}
$$

where C_i is the i-th row of matrix C. Since at t_0 all opinions are within distance ε, in any round $t > t_0$, all agents take the average of all opinions in their social neighborhood (including their opinion). Hence, after round t_0, we have essentially an instance of DeGroot's model on the undirected connected network G (enhanced with self-loops). Since G (with self-loops) defines an irreducible and aperiodic process, all agents converge to a single opinion [19]. Thus, in order to prove the convergence of the Network HK-Model, we just need to prove that $\lim_{t\to\infty} \tau(A_t \cdots A_0) = 0$.

Lemma 2. *Let $(G, \vec{x}(0), \epsilon)$ an instance of the Network HK-Model s.t. V is weakly connected. Then there exists $\ell(t) \in \mathbb{N}$ s.t. $\tau(A_{\ell(t)} \cdots A_0) \leq 1 - (1/n)^{n^2}$.*

Proof. At first, we may prove that there exists $\ell(t) \in \mathbb{N}$ s.t. $\tau(A_{\ell(t)} \cdots A_0) < 1$. To this end, we first show that since V remains weakly connected, for any round t, there is a round $\ell(t)$, such that the matrix $C_{\ell(t)} = A_{\ell(t)} A_{\ell(t)-1} \cdots A_0$ has all its elements positive, and thus, by the properties of coefficient of ergodicity $\tau(C_{\ell(t)}) < 1$.

An element $C_t(i,j)$ is positive iff there is a (time-respecting) walk $(i, u_1, \ldots, u_{t-1}, j)$ from agent i to agent j such that (i) the first edge $\{i, u_1\}$ exists in A_0, (ii) the edge $\{u_{k-1}, u_k\}$ exists in A_k, (iii) the last edge $\{u_{t-1}, j\}$ exists in A_t. Recall that any matrix A_t has positive diagonal elements. Thus, if $C_{t-1}(i,j) > 0$ then $C_t(i,j) > 0$, since the walk can use the self loop of A_t. Let $\mathrm{Pos}_i(t)$ the positive elements of at the i-th row of C_t (equivalently, the agents reachable from i in t steps). Since V is weakly connected there exists a time step $t' > t$ s.t. $A_{t'}$ contains an edge traversing $\{j, m\}$ the cut $(\mathrm{Pos}_i(t), V \backslash \mathrm{Pos}_i(t))$. Since j is reachable form i in t steps, j is reachable form i in $t' - 1$ steps (using the self loops) and using the edge $\{j, m\}$ of $A_{t'}$, m is reachable from i in t' steps. Finally, $\mathrm{Pos}_i(t) + 1 \leq \mathrm{Pos}_i(t')$ and repeating the same argument for all the rows of C_t proves that there exits $l(t)$ s.t. all the elements of the product $C_{\ell(t)} = A_{\ell(t)} \cdots A_0$ become positive.

Up next, we prove that $\tau(C_{\ell(t)}) \leq 1 - (1/n)^{n^2}$. Observe that in tha matrix product

$A_{\ell(t)} \cdots A_0$, there are some $expanding$ martices, that augment the number of positive elements of the product. Let $B_1 \cdots B_k$ these matrices. Observe that $k \leq n^2$ since the number of positive elements in the final matrix product is n^2. We may rewrite $C_{\ell(t)}$ as follows:

$$C_{\ell(t)} = B_k \cdot A_{\ell(t)-1} \cdots A_{i_k} \cdot B_{k-1} A_{i_{k-2}} \cdots A_1 B_0$$

Clearly, $\tau(B_k \cdot B_{k-1} \cdots B_0) < 1$, since all the elements of this product are positive.

Let's study the product $B_1 \cdot A_m \cdots A_1 \cdot B_0$ (A_0 is always an $expanding$ matrix, so $B_0 = A_0$). Let Pos(A) is the set of positive elements of matrix A. Then, Pos(A_q) $\subseteq$ Pos($A_{q-1} \cdots B_0$). Otherwise, A_i would be an $expanding$ matrix. Using the last property we prove that the minimum positive element of the matrix B_0 doesn't decrease during the $non-expanding$ steps.

For simplicity we denote $A = A_q$ and $B = A_{q-1} \cdots B_0$. We prove that the minimum positive element of $A \cdot B \geq$ the minimum positive element of B. Let $B_{ij} > 0$ then

$$(AB)_{ij} = \sum_{l=1}^{n} A_{il} B_{lj} = \sum_{l:B_{lj}>0} A_{il} B_{lj}$$

In order to prove our claim, we just need to show that $\sum_{l:B_{lj}>0} A_{il} = 1$. Let us assume that $\sum_{l:B_{lj}>0} A_{il} < 1$. This means that there exists k s.t. $A_{ik} > 0$ and $B_{kj} = 0$. Since A is $non-expanding$ matrix if $B_{kj} = 0$ then $(AB)_{kj} = 0$ (otherwise A would add a positive element). Observe that $(AB)_{kj} \geq A_{ki} \cdot B_{ij}$. Since $(AB)_{kj} = 0$ and $B_{ij} > 0$ then $A_{ki} = 0$. We have concluded that $A_{ki} = 0$ and $A_{ik} > 0$, something that can not be true because $A_{ik} > 0$ implies that $\{i, k\} \in E$ and $|x_i(q) - x_j(q)| \leq \epsilon$, which implies that $A_{ki} > 0$.

Inductively, we may prove that the minimum positive element of the matrix product $A_t \cdots A_0$ decreases only during the expanding steps [5, 23]. Since there are at most n^2 expanding steps and the minimum positive element of each $expanding$ matrix is at least

$1/n$ then the minimum positive element of $C_{\ell(t)} = A_{\ell(t)} \cdots A_0$, is greater than $(1/n)^{n^2}$. Combining this with the faxt that $\tau(C_{\ell(t)}) < 1$ then $\tau(C_{\ell(t)}) \leq 1 - (1/n)^{n^2}$. $\square$

Using Lemma 2 and the following simple algorithm, we can establish the fact that $\lim_{t\to\infty} \tau(A_t \cdots A_0) = 0$. Observe that Lemma 2 ensures that the algorithm always gets out of the **While Loop** in step 6, something that ensures the termination of the algorithm and that $\tau(C_i) \leq (1 - (1/n)^{n^2})$. Once the algorithm is terminated we get:

$$\tau(A_t \cdots A_0) = \tau(C_k \cdots C_1) \leq \tau(C_k) \cdots \tau(C_1) \leq (1 - (1/n)^{n^2})^k$$

Since integer k can be arbitrarily large then $\lim_{t\to\infty} A_t \cdots A_0 = 0$, which completes the proof of convergence of Network HK-Model.

Algorithm 1

1: **Input:** An instance of the **Network HK-Model** and an integer k

2: $t \leftarrow 0$

3: $i \leftarrow 1$

4: **while** $i \leq k$ **do**

5:     $C_i \leftarrow I$

6:     **while** $\text{Pos}(C_i) < n^2$ **do**

7:         $C_i \leftarrow A_t \cdot C_i$

8:         $t \leftarrow t + 1$

9:     $i \leftarrow i + 1$

10: **Output:** $(C_1, \cdots, C_k), t$

Before procceding to the next section, we mention that this proof can be generalized to prove convergence of the d-dimensional HK-Model on a Social Network. In this model each agent i maintains a d-dimensional opinion vector $\vec{x}_i(t) \in [0,1]^d$ and the update rule is defined respectively by the d-dimensional HK model (see e.g.) and a social network G. The proof is essentially identical, with the only difference that we need to prove the existence of a time step t_0 such that $\tau(C) \le \varepsilon/(2\sqrt{d})$, where $C = A_{t_0} \cdots A_0$. But, we have already proven that $\lim_{t\to\infty} A_t \cdots A_0 = 0$.

2.3 Random HK-Model

In this section we present our second variation of the Hegelsmann-Krause Model called Random Hegelsmann-Krause Model. Before procceding to the definion of our current model, we give a small motivation. We may imagine a small town in which each resident knows all the other resident. We may also imagine that at each day each of the residents meets some others residents each day and learns their opinions concerning a specific subject. At the end of the day, each resident is influenced by the opinion that he learned under HK-assumption. In other words, he takes account only the opinions that are similar to his. Now, we define the model in a more formal way: Let $\vec{x}(0) \in [0,1]^n$ the initial opinion vector, $\epsilon > 0$ and $K \in N$. At $t \ge 1$, agent i:

1. picks K other agents uniformly at random. Let $R_i(t)$ be this random set.

2. finds all the agents $j \in R_i(t)$ s.t. $|x_i(t) - x_j(t)| \le \epsilon$. Let $S_i(t)$ be this random set.

3. $x_i(t+1) = \frac{\sum_{j\in S_i(t)} x_j(t) + x_i(t)}{|S_i(t)|+1}$

where $\epsilon > 0$ again denotes how open minded the agents are and K denotes how many other agents each agent meets. As in the previous section, we want to prove that each instance $(\vec{x}(0), \epsilon, K)$ of the Random HK model converges to equilibrium. The ideas of the proof are quite similar with the proof in the previous section, however there are some major differences. At first, Random HK-model is a non-deterministic system and thus, we have to prove convergence in a probabilistic framework. Apart from that, in the Network HK-Model interaction between the agents at each time step was symmetric. On the contrary, in this model interaction is asymmetric and as we will see, this causes some difficulties in proving the convergence.

Let $(\vec{x}(0), \epsilon, K)$ be an instance of the Random HK-Model. Again we use the matrix description of the above procedure to prove the convergence of the system.

$$\vec{x}(t+1) = A_t \cdots A_0 \cdot \vec{x}(0)$$

Each of the matrices A_i is produced by the update rule defined in the beginning of this section. We again refer to A_i either as matrix or graph, giving it the respective properties. It is easy to observe that A_i is a random directed graph, with vertices the agents and acres the interaction between them at time step i. Note that again A_i has positive diagonal elements no matter what the random choice of each agent is.

Firstly, let's assume that there exists a partition $(S, V \backslash S)$ and a time step t_0 such that for all $i \in S$ and $j \in V \backslash S$, $|x_i(t_0) - x_j(t_0)| > \epsilon$. Clearly, after time t_0 the agents in S are not influenced by the agents in $V \backslash S$, thus the system can be divided into two independent sub-systems. This simple observation lead us to the following definitions.

Definition 3. *Let S_1, S_2 two disjoint sets of agents, we denote their distance at round t as* $d^t(S_1, S_2) = \min_{i \in S_1, j \in S_2} |x_i(t) - x_j(t)|$.

Definition 4. *A set of agents S is ε-*connected *at round t, if and only if for any non-empty set $S' \subset S$, $d^t(S', S \setminus S') \leq \varepsilon$.*

Definition 5. *A set of agents S **breaks at round** t if and only if S is ε-connected at round $t-1$ and is not ε-connected at round t.*

With the last definition, we introduce the notion of the **break**. As we have already discussed, once S' and $S \setminus S'$ break, they behave as independent subinstances in the future and that's why this is a key notion to our proof. Notice that independently, of what the random choices of each agent at each time step may are, at most $n-1$ breaks can occur. The latter means that when a break occurs at specific time step, this event automatically reduces the number of the probable future breaks. Intuitively, if we run the system for all long period all the possible breaks will occur at the end of this period and after that it will be certain that no break occurs. As a first step, to check whether this intuition is true, we present an instance in which no break occurs for all the possible random choices. For example, take the instance $(\vec{x}(0), \epsilon, K)$ such that for all i, j, $|x_i(0) - x_j(t)| \leq \epsilon$. It is obvious that in this instance no **break** ever occurs. Following the intuition presented above, we prove that in instances in which no **break** is possible, all agents converge with high probability to a single opinion. To simplify notation we provide the following definitions.

Definition 6. *We denote as Γ_l the set of all opinion vectors $\vec{y}$ such that for all rounds $t \geq 0$, $\mathbb{P}\mathrm{r}[$at most l breaks occur in $\{0, t\} \mid \vec{x}(0) = \vec{y}] = 1$.*

Namely, Γ_l consists of all vectors $\vec{y}$ such that if the initial opinions are $\vec{y}$, then no matter the random choices, at most l breaks occur.

Definition 7. *We say that agents i and j are (ε, t)-connected if there is a "path" $(i, i_1, \ldots, i_{k-1}, j)$ so that for each "step" q, $|x_q(t) - x_{q+1}(t)| \leq \varepsilon$.*

Definition 8. *The* diameter *at some time step t, denoted $Diam(t)$, is the maximum distance $|x_i(t) - x_j(t)|$ overall (ε, t)-connected pairs of agents i, j.*

Up next, we present our first major Lemma of this section. By the following Lemma it is ensured that if **no break** occurs then all agents converge to a single opinion.

Lemma 3. *Let $(k, \varepsilon, \vec{x}(0))$ be an instance of the random-HK model with $\vec{x}(0) \in \Gamma_0$. For any $\gamma, \delta > 0$, there is a round t_0 such that,*

$$\Pr[Diam(t) \leq \gamma] \geq 1 - \delta$$

Proof. Without loss of generality, we assume that there exists a single ε-connected component (otherwise, the lemma applies to each ε-connected component separately). Since $\vec{x}(0) \in \Gamma_0$ no break occurs and the agents are (ε, t)-connected for all t and all random choices.

Let $p = 1-(1-1/n)^k$ be the probability that an agent j is not in the sample set of agent i in a round t. For any round ℓ, we denote $C_\ell = A_{\ell+2n^2/p} \cdots A_\ell$ and $D_\ell = A_{\ell-1} \cdots A_1$. The important step is to show that there is some fixed $\eta > 0$ such that for any fixed value of D_ℓ, $\mathbb{E}[\tau(C_\ell)|D_\ell] \leq 1 - \eta/2$.

For any round $t' \geq \ell$, $\mathrm{pos}(t')$ (resp. $\mathrm{pos}_i(t')$) denotes the number of positive entries in (resp. the i-th row of) matrix $A_{t'} \cdots A_\ell$. We have $0 \leq \mathrm{pos}(t') \leq n^2$ and $\mathrm{pos}(t'+1) \geq \mathrm{pos}(t')$. As long as $\mathrm{pos}(t') < n^2$, there is some agent i with $\mathrm{pos}_i(t') < n$. As in the proof of Lemma 2, $\mathrm{pos}_i(t')$ is the number of agents reachable from i, between rounds ℓ and t', by time-respecting walks. Since V is ε-connected and **no break** occurs, if $\mathrm{pos}_i(t') < n$, there is at least one new agent reachable from i in round $t'+1$, with probability at least p. Hence, for any round t' with $\mathrm{pos}(t') < n^2$, $\mathrm{pos}(t'+1) > \mathrm{pos}(t')$ with probability at least p, and the expected number of rounds before it becomes $\mathrm{pos}(t') = n^2$ for the first time is at most n^2/p. By Markov's inequality, $\mathbb{P}\mathrm{r}[\mathrm{pos}(\ell + 2n^2/p) < n^2 \mid D_\ell] \leq 1/2$. Moreover, since

$\mathrm{pos}(\ell + 2n^2/p) = n^2$ implies that $\tau(C_\ell) < 1$,

$$\mathbb{P}\mathrm{r}[\tau(C_\ell) < 1 \mid D_\ell] \geq 1/2$$

As in the proof of Lemma 2, since C_ℓ is the product of $2n^2/p$ matrices, there exists a fixed fractional $\eta > 0$ such that if $\tau(C_\ell) < 1$, then $\tau(C_\ell) \leq 1 - \eta$. Thus, we obtain that for any fixed value of D_ℓ, $\mathbb{E}[\tau(C_\ell)|D_\ell] \leq 1 - \eta/2$.

Now, we can work as in the proof of Lemma 2. Taking an appropriatelly large number of rounds, we obtain a t_0 and a matrix $C = A_{t_0} \cdots A_1$ such that $\tau(C) \leq \gamma/2$ with probability at least $1 - \delta$. Then, the Lemma follows from the properties of the coefficient of ergodicity. $\square$

We proceed to show that the random-HK model converges asymptotically, with probability that tends to 1. We recall that if there exists a round t^* with $Diam(t^*) \leq \varepsilon$, then $\vec{x}(t^*) \in \Gamma_0$ and Lemma 3 again implies convergence in each ε-connected component separately. The following lemma establishes the existence of such a round t^* with probability that tends to 1.

Theorem 2. *3.2 Let $(k, \varepsilon, \vec{x}(0))$ be any instance of the random-HK model. For any $\delta > 0$ there is a round t^* such that*

$$\mathbb{P}\mathrm{r}[Diam(t^*) \leq \varepsilon] \geq 1 - \delta$$

Proof. Intuitively, if $\vec{x}(0) \in \Gamma_l$, there are constants p and t_0 such that $\mathbb{P}\mathrm{r}[\vec{x}(t_0) \in \Gamma_{l-1}] \geq p$. Moreover, there is a constant m such that $\mathbb{P}\mathrm{r}[\vec{x}(mt_0) \in \Gamma_0] \approx 1$, i.e., with almost certainty, all possible breaks have occurred by round mt_0. Then, the proof follows easily from Lemma 3.

In the following, we let t_0 be the number of rounds in Lemma 3 for $\gamma = \varepsilon$. Namely, t_0 is such that if $\vec{x}(0) \in \Gamma_0$ then $\mathbb{P}\mathrm{r}[Diam(t_0) \leq \varepsilon] \geq 1 - \delta$. For brevity, we let $p = 1/n^{knt_0}$ and let $P(\vec{y}, m) = \mathbb{P}\mathrm{r}[Diam((m+1)t_0) \leq \varepsilon \,|\, \vec{x}(0) = \vec{y}]$. In other words, $P(\vec{y}, m)$ is the probability that the diameter is at most ε after $(m+1)t_0$ rounds, given that the initial opinion vector is $\vec{y}$. At first, we consider the case where $\vec{x}(0) \in \Gamma_1$ and prove that:

$$P(\vec{x}(0), m) \geq (1-\delta)(1-(1-p)^m) \tag{2.4}$$

We can verify (2.4) is true for $m = 1$. We inductively assume that m satisfies (2.4) and consider the following cases for $m + 1$.

$\mathbb{P}\mathrm{r}[\vec{x}(t_0) \in \Gamma_0] = 0:$ Therefore, since $\vec{x}(0) \in \Gamma_1$, no break occurs in $\{0, t_0\}$ for all random choices. Thus, $\vec{x}(0)$ satisfies the hypothesis of Lemma 3 and $P(\vec{x}(0), 0) \geq 1 - \delta$. As a result, $P(\vec{x}(0), m+1) \geq 1 - \delta$.

$\mathbb{P}\mathrm{r}[\vec{x}(t_0) \in \Gamma_0] > 0:$ There is an opinion vector $\vec{y} \in \Gamma_0$ such that $\mathbb{P}\mathrm{r}[\vec{x}(t_0) = \vec{y}] \geq p$. Since $\vec{x}(0) \in \Gamma_1$, if $\vec{x}(t_0) \neq \vec{y}$, then $\vec{x}(t_0) \in \Gamma_1$. Hence, we obtain that:

$$\begin{aligned}
&P(\vec{x}(0), m+1) = \\
&\mathbb{P}\mathrm{r}[\vec{x}(t_0) = \vec{y}]\, P(\vec{y}, m) + \sum_{\vec{a} \in \Gamma_1} \mathbb{P}\mathrm{r}[\vec{x}(t_0) = \vec{a}]\, P(\vec{a}, m) \\
&\geq (1-\delta)[p + (1-p)(1-(1-p)^m)] \\
&\geq (1-\delta)(1-(1-p)^{m+1})
\end{aligned}$$

Figure 3.2 provides a graphical representation of this induction. Now we extend the proof to the case where $\vec{x}(0) \in \Gamma_l$, for any $2 \leq l \leq n-1$. We recursively define the functions $f_l(m) = pf_{l-1}(m-1) + (1-p)f_l(m-1)$, for all $m, l \geq 2$, with $f_1(m) = (1-\delta)(1-(1-p)^m)$. Using induction and the same arguments as above, we can show

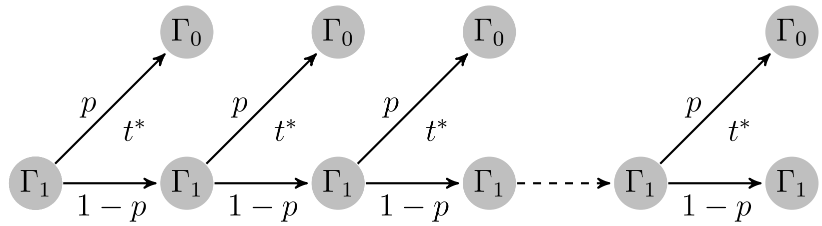


Figure 2.1

that if $\vec{x}(0) \in \Gamma_l$, then $P(\vec{x}(0), m) \geq (1-\delta) f_l(m)$. We observe that $\lim_{m\to\infty} f_1(m) = 1$. Then, by the definition of f_l, we can show inductively that $\lim_{m\to\infty} f_{n-1}(m) = 1$. Since at most $n-1$ breaks can occur, we conclude that $P(\vec{x}(0), m) \geq (1-\delta) f_{n-1}(m)$. Finally, $\lim_{m\to\infty} P(\vec{x}(0), m) = 1$ and as a result for all $\delta > 0$ there exists t^* such that $\mathbb{P}\mathrm{r}[Diam(t^*) \leq \varepsilon] \geq 1 - \delta$. $\square$

Before finishing with this section, we summarize the proof of convergence of the Random Hegelsmann-Krause Model. At first, Theorem 3.2 ensures that with high probability there exists $t^* \in \mathbb{N}$ such that $Diam(t^*) \leq \epsilon$, which means that the maximum distance at time step t^* in each ϵ-connected component is at most ϵ. The latter implies that after t^* no break occurs and by Lemma 3 convergence to a single opinion in each ϵ-component is ensured.

Chapter 3

The Friendkin-Johnson with Limited Information

3.1 The Friendkin-Johnson Model

In this section we present one of the most influential and well studied models in Opinion Dynamics, the Friendkin-Johnson Model [15]. There are many important works concerning the study of FJ Model [3, 4, 16], in this section we present some of these results. At first, we provide the definition of the model. We are given a weighted network $G(V, E)$ and the agents' initial opinions $\vec{x}(0)$. Each agent i corresponds to vertex $i \in V$, has weight $w_{ii} \in (0, 1]$ for her initial opinion and weight $w_{ij} \in [0, 1)$ for the current opinion of each other agent j. At any round $t \geq 1$, each agent i updates her opinion to:

$$x_i(t) = \sum_{j \neq i} w_{ij} x_j(t-1) + w_{ii} x_i(0) \tag{3.1}$$

As always we are interested in the existence of equilibrium points and in the convergence of the sytem to them. Let A denote the adjacency matrix of G with 0 on its main diagonal and let B be the diagonal matrix with $B_{ii} = w_{ii}$, for all i. Then, (3.1) can be written in matrix form as:

$$\vec{x}(t) = A\vec{x}(t-1) + B\vec{x}(0) \tag{3.2}$$

Apparently, an equilibrium point $x^* \in [0,1]^n$ must satisfy the following equation.

$$x^* = A \cdot x^* + B \cdot \vec{x}(0)$$

Since $w_{ii} > 0$, A is a substochastic matrix and $\rho(A) = |A|_\infty < 1$, then all the eigenvalues of matrix $I - A$ are not zero. Consequently, the matrix $I - A$ is reversible and $x^* = (I-A)^{-1}B \cdot x(0)$ something that ensures both the existence and the uniqueness of the equilibrium point. For more details, you may see [4, 16]. The fact that A is a substochastic matrix also ensures the convergence to x^*.

Theorem 3. *Let an instance of the FJ Model and x^* its equilibrium point. Then, for all $\gamma > 0$ there exists t such that $||x(t) - x^*||_\infty \leq \gamma$ and $t = O(\frac{\ln(n/\gamma)}{1-\rho(A)})$.*

Proof. Let $e(t) = ||x(t) - x^*||_\infty$ we prove that $e(t) = \rho(A)^t \cdot e(0)$. By definition of $x(t)$ we have that $x(t) = A \cdot x(t-1) + B\vec{x}(0)$ and $x^* = A \cdot x^* + B \cdot x(0)$, thus:

$$\begin{aligned} e(t) &= ||x(t) - x^*||_\infty \\ &= ||A(x(t-1) - x^*)||_\infty \\ &\leq \rho(A)||x(t-1) - x^*||_\infty \\ &\leq \rho(A)^t e(0) \end{aligned}$$

As a result, we can bound the convergence time by finding the smallest t s.t. $\rho(A)^t e(0) \leq \gamma$. Using the above inequality, we get $t \leq \frac{\ln(e(0)/\gamma)}{\ln(1/\rho(A))} \leq \frac{\ln(n/\gamma)}{1-\rho(A)}$. The last bound follows from the fact that $e(0) \leq 1$ since $x_i(0) \in [0,1]$ and that $e^{-x} + x - 1 \geq 0$, if $x \geq 0$. This bound can be impoved with a similar but more rigorous analysis, using as $\rho(A)$ the spectral radious of A instead of $||A||_\infty$ [16]. $\square$

3.2 The Friendkin-Johnson Model with Limited Information

A natural question about the FJ model is whether we can simulate the opinion formation process by simple protocols where agents consult the opinions of a small subset of their neighbors in each round. In this section, we present the *Limited Information Protocol*, or *LIP-FJ*, in brief, and discuss its convergence properties. Let $(G(V, E), \vec{x}(0))$ be an instance of the FJ model. At any round $t \geq 1$, each agent i randomly selects one index $j \in V$. Let $s_i(t)$ be a random variable, denoting the random choice of agent i at time step t, with distribution:

$$\mathbb{P}\mathrm{r}[s_i(t) = y] = \begin{cases} w_{ij} & \text{if } y = x_j(t-1) \\ w_{ii} & \text{if } y = x_i(0) \end{cases}$$

Then, agent i updates her opinion as follows:

$$x_i(t) = \lambda(t) x_i(t-1) + (1 - \lambda(t)) s_i(t)$$

where $\lambda : \mathbb{N} \mapsto (0, 1]$ is a decreasing function. As we see in the **LIP-FJ Model** the information exchange between the agents is vastly reduced, since at each time step each agents needs to learn only one opinion of her neighbors. In the original **FJ Model** each agent learns the opinions of all of her neighbors. In the rest of the section, we will try to appropriately selects the function $\lambda(t)$ such that the **LIP-FJ Model** converges with high probability to the equilibrium point x^* of the original **FJ Model**. A very interesting question is the following. **Is there a function $\lambda : \mathbb{N} \mapsto (0, 1]$ such that LIM-FJ Model converges with high probability to the equilibrium point of FJ Model?** Although, we don't provide an answer to the above question, in the rest of the section we present theoretical and experimental results that provide strong evidence about the form of the $\lambda(t)$.

Lemma 4. $\mathbb{E}[\vec{x}(t)] = \lambda(t)\mathbb{E}[\vec{x}(t-1)] + (1 - \lambda(t))(A\,\mathbb{E}[\vec{x}(t-1)] + B\vec{x}(0))$

Proof. We observe that (i) for any fixed instance and any fixed round t, the set Ω^t of possible values of the random variable $\vec{x}(t)$ is finite; and that (ii) for any possible value $\vec{y}$ of $\vec{x}(t-1)$,

$$\mathbb{E}[\vec{x}(t)|\vec{x}(t-1) = \vec{y}] = \lambda(t)\vec{y} + (1 - \lambda(t))(A\vec{y} + B\vec{x}(0))$$

For brevity, let $p_{t-1}(\vec{y}) \equiv \mathbb{P}\mathrm{r}[\vec{x}(t-1) = \vec{y}]$ be the probability that the opinions at round $t-1$ are as in $\vec{y}$. Then, using (i) and (ii), we obtain that for any fixed $t \geq 1$,

$$\begin{aligned} \mathbb{E}[\vec{x}(t)] &= \sum_{\vec{y}\in\Omega^{t-1}} p_{t-1}(\vec{y})\,\mathbb{E}[\vec{x}(t)|\vec{x}(t-1) = \vec{y}] \\ &= \lambda(t) \sum_{\vec{y}\in\Omega^{t-1}} p_{t-1}(\vec{y})\,\vec{y}\, + \\ &\quad + (1 - \lambda(t)) \sum_{\vec{y}\in\Omega^{t-1}} p_{t-1}(\vec{y})\,(A\vec{y} + B\vec{x}(0))) \end{aligned}$$

Using linearity of expectation, we conclude that any $t \geq 1$,

$$\mathbb{E}[\vec{x}(t)] = \lambda(t)\mathbb{E}[\vec{x}(t-1)] + (1 - \lambda(t))(A\,\mathbb{E}[\vec{x}(t-1)] + B\vec{x}(0))$$

□

Then, using induction on t and standard properties of the matrix infinity norm $\nu(A) \equiv \|A\|_\infty \in (0, 1)$, we show that for an appropriate choice of $\lambda(t)$, LIP-FJ converges in expectation to the stable state $\vec{x}^* = (I - A)^{-1}B\vec{x}(0)$ of the FJ model.

Theorem 4. *For any instance $(G(V,E), \vec{x}(0))$ and any round $t \geq 1$, the opinions maintained by LIP-FJ satisfy*

$$\|\mathbb{E}[\vec{x}(t)] - \vec{x}^*\|_\infty \leq e^{-(1-\nu(A))\sum_{q=1}^{t}(1-\lambda(q))}\|\vec{x}(0) - \vec{x}^*\|_\infty$$

where $\vec{x}^*$ is the stable state of the FJ model.

Proof. From Lemma 4 we know that

$$\mathbb{E}[\vec{x}(t)] = \lambda(t)\mathbb{E}[\vec{x}(t-1)] + (1-\lambda(t))(A\,\mathbb{E}[\vec{x}(t-1)] + B\vec{x}(0))$$

and since x^* is the equilibrium point of the FJ-Model we have:

$$x^* = \lambda(t)x^* + (1-\lambda(t))(A \cdot x^* + Bx(0))$$

Using the above equation, we get:

$$\begin{aligned}
||\mathbb{E}[\vec{x}(t)] - x^*||_\infty &= ||[\lambda(t) \cdot I + (1-\lambda(t)) \cdot A](\mathbb{E}[\vec{x}(t-1)] - x^*)||_\infty & (3.3)\\
&\leq ||\Pi_{i=1}^{t}[\lambda(i) \cdot I + (1-\lambda(i)) \cdot A])||_\infty \cdot ||\vec{x}(0) - x^*||_\infty & (3.4)\\
&\leq \Pi_{i=1}^{t}[1 - (1-\lambda(i))(1-\nu(A))] \cdot ||\vec{x}(0) - x^*||_\infty & (3.5)\\
&\leq e^{-(1-\nu(A))\sum_{q=1}^{t}(1-\lambda(q))}\|\vec{x}(0) - \vec{x}^*\|_\infty & (3.6)
\end{aligned}$$

Inequalities (3.4),(3.5) are directly implied by standard properties of the matrix norm. Using the inequality $1 - x \leq e^{-x}$ if $x \in [0,1]$ we get inequality (3.6).

□

Using the Theorem 3, we are ready to derive some intuition about the form of $\lambda(t)$. At first, we may observe that if $\lambda(t) = \frac{t}{t+1}$, then $\|\mathbb{E}[\vec{x}(t)] - \vec{x}^*\|_\infty \leq e^{1-\nu(A)}/t^{1-\nu(A)}$, by Theorem 4, and LIP-FJ converges in expectation in time exponential in $1/(1-\nu(A))$.

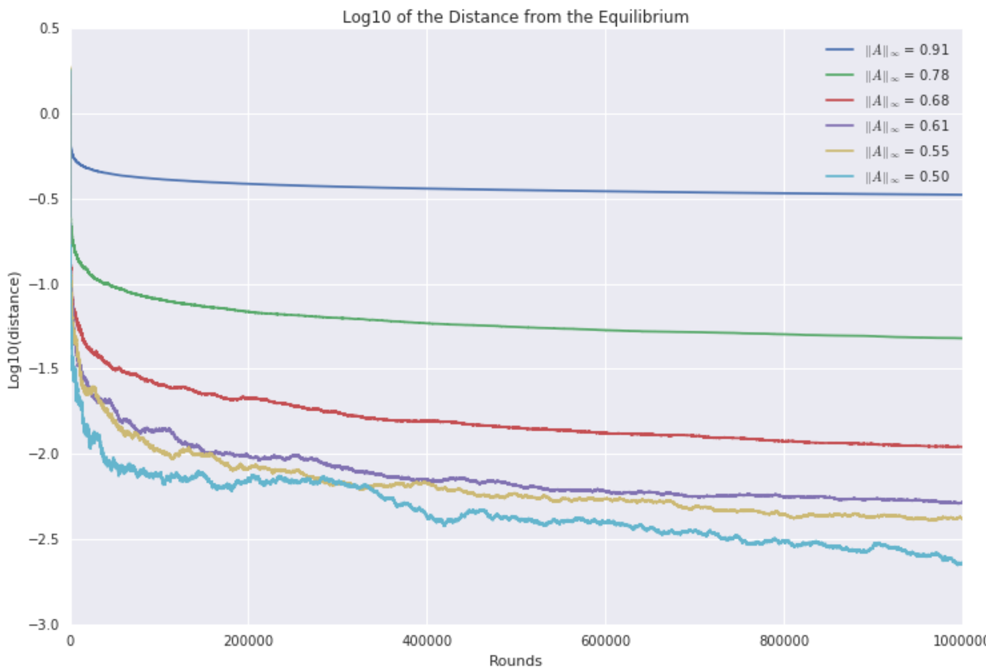
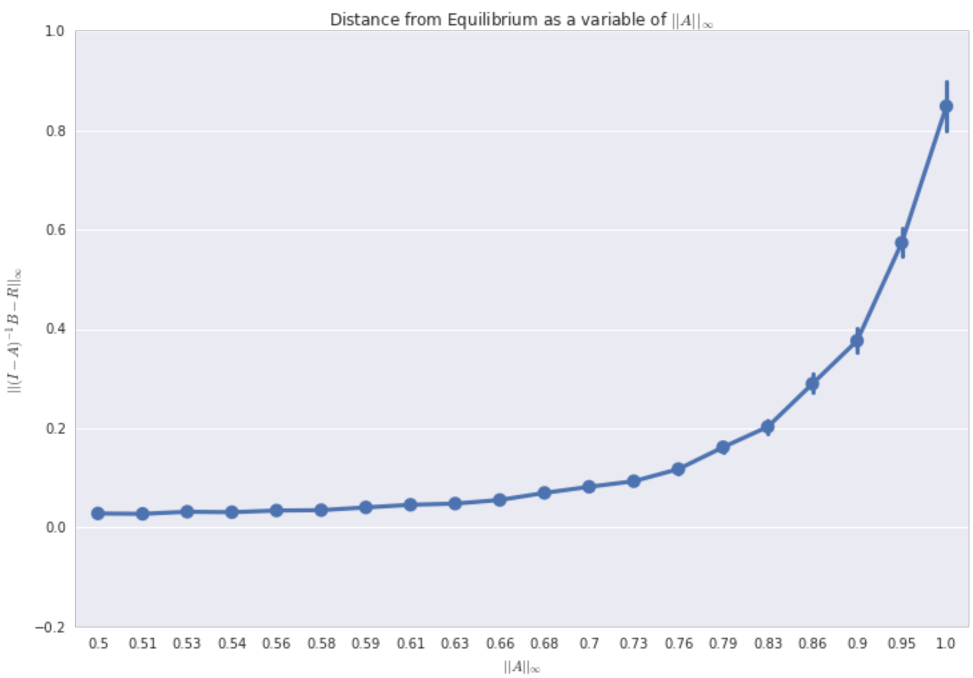


Figure 3.1

The experiments indicate that for such values of $\lambda(t)$, LIP-FJ indeed converges with high probability to $\vec{x}$ at a very slow rate. For larger values of $\lambda(t)$, e.g., for $\lambda(t) = 1 - \frac{1}{t^2}$, $\sum_{q=1}^{t}(1 - \lambda(q))$ converges to a constant value. Therefore, the expected distance to $\vec{x}^*$ stops decreasing after a finite number of rounds. If we set $\lambda(t)$ to some constant, aiming at improving the convergence time, LIP-FJ does not converge asymptotically to $\vec{x}^*$, due to the high variance of the stochastic process. Finally, we present our experimental work showing that in case that $\lambda(t) = 1 - 1/t$ the **LIM-FJ Model** converges with high probability to the equilibrium point at a very slow rate.



Distance from Equilibrium as a variable of $||A||_\infty$
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
$||(I-A)^{-1}B-R||_\infty$
0.5
0.51
0.53
0.54
0.56
0.58
0.59
0.61
0.63
0.66
0.68
0.7
0.73
0.76
0.79
0.83
0.86
0.9
0.95
1.0
$||A||_\infty$

Figure 3.2

Bibliography

[1] A.L. Barabási and R. Albert. *Emergence of scaling in random networks.* Science, 286(5439):509–512, 1999.

[2] A. Bhattacharyya, M. Braverman, B. Chazelle, and H.L. Nguyen. *On the convergence of the Hegselmann-Krause system*. *I*n *Proc. of 4th Innovations in Theoretical Computer Science Conference (ITCS '13), pages 61–66, 2013.*

[3] K. Bhawalkar, S. Gollapudi, and K. Munagala. Coevolutionary opinion formation games. In Proc. of the 45th ACM Symposium on Theory of Computing (STOC '13)*, pages 41–50, 2013.*

[4] D. Bindel, J.M. Kleinberg, and S. Oren. How bad is forming your own opinion? In Proc. of the 52nd IEEE Symposium on Foundations of Computer Science (FOCS '11)*, pages 57–66, 2011.*

[5] V.D. Blondel, J.M. Hendrickx, and J.N. Tsitsiklis. On Krause's Multi-Agent Consensus Model With State-Dependent Connectivity. IEEE Transactions on Automatic Control*, 54(11):2586–2597, 2009.*

[6] B. Bollobas. Random Graphs. *Cambridge University Press, 2001.*

[7] *A. Carvalho and K. Larson. A consensual linear opinion pool. In* Proc. of the 23rd International Joint Conference on Artificial Intelligence (IJCAI '13)*, pages 2518–2524, 2013.*

[8] *B. Chazelle and C. Wang. Inertial Hegselmann-Krause systems.* CoRR*, abs/1502.03332, 2015.*

[9] *Bernard Chazelle. The total s-energy of a multiagent system.* SIAM J. Control and Optimization*, 49(4):1680–1706, 2011.*

[10] *B. Chazelle. The Dynamics of Influence Systems.* 53rd Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages: 311–320, 2012.*

[11] *B. Chazelle. How Many Bits Can a Flock of Birds Compute?* Theory of Computing, vol 10*, pages: 421-451, 2014.*

[12] *M.H. DeGroot. Reaching a consensus.* Journal of the American Statistical Association*, 69:118–121, 1974.*

[13] *G. Deffuant, D. Neau, F. Amblard, G. Weisbuch. Mixing beliefs among interacting agents.* Adv. Complex Syst*, 3 (1–4), 87–98., 2000.*

[14] *D. Ferraioli, P.W. Goldberg, and C.Ventre. Decentralized dynamics for finite opinion games. In* Proc. of the 5th Symposium on Algorithmic Game Theory (SAGT '12)*, volume 7615 of* LNCS*, pages 144–155, 2012.*

[15] *N.E. Friedkin and E.C. Johnsen. Social influence and opinions.* Journal of Mathematical Sociology*, 15(3-4):193–205, 1990.*

[16] *J. Ghaderi and R. Srikant. Opinion dynamics in social networks with stubborn agents: Equilibrium and convergence rate.* Automatica*, 2014.*

[17] *Julien M. Hendrickx and Vincent D. Blondel. CONVERGENCE OF LINEAR AND NON-LINEAR VERSIONS OF VICSEKS MODEL. In* Proc. of the 17th International Symposium on Mathematical Theory of Networks and Systems (MTNS 06) *,page 12291240, 2006*

[18] *R. Hegselmann and U. Krause. Opinion dynamics and bounded confidence models, analysis, and simulation.* Journal Artificial Societies and Social Simulation*, 5, 2002.*

[19] *M.O. Jackson.* Social and Economic Networks. *Princeton University Press, 2008.*

[20] *D. Kempe, J.M. Kleinberg, and A. Kumar. Connectivity and inference problems for temporal networks.* Journal of Computer and System Sciences*, 64(4):820–842, 2002.*

[21] *J. Leskovec. Stanford network analysis package (snap).* http://snap.stanford.edu*, 2012.*

[22] *C.R. Lever.* Strategic Competition over Networks. *PhD Thesis, Stanford University, 2010.*

[23] *J. Lorenz. A stabilization theorem for dynamics of continuous opinions.* Physica A*, 355(1):217-223, 2005*

[24] *E. Seneta. Coefficients of ergodicity: Structure and applications.* Advances in Applied Probability*, 11(3):576–590, 1979.*

[25] *A. Tsang and K. Larson. Opinion dynamics of skeptical agents. In* Proc. of the 13th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS '14)*, pages 277–284, 2014.*

[26] *E. Wedin and P. Hegarty. A Quadratic Lower Bound for the Convergence Rate in the One-Dimensional Hegselmann-Krause Bounded Confidence Dynamics.* Discrete and Computational Geometry*, 53(2):478–486, 2015.*

[27] *E. Yildiz, A. Ozdaglar, D. Acemoglu, A. Saberi, and A. Scaglione. Binary opinion dynamics with stubborn agents.* ACM Transactions on Economics and Computation*, 1(4):19:1–19:30, 2013.*