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Variants of Stable Marriage
Algorithms, Complexity and Structural Properties

Georgios K. Stathopoulos

Supervisors
Ioannis Mourtos · Dimitrios Thilikos

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$\mu\Pi\lambda\forall$

To the memory of my brother, Nikos,
and of my father, Constantine.

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Georgios K. Stathopoulos with registry number 201004, is the author. The Lecturer, of the Department of Management Science and Technology (DMST) located in the Athens University of Economics and Business (AUEB), Ioannis Mourtos, and the Associate Professor of the Mathematics Department located in the National and Kapodistrian University of Athens (UOA), Dimitrios Thilikos are Supervisors of this thesis. Dr Mourtos and Dr Thilikos are also members of the four-member comitee whose members are also, Professor Lefteris Kirousis, of the Mathematics Department (UOA) and the Associate Professor, Stavros Kolliopoulos, of the Department of Informatics and Telecommunications. The committee approved this thesis on the 5^{th} of December, 2011.

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Foreword

I consider myself lucky to have been able to study for this graduate program that combines two undoubtedly exciting scientific fields: Mathematics and Computer Science.

Taking this opportunity, I would like to genuinely thank all the professors of the program for the stimulating lectures that they gave and continue to give, my fellow students for their help and friendliness and, the secretary of the program Mrs Chrysafina Hondrou for her readiness in dealing with all requests.

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Finally I would like to thank my mother, for her support during all the years of my studies which have been hard, but still, rewarding.

Abstract

Since 1962, when the Stable Marriage Problem was first proposed by David Gale and Lloyd Sharpley, there have been many theoretical results as well as numerous applications. Also, numerous variants of the problem have been proposed, thus establishing a very interesting class of problems. This thesis is a thorough review of the literature, concerning Stable Marriage and its variants, concentrating on the algorithmic, complexity and structural results that have been published so far. Also, a new result is presented, namely, an upper bound on the maximum number of stable matchings for the Stable Marriage problem and, via reductions this bound is extended to the Hospital-Residents and Stable Roommates problems.

KEYWORDS: Stable Marriage, Variants of Stable Marriage, Maximum number of stable matchings, Complexity, Structure

Περίληψη

Το πρόβλημα του Σταθερού Γάμου, προτάθηκε το 1962 από τους David Gale και Lloyd Sharpley. Από τότε, έχουν υπάρξει πολλά θεωρητικά αποτελέσματα αλλά και πολλές εφαρμογές. Επίσης, έχουν προταθεί αρκετές παραλλαγές του προβλήματος, δημιουργώντας μία πολύ ενδιαφέρουσα κλάση προβλημάτων. Η διπλωματική αυτή είναι μία επισκόπηση της βιβλιογραφίας, σχετικά με το πρόβλημα του Σταθερού Γάμου και των παραλλαγών του και, επικεντρώνεται στα αλγοριθμικά αποτελέσματα αλλά και στα αποτελέσματα πολυπλοκότητας και δομής που έχουν δημοσιευτεί ως τώρα. Επιπλέον, ένα καινούριο αποτέλεσμα παρουσιάζεται. Συγκεκριμένα, υπολογίζεται ένα άνω φράγμα για το μέγιστο πλήθος σταθερών ταιριασμάτων για το πρόβλημα του Σταθερού Γάμου και, μέσω αναγωγών αυτό το φράγμα επεκτείνεται στα προβλήματα Hospital-Residents και Stable Roommates.

ΛΕΞΕΙΣ-ΚΛΕΙΔΙΑ: Σταθερός Γάμος, Παραλλαγές του Σταθερού Γάμου, Μέγιστο πλήθος σταθερών ταιριασμάτων, Πολυπλοκότητα, Δομή

Contents

Chapter 1

Introduction

The Stable Marriage problem (or SM) was first introduced in the seminal paper by Gale and Sharpley [24] in order to approach the more general setting of the College Admissions problem (or Hospital - Residents problem). The problem involves two sets A, B, of equal size and for each member of each set a strict preference over each member of the other set. The question that arises is to find a matching between the two sets (i.e. a one-to-one mapping) so that no pair (a, b), $a \in A$ $b \in B$ not matched can improve its position (i.e. a prefers its partner to b or b prefers its partner to a). The general setting often assumes the sets being men and women and a stable matching being a collection of marriages, hence the name Stable Marriage problem. A polynomial algorithm for SM was presented in [24] and it was shown to produce an optimal solution under a natural dominance relation.

Since the publication of [24] there have been many developments both in the original problem and in the numerous variants and generalizations that have been proposed. For example the set of solutions of the original problem (i.e. stable matchings) has proved to possess a rich structure which can be exploited for the solution of many related problems. These developments have many applications in the economic, social and scientific fields.

The contribution of this thesis is a thorough review of known structural, algorithmic and complexity results concerning the Stable Marriage problem and its variants. Given the rich literature on the subject some results might be missed; still the results presented here are at least a coherent set. Another key outcome is something that to the best of my knowledge, has not appeared in the literature so far, namely, upper bounds on the maximum number of stable matchings for the Stable Marriage and the Stable Roommates (or SR) problem. In particular it is shown that if $f(n)$ and $r(n)$ is the maximum number of stable matchings for these two problems, then for all $n \geq 4$, $f(n) \in O\left(\frac{n!}{2^n}\right)$ and, for all $n = 2k \geq 8$, $r(n) \in O\left(\frac{n!}{\frac{n}{2}!2^n}\right)$.

Also, via reductions to SM and SR, upper bounds for the Hospital-Residents and other problems can be derived.

Let us highlight the structural elements in the definition of SM, so that modifying certain of them, leads to the definition of interesting variants. We repeat the definition of SM so that these elements become more apparent. SM involves

- a collection of (two) sets;
- exogenously defined, strict, preferences of the elements of one set over *all* elements of the other;
- matchings;
- a stability criterion.

The material is organized in three chapters. In the second chapter, titled Stable Marriage, the classical Stable Marriage problem is presented together with all two-sided generalizations and variants of it. That is, the following problems result from SM by dropping some constraints, but keeping the constraint of looking at matchings between two distinct sets.

- In the Stable marriage with incomplete lists (or SM-IL), we drop the constraint that all the preference lists contain all the elements of the other sex i.e. we allow unacceptable partners.
- In the Stable Marriage with ties (or SM-T), we drop the constraint that the preference lists are strict i.e. we allow *indifference*. Another form of indifference is to allow preferences to take the form of partially ordered sets (or posets). We denote this problem by SM-P.
- In the problem denoted by SM-ILT we allow incomplete lists and ties.
- In the Hospital - Residents problem (or HR), we drop the constraint that the two sets are of equal size, we allow unacceptable partners and, in our setting,we allow each hospital to have one or more places to fill while each resident can be assigned to at most one hospital. A matching is a partial mapping from the set of the residents to the set of the hospitals. The definition of a stable matching changes accordingly.
- In the Hospital - Residents with couples (or HR-C), we allow couples of residents to be married, thus having a joint preference list for pairs of hospitals.
- We also look at the Hospital-Residents with ties (or HR-T) and the HR problem where preferences are posets (HR-P).
- In the Many-to-Many problem (or MM), which is a generalization of HR, we allow all members of both sets to have one or more places to fill. Again we look at the version with ties (MM-T).

- In the stable allocation problem (or SAL), a generalization of MM, we modify the definition of matching so that it doesn't involve the notion of "matched or not matched" between two members of the distinct sets[1] but rather an arbitrary integer[2] or even a real number.

- Furthermore the above problems admit generalizations, inspired from economics, where the preferences are more elaborate. In the paragraph titled Money Markets we give references for these generalizations.

- Changing the stability criterion results in another variant which we discuss in the section titled Exchange Stability.

- Adding the constraint that only one side has preferences produces another two-sided variant.

In the third chapter, starting with SM, we drop the constraint of the two distinct sets thus having only one set of even order, where every member of the set has a strict preference list over all other members of the set. The problem of matching all the members of the set into couples so that the matching is stable is the Stable Roommates problem (or SR). If SM is the "stable" version of bipartite matching, SR is the stable version of a matching in an arbitrary graph.

Here is the list of variants of SR examined here:

- Naturally enough, in the Stable Roommates with ties problem (or SR-T) we allow indifference in the lists i.e. non strict preference; of course there is the case of incomplete lists (SR-IL) also.

- The Stable Partnership problem is another generalization of SR where each member of the set has a so-called substitutable choice function (which serves as a preference function) and stability is interpreted in a natural way.

- In the 3-Person Stable Assignment problem (or 3PSA), we wish to partition the original set into groups of 3 persons.

- The Stable Activities problem (or SAC) is a generalization where much like in SAL we allow multiple edges between two partners in a matching. In the Stable Multiple Activities problem (or SMA) we allow multiple partners in a matching, much like in MM.

[1] this can be expressed by an edge between the members or even by the number 0 or 1
[2] or multiple edges

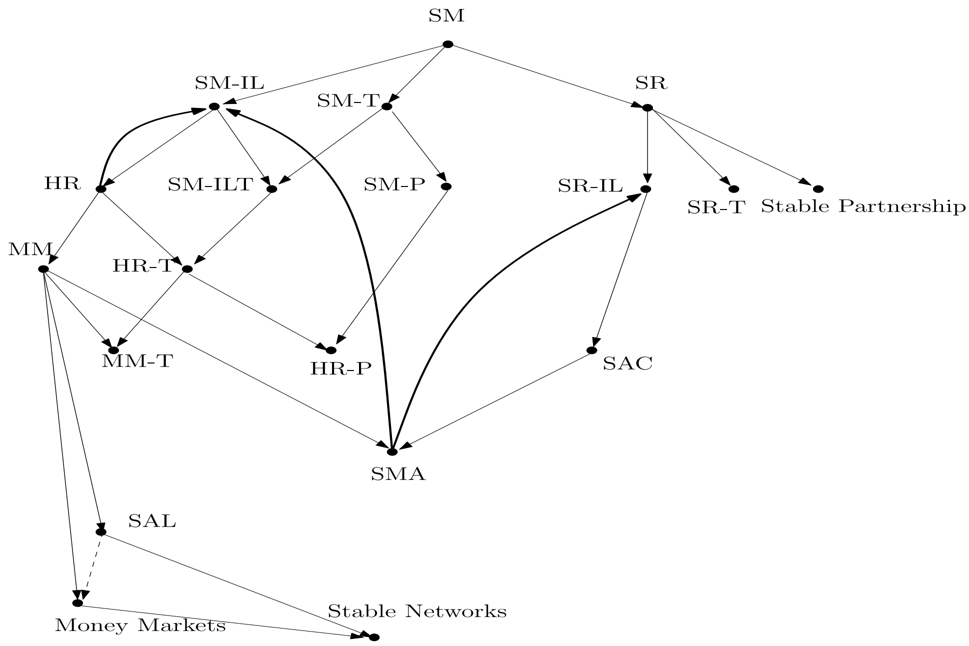

Figure 1.1: Generalization Graph.

In the last chapter the generalizations of SM go towards another direction. In the Three Sided SM (or 3DSM) we allow three sets with various preference schemes and natural stability criteria.

Finally in the Stable Networks problem (or SN) we generalize the notion of three sets into an arbitrary number of sets.

Detailed definitions for all the problems are provided in the corresponding sections.

Figure 1.1 shows some relations between the problems. In particular if there is a directed edge (downstream arcs) from problem B to problem A, then B reduces to A with a usually obvious reduction which shows that A is a generalization of B. The dashed line means that the generalization treats a special case of Stable Allocation (the discrete version). In some cases there are reductions that go upstream, represented in the graph by thicker, curved, upstream arcs (these aren't generalizations). For example in Section 2.2.2.1 we show that HR reduces to SM. The rest of the upstream reductions can be found in Section 3.2.4. An open problem posed by Gusfield and Irving is whether there is such a reduction from SR to SM.

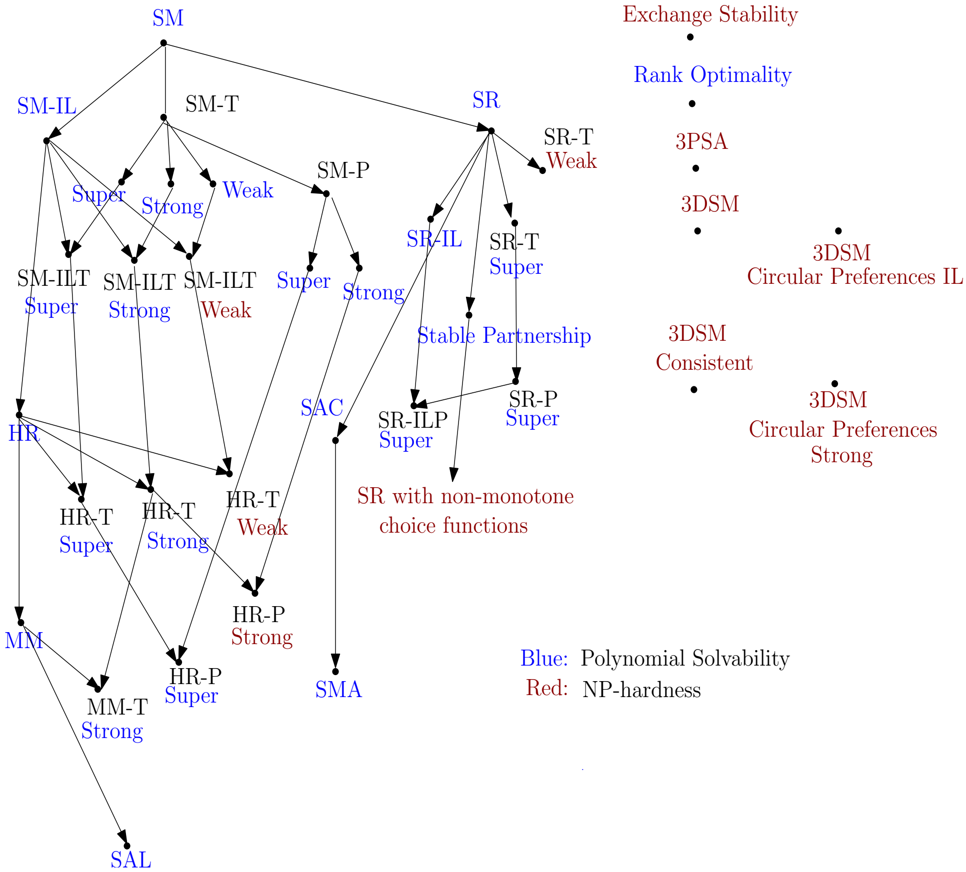

Figure 1.2: Classification Graph.

In Figure 1.2 we illustrate a classification, of most of the problems we discuss in this thesis, into polynomial-time solvable and NP-hard. All the problems mentioned in the figure ask whether we can find a stable matching except for SM-ILT(weak) and HR-T(weak) where we ask whether we can find a stable matching of maximum cardinality. At this point, it might not be obvious what we mean with each node since not everything has been defined and after reading all the chapters should remove any vagueness. The edges, as in Figure 1.1, show the obvious reductions between problems.

Finally, in Figure 1.3, we present compactly for which problems it is known that the set of solutions forms a lattice and for which problems it is known that the set of solutions does not.

All three figures are depicted early enough to allow an initial understanding of the broad spectrum of SM variants, hence showing the motivation for working on this thesis. In addition, these figures may well serve as a reference point while the reader is gradually exposed to a more detailed discussion of each SM variant.

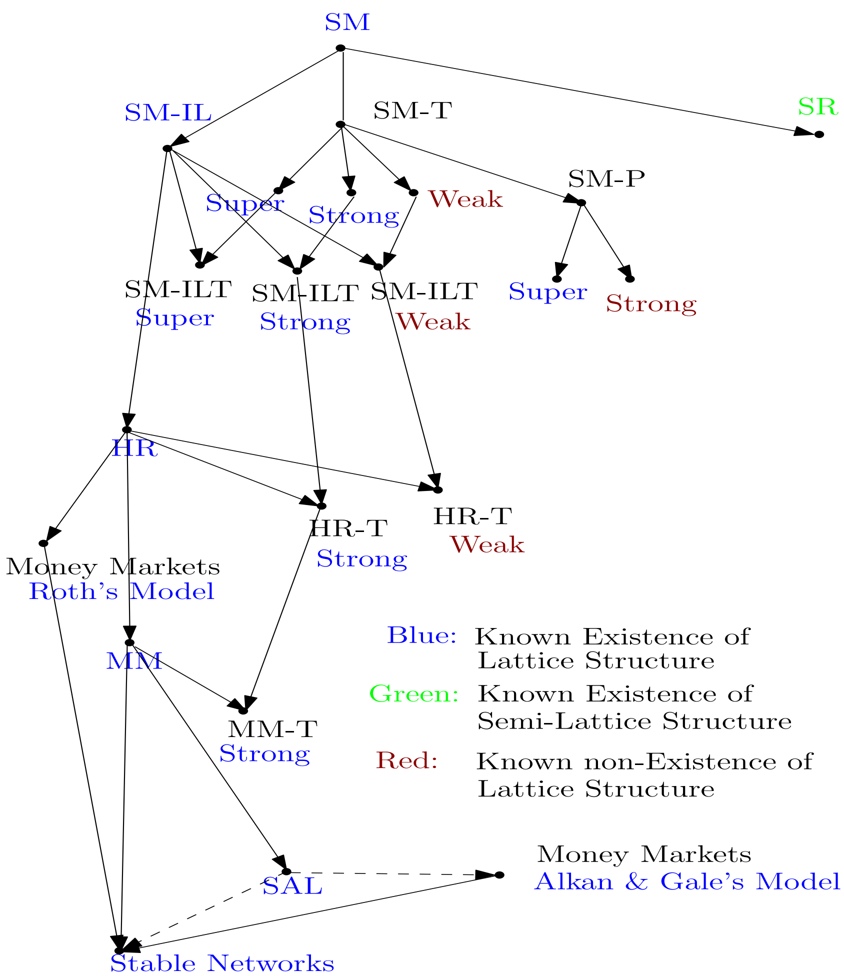

Figure 1.3: Lattice Graph.

Chapter 2

Stable Marriage

2.1 Stable Marriage Problem

2.1.1 Definitions

As stated in the introduction, SM comprises two sets of equal and finite size, namely, the men and women. For each individual there is a strict preference list containing all the individuals of the opposite sex. If the person x is above y on z' s list we say that z prefers x to y and we denote this fact by $x >_z y$. If an instance, I, of SM involves n men and n women we say that it is of order (or of size) n. In this setting, a matching is a one-to-one correspondence between men and women. We denote a pair of man x with woman y, by (x, y). If (x, y) is a matched pair in matching M we denote x by $p_M(y)$ and y by $p_M(x)$, i.e. we use the same symbol, p_M, for the one-to-one correspondence and for its inverse.

Definition 2.1. A stable matching, M, is a matching satisfying the requirement that, for every unmatched pair (x, y), one of the following holds

(a) $p_M(x) >_x y$

(b) $p_M(y) >_y x$

i.e. at least one of x, y prefers its partner in the matching to y, x respectively.

In the following section, we will prove that a stable matching always exists and that it can be found in linear time (with respect to the input, that must contain all the preference lists). In a matching, M, any unmatched pair (x, y) for which neither of $(a), (b)$ in the above definition holds is called a blocking pair for M. If a blocking pair exists for M, then M is called unstable. If a pair is matched in some stable matching, then it is called a stable pair.

A person x is said to prefer a matching M to the matching M' if $p_M(x) >_x p_{M'}(x)$. Let M, M' be two stable matchings for the same SM instance, I. Then M is said to dominate M' from the man's point of view, if for every man x, either x prefers M to M' or $p_M(x) = p_{M'}(x)$. We use the notation $M \preceq M'$ and refer to this relation as the man dominance relation. M is said to strictly dominate M' if it dominates M' and exists a man x so that x prefers M to M'. This is denoted by $M \prec M'$. The definition of the woman dominance relation is analogous. The man and woman dominance relations are partial orders on the set $\mathcal{M}$ of stable matchings of I. In fact, $(\mathcal{M}, \preceq)$ is a distributive lattice.

2.1.2 Problems

Many problems arise from just the definitions of the previous section. The first and foremost is whether a stable matching always exists and, if it does, whether it is unique. We will prove that it does exist through a constructive proof presented already in [24] and we will give the necessary and sufficient conditions for its uniqueness. We shall refer to these problems as the *Existence* and *Uniqueness* problems.

When the stable matching is not unique a natural question is to find the most or the least dominant stable matching under the man or woman dominance relation. We shall refer to this problem as the *Dominance* problem.

In the weighted version of SM, i.e. where we assign a weight to each man-woman pair and the weight of a matching is the sum of the weights of the matched pairs, the question is to find a stable matching of minimum weight. We shall refer to this problem as the *Min-Weight* problem. The case where the weight of a pair (x, y) is the sum of y's position on x's list and of x's position on y's list is known as *Egalitarian* SM since it treats men and women equally.

Another natural question is to count in an efficient way the cardinality of $\mathcal{M}$[1] and to efficiently generate all the elements of $\mathcal{M}$. We shall refer to these problems as the *Cardinality* and *Enumeration* problems.

The problem of finding the maximum cardinality of $\mathcal{M}$ for an instance of order n is an open problem first posed by Knuth in [42] (problem 5). We shall refer to this problem as the *Maximum Cardinality* problem.

Other problems are finding all stable pairs efficiently and checking if a pair is stable, henceforth referred to as the *Pair Stability* problem.

[1] We will show that the number of stable matchings can grow exponentially as the order of the instance increases, so this question makes perfect sense.

2.1.3 Complexity Results

We now mention the main complexity results for the problems appearing in the previous section. The corresponding algorithms will be presented in the next section.

Finding a stable matching for an SM instance of order n takes $O(n^2)$ time and of course it is linear and optimal on the size of the instance since the input must contain the preference lists and thus we need n^2 time just to read the input. The Gale-Sharpley algorithm that achieves this optimal result by constructing a solution, actually constructs the optimal solution under the man dominance relation.

Exploiting the structure of $\mathcal{M}$, Irving and Leather proved in [37] that counting the number of stable matchings is #P complete. Gusfield and Irving in [27] present an algorithm for finding all stable matchings in $O(n^2 + n|\mathcal{M}|)$ time.

A lower bound for the maximum cardinality of $\mathcal{M}$ is given in [37]. If $f(n)$ is the maximum number of stable matchings in an instance of order n, where n is a power of two, then it is shown that $f(n) > \frac{2.28^n}{1+\sqrt{3}}$. This result is generalized for arbitrary n's by Thurber in [66]. In this thesis a relevant new result is presented, namely, an upper bound for $f(n)$, for all $n \geq 4$. Specifically, it is proved that $f(n) \in O(\frac{n!}{2^n})$ where $n!$ is the cardinality of the set of all matchings (stable or not).

By exploiting the structure of $\mathcal{M}$, we can find all stable pairs in $O(n^2)$ time as presented in [27]. Combined with the result, established in [52] by Ng, that checking a pair for stability takes $\Omega(n^2)$ time the above result is quite remarkable.

A minimum weight stable matching can be found in $O(n^3(\log n)^2)$ time as shown in [19] by Eirinakis et al.

2.1.4 Structure and Algorithms

2.1.4.1 Existence and uniqueness problems

The fundamental algorithm by Gale and Sharpley may be expressed in an informal way, in terms of proposals, engagements and marriages and that makes it easy to grasp. Unless otherwise specified, the setting is an SM instance of order n.

We make the following assumptions. At any time, a person can be engaged or free and once a woman is engaged she can never be free. Now the algorithm can be viewed as a series of proposals from men to women until the desired result occurs. In particular, we assign each person to be free, and while a man is still free we choose such a man[1] and this man proposes to the first woman on his list that he has not yet proposed to.

[1] This nondeterminism will be proved to be of no consequence

A woman that is free always accepts a proposal and thus becomes engaged to the corresponding man. If at any time she receives a proposal from a man whom she prefers to her fiancée, she breaks off her current engagement and becomes engaged to the more gifted man else she rejects the proposer and he moves to the next choice on his list. We will prove that this algorithm always terminates, that it runs in $O(n^2)$ time, that the result is a stable matching and that the result is the man-optimal stable matching.

Theorem 2.2. *The Gale-Sharpley Algorithm always terminates in $O(n^2)$ time and upon termination the engaged pairs constitute a stable matching.*

Proof. It is immediate that no man can be rejected by all the women since that would mean that when this man proposed to the last woman on his list all women were engaged. Indeed this man would subsequently propose to all the women in his list and a woman only rejects(also counting the breaking of an engagement as a rejection) when she is engaged. Each step of the algorithm involves one proposal and no man proposes twice to same woman so termination is guaranteed in $O(n^2)$ steps. We accept that all the other operations such as a woman comparing two men on her list take constant time[1] and so the total time is $O(n^2)$. No person is ever engaged to two persons and the algorithm terminates when no man is free so the engaged pairs upon termination constitute a matching. Assuming that that there is a blocking pair, (x, y), it is immediate that y must have rejected x at some point. But then from that point on y can have only better partners than x, a contradiction. □

Theorem 2.3. *All possible executions of the Gale-Sharpley algorithm yield the same stable matching (thus the nondeterminism during the execution is of no importance) and this is the man-optimal stable matching i.e. every man has the best partner he can have in any stable matching.*

Proof. Suppose that an arbitrary execution of the algorithm yields the stable matching M. We will prove that M is man optimal and since the execution was arbitrary this will mean that all executions yield the same result.[2] Suppose that in stable matching M' man x has a better partner than $p_M(x)$. Then at some point during the execution of the algorithm, $p_{M'}(x)$ rejected x for man y. We assume without loss of generality that this was the first occasion that a woman rejected a stable partner. This means that y prefers $p_{M'}(x)$ to $p_{M'}(y)$ since no other stable partner had been rejected before the matching of y to $p_{M'}(x)$ in M. But since $p_{M'}(x)$ rejected x for y she prefers y to x. So M' is unstable, a contradiction. □

It is immediate from the above that under the man dominance relation, the stable matching that the Gale-Sharpley algorithm yields, M_0, is the most dominant stable matching. That is, if M is a stable matching then $M_0 \preceq M$.

[1] For a detailed analysis on these subjects see paragraph 1.2.3 of [27]
[2] Proving the uniqueness of the man optimal stable matching is trivial

Theorem 2.4. *The man-optimal stable matching is woman-pessimal i.e. each woman has the worst partner she can have in any stable matching*

Proof. Assuming that this does not hold, let M be a stable matching and x a woman such that $p_{M_0}(x) >_x p_M(x)$ then unless

$$p_M(p_{M_0}(x)) >_{p_{M_0}(x)} x$$

M is not stable. But M_0 is man-optimal and thus the converse inequality holds, a contradiction. □

Running the Gale-Sharpley algorithm with the women as the proposers we can get corresponding results about the woman-optimal and man-pessimal stable matching, M_z. In particular if M is a stable matching and $\preceq$ is the man dominance relation then $M \preceq M_z$. Now the condition for the uniqueness of a stable matching is immediate, i.e. $M_0 = M_z$.

We state now a theorem that we use later without giving its proof (for a proof see [27] or [37]).

Theorem 2.5. *Let I be an SM instance. If (x, y) is a matched pair in some stable matching M of I then there is no stable matching, N, of I where both x, y, prefer M to N or both x, y, prefer N to M.*

2.1.4.1.1 Extended Gale-Sharpley

The fact that the Gale-Sharpley algorithm is independent of the occasional sequence of proposals, and that its outcome is the man-optimal and woman-pessimal stable matching, allows us to make some modifications to the algorithm resulting to the known as extended Gale-Sharpley algorithm. In particular, while running the algorithm we observe that a proposal from man x to woman y means that there is no stable partner of x better than y and no stable partner of y worse than x. This means that if at the moment that x proposes to y we delete all entries above y in x's list and all entries below x in y's list the algorithm may proceed without any loss of information, i.e. it still yields the man-optimal stable matching. The final form of the preference lists after the execution of the above algorithm will be called GS-lists. It is trivial to see that the man-optimal stable matching results if we assign each man to the first woman in his GS-list and that if y is the first woman in x's GS-list then x is the last man in y's GS-list.

2.1.4.2 Maximum Cardinality Problem

Let $f(n)$ denote the maximum number of stable matchings for instances of size n. Then we can show that $f(n) \geq 2f^2(\frac{n}{2})$ thus establishing with an inductive argument the exponential growth of f for $n = 2^k$. Indeed, we can easily construct an instance of size 2 with two stable matchings. This instance (denoted F_2) is

men			women		
1	1	2	1	2	1
2	2	1	2	1	2

i.e. in the left table we have the preferences of men 1,2 over women 1,2 and in the right table we have the preference lists of the women over the men. Using this instance, we can derive the desired result. Indeed, let an instance, I, of order n have $f(n)$ stable matchings. Then it can be expressed by two ranking arrays R_I^m, R_I^w; the preferences of men over the women and of the women over the men. Let $M_I, M_{I'}$ be the sets of men for instances I, I' and $W_I, W_{I'}$ the women, where I' is defined to be a duplicate of I. That is, for every man x in I we define a man x' in I' and we do the same for the women. Also we define the ranking arrays of I' to be $R_{I'}^m$ and $R_{I'}^w$. Then the following tables capture the intricacy of our construction.

men			women		
M_I	R_I^m	$R_{I'}^m$	W_I	$R_{I'}^w$	R_I^w
$M_{I'}$	$R_{I'}^m$	R_I^m	$W_{I'}$	R_I^w	$R_{I'}^w$

What we mean by these tables is that the men's preferences over the women in the new instance result from putting for M_1 the ranking array R_I^m and next to it the isomorphic array of R_I^m , namely, $R_{I'}^m$ etc. Now combining any stable assignment of I with any stable assignment of I' gives us a stable assignment for the derived instance. Indeed no pair $(x, y) \in I$ or I' can be a blocking pair since the ranking arrays of I, I', have been used as they are. Also no pair (x, y) with $x \in M_I$ and $y \in W_{I'}$ (or $x \in M_{I'}$ and $y \in W_I$, respectively) can be a blocking pair since x prefers all women in W_I to all women in $W_{I'}$ (x prefers all women in $W_{I'}$ to all women in W_I respectively). Also if we match any man x in any stable matching, M, of I with $(p_M(x))'$ and any man x' in any stable matching, M', of I' with $(p_{M'}(x'))'$[1] then the result is a stable matching for the derived instance.(The proof is similar to the one proceeding it.) This means that we can construct $2f^2(n)$ stable matchings for the derived instance of order $2n$. From this point we can inductively double the instance F_2 in order to derive a size-n instance with at least 2^{n-1} stable matchings. In fact a detailed proof appears in [66], by Thurber, of why these instances have approximately $\frac{2.28^n}{1+\sqrt{3}}$ stable matchings. It is conjectured that this is approximately the value of f(n) at least for $n = 2^k$.

[1] we use the notation $(x')' = x$

We shall now prove that $f(n) \in o(n!)$ for $n = 2^k$. First let us mention that Eilers [18] has proved through exhaustive computer calculations that $f(4) = 10 = \frac{5}{12}4!$.

Definition 2.6. Let I be an instance of order n. Then for any pair (i, j) we define the (i, j) projection of I, denoted $I(i, j)$, to be all the preference lists of I, from which we have removed the entries of man i and woman j, apart from the preference lists of man i and woman j which we remove completely.

Definition 2.7. Let $\mathcal{M}$ be the set of stable matchings of I. Then we define the (i, j) projection of $\mathcal{M}$, denoted $\mathcal{M}(i, j)$, to be all the stable matchings that contain the matched pair (i, j), from which we remove (i, j).

Proposition 2.8. *If $N \in \mathcal{M}(i, j)$ then N is a stable matching for the instance $I(i, j)$.*

Proof. If N is not a stable matching for $I(i, j)$ then

$$\exists (m, k), (n, l) \in N : k <_m l \text{ and } n <_l m.$$

But then, since m, k, l, n are contained in the preference lists of $I(i, j)$ they will be contained in the preference lists of I with the same order, meaning that (m, l) will be a blocking pair for the stable matching $N \cup \{(i, j)\}$ of I, a contradiction. □

In the same manner we can define projections of $I, \mathcal{M}$ of higher order.

Let n be a power of 2. Then considering a random $\frac{n}{2}$-projection, i.e. $I(i_1, j_1)(i_2, j_2)...(i_{n/2}, j_{n/2})$ we have that

$$|\mathcal{M}(i_1, j_1)(i_2, j_2)...(i_{n/2}, j_{n/2})| \le f(\frac{n}{2}).$$

We also observe that $I(i_1, j_1), ...(i_{n/2}, j_{n/2}) = I(i_1, j_{\sigma(1)})...(i_{n/2}, j_{\sigma(n/2)})$ for every permutation, σ, of the elements $1, .., \frac{n}{2}$. Meaning that

$$\left| \bigcup_{\sigma \text{ permutation}} \mathcal{M}(i_1, j_{\sigma(1)})...(i_{n/2}, j_{\sigma(n/2)}) \right| \le f\left(\frac{n}{2}\right).$$

Indeed $M(i_1, j_{\sigma(1)})...(i_{n/2}, j_{\sigma(n/2)})$ are sets of stable matchings for the same instance so their total cardinality will be at most $f(\frac{n}{2})$.

Taking the complements $\{i_{\frac{n}{2}+1}, ..., i_n\} = \{1, ..., n\} \backslash \{i_1, ..., i_{\frac{n}{2}}\}$ and $\{j_{\frac{n}{2}+1}, ..., j_n\} = \{1, ..., n\} \backslash \{j_1, ..., j_{\frac{n}{2}}\}$ and repeating the above process we have

$$\left| \bigcup_{\sigma \text{ permutation}} \mathcal{M}(i_{\frac{n}{2}+1}, j_{\sigma(\frac{n}{2}+1)})...(i_n, j_{\sigma(n)}) \right| \le f\left(\frac{n}{2}\right).$$

All of the above mean that in $\mathcal{M}$, at most $f(n/2)$ permutations of $j_1, .., j_{n/2}$ (corresponding to $i_1, ..., i_{n/2}$) appear. Also at most $f(n/2)$ permutations of $j_{n/2+1}, .., j_n$ (corresponding to $i_{n/2+1}, ..., i_n$) appear. This means that at most $f^2(\frac{n}{2})$ stable matchings exist such that $j_1, .., j_{n/2}$ are mapped to $i_1, ..., i_{n/2}$ and $j_{n/2+1}, .., j_n$ are mapped to $i_{n/2+1}, ..., i_n$. We observe that there are $\binom{n}{\frac{n}{2}}$ different ways to do the above process and cover all matchings in $\mathcal{M}$, so that

$$f(n) \leq f^2\left(\frac{n}{2}\right)\frac{n!}{\frac{n}{2}!\frac{n}{2}!}$$

Now we can easily show that for $n = 2^k,\ n \geq 4 : f(n) \leq \left(\frac{5}{12}\right)^{\frac{n}{4}}(n!)$

- Indeed for $k = 2$ it is true.
- If it is true for $n = 2^k$ then for $2n = 2^{k+1}$ we have

$$f(2n) \leq f^2(n)\frac{2n!}{n!n!} \leq \left(\left(\frac{5}{12}\right)^{\frac{n}{4}}\right)^2 (n!)^2\frac{2n!}{n!n!} = \left(\frac{5}{12}\right)^{\frac{2n}{4}}(2n!).$$

It is now immediate that $f(n) \in o(n!)$.

2.1.4.3 The Structure of $\mathcal{M}$

It is well known (a detailed proof can be found in [27]) that $(\mathcal{M}, \preceq)$ is a distributive lattice[1] where if $M, N \in \mathcal{M}$ then $M \vee N$ is the stable matching that results if we assign to each man the worst of its partners in M, N, and $M \wedge N$ the stable matching that results if we assign to each man the best of its partners in M, N. We omit proofs of these facts and we move on to some highly interesting and widely used structural results concerning the rotations and the rotation poset. Of course, these results are made possible because of the lattice structure of $\mathcal{M}$, even though this will not be obvious in our approach.

We have already mentioned that, in M_0, each man is partnered with the first woman on his GS-list. Any other stable matching is strictly dominated by M_0 so that in any other matching at least one man, x, must sacrifice his man-optimal stable partner, y, for a woman further down on his GS-list, say y'. This means that $p_{M_0}(y')$ must also move down on his GS-list and continuing in the same manner we create a sequence of men that either cycles or reaches a man who cannot move down on his GS-list (for example, if his GS-list contains only one element). If the case is that the men chosen this way move down only one place in their list and the sequence cycles, we call such a sequence

[1] A distributive lattice is a partial order in which each pair of elements a, b, has a greatest lower bound denoted $a \wedge b$ and a least upper bound denoted $a \vee b$. Also, the distributive laws hold, i.e.

- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
- $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

of men a *rotation*. The above observations suggest a way of generating stable matchings from M_0: successively identify and *eliminate* rotations (a precise definition of a rotation elimination will be given shortly). A next goal would be to generalize the concept of the GS-lists and the rotations in order to try to generate all the stable matchings. This is the idea that is explored successfully in [37] resulting in a compact representation of $\mathcal{M}$ and an efficient way to generate all stable matchings. In the rest of this section we follow [37] , outlining the basic features of this construction.

Definition 2.9. We define a set of reduced preference lists to be a set of lists obtainable from the original preference lists by one ore more deletions so that no list is empty and woman y is deleted from man's x list iff x is deleted from y's list. Relative to such a set we define $f(x), s(x), l(x)$ to be the first, second and last entry on person's x list respectively.

Definition 2.10. A set of reduced preference lists will be called stable[1] if for every man x and woman y

- $y = f(x)$ iff $x = l(y)$
- y is absent from x's list iff x is below $l(y)$ on y's original preference list.

It can be proved that the GS-lists form a stable set. The notion of a stable set generalizes that of the GS-lists and naturally enough the following lemma holds.

Lemma 2.11. *If relative to a stable set, each man, x, is partnered by $f(x)$ then the result is a stable matching.*

Proof. It is immediate that it is a matching since if for men x, x', we have $y = f(x) = f(x')$, then $x = l(y) = x'$, from the definition of the stable set.

Also if x prefers y to $f(x)$ then y has already been deleted from x's list, therefore, y prefers $l(y)$ to x from the definition of the stable set, meaning that there is no blocking pair. □

The following lemma establishes that any deletions resulting to a stable set are inconsequential.

Lemma 2.12. *If relative to a stable set, woman $f(x)$ is above y in x's original preference list and y has been deleted from x's list then there is no stable matching where x and y are partners.*

Proof. By the above lemma $(x, f(x))$ is a stable pair appearing in the stable matching resulting from the corresponding stable set, say M. If (x, y) is also a stable pair, say in stable matching M', then x prefers M' to M. But also y has been deleted from x's list in that stable set, meaning that y also prefers M' to M. Using theorem 2.5 we see that this is a contradiction. □

[1] for brevity called a stable set

Now, we generalize the notion of rotation to cover all stable sets.

Definition 2.13. An ordered sequence $(m_0, w_0), ..., (m_{r-1}, w_{r-1})$ of pairs is said to be a rotation if, relative to a stable set,

$$w_{i+1} = f(m_{i+1}) = s(m_i)$$

where $i + 1$ is taken modulo r. The rotation is said to be exposed in the corresponding stable set.

The following lemma demonstrates the usefulness of the rotations.

Lemma 2.14. *Let $\mathcal{L}$ be a stable set and let S be the corresponding stable matching. If S' is a stable matching in which a man m has a worse partner than $f(m)$ then there is a rotation exposed in $\mathcal{L}$ all of whose male members prefer S to S'.*

Proof. By lemma 2.12, if $f(m)$ is the only entry in m's $\mathcal{L}$-list then we have nothing to prove. Otherwise, we form the sequence $\{(m_i, w_i)\}$ where

- $m_0 = m$
- $w_i = f(m_i)$
- $m_{i+1} = l(s(m_i))$.

Since $w_0 = f(m_0)$, it follows that $(m_0, w_0) \in S$. By lemma 2.12 and our lemma's assumptions, $p_{S'}(m_0) \leq_{m_0} s(m_0) = f(m_1) = w_1$. Hence for the stability of S', $p_{S'}(w_1) \geq_{w_1} m_0 >_{w_1} m_1$, where the last inequality holds because $m_1 = l(w_1)$ and w_1 has not been deleted from m_0's list. So, by theorem 2.5, $p_{S'}(m_1) <_{m_1} w_1$ and therefore by lemma 2.12, $s(m_1)$ is defined. This argument can be repeated, establishing that $s(m_i)$ is defined for all i and that all the m_i have worse partners in S' than in S. Now the constructed sequence must cycle eventually, meaning that there is a subsequence that forms a rotation exposed in $\mathcal{L}$ with the required property. □

An immediate corollary of the above lemma is that in any stable set either at least one rotation is exposed or the corresponding stable matching is M_z. Another consequence is that for an exposed rotation, in any stable matching that one man of the rotation has a worse partner, all men in the rotation have worse partners.

We now introduce the concept of a rotation elimination.
Let $((m_0, w_0), ..., (m_{r-1}, w_{r-1})$ be a rotation exposed in the stable set $\mathcal{L}$. If all successors of m_i are deleted from w_{i+1}'s list in $\mathcal{L}$ and w_{i+1} is deleted from the corresponding men's lists we say that the rotation has been eliminated. It can be proved that if a rotation is eliminated then the resulting set of reduced preference lists forms a stable set. We now state a crucial lemma.

Lemma 2.15. *Let I be an SM instance. There is an one-to-one correspondence between the stable sets and the stable matchings.*

Proof. Given a stable set we can construct a unique stable matching as in Lemma 2.11.

Given a stable matching $S = \{(m_0, w_0), ..., (m_{n-1}, w_{n-1})\}$ we construct a stable set as follows. We start from the stable set $\mathcal{L}_0$ which consists of the GS-lists and corresponds to M_0. Let S be different from M_0. Then for some i, $w_i \neq f_{\mathcal{L}_0}(m_i)$. By lemma 2.12, w_i is in m_i's GS-list and by lemma 2.14, the exposed rotation generated by m_i is such that all of its male members prefer M_0 to S. Eliminating this rotation and repeating the argument as many times as needed we can produce a stable set where $w_i = f(m_i)$ for all i. We observe that we constructed a stable set corresponding to a specified matching through a sequence of rotation eliminations and the matching was arbitrary. □

Now it can be proved that

Lemma 2.16.

1. *a pair (x, y) can belong to at most one rotation;*
2. *if (x, y) belongs to a rotation then it is a stable pair;*
3. *if (x, y) belongs to a rotation then in a stable set obtained by a sequence of rotation eliminations y is absent from x's list iff the rotation containing it has been eliminated.*

We can observe that a rotation ρ might be exposed in several stable sets, all of which can be constructed by a sequence of rotation eliminations as stated in Lemma 2.15.

Definition 2.17. If rotation ρ cannot be exposed in a stable set constructed by a sequence of rotation eliminations unless rotation π has been eliminated we say that π is a predecessor of ρ and denote it by $\pi < \rho$.

The above relation defines a partial order on the set of rotations and we call this partial order the rotation poset for the corresponding SM instance.

We define π to be an immediate predecessor of ρ if there exists no σ such that $\pi < \sigma < \rho$ and $\pi < \rho$ holds. A subset R of a poset is closed if $\pi \in R \Rightarrow \forall \sigma < \pi$, $\sigma \in R$.

An alternative representation for the poset is by an acyclic directed graph with a node representing each rotation and an arc from π to ρ if π is an immediate predecessor of ρ.

It can be proved that there is an one-to-one correspondence between the stable matchings and the closed subsets of the rotation poset. Also all the rotations can be found in $O(n^2)$ time from just the preference lists and an even more sparse graph (a digraph actually with $O(n^2)$ edges denoted $G(\mathcal{M})$) than the rotation poset, that still maintains the one-to-one correspondence between its closed subsets and the stable matchings, can be constructed in $O(n^2)$ time. These results combined with Lemma 2.16(1) that implies that there are at most $\frac{n(n-1)}{2}$ rotations justifies our claim that a compact representation of $\mathcal{M}$ exists (these results can be found in [27]).

Definition 2.18. An antichain, A, in a poset $(P, \leq)$ is a subset of P containing no elements π, ρ such that $\pi < \rho$. Given such an antichain, the closure A^* is defined as

$$A^* = \{\pi \in P \mid \exists \rho \in A : \pi < \rho\}$$

For any closed subset C of P, the unique antichain A such that $A^* = C$ is called the spanning antichain of C.

Theorem 2.19. *For any SM instance there is an one-to-one correspondence between stable matchings and antichains.*

Proof. Given an antichain, A, we can eliminate all the rotations in A^*, to produce a stable set and its corresponding stable matching. We now observe that from Lemma 2.16(3) results that eliminating two different sets of rotations results in two different stable sets and thus to two different stable matchings.

For the other direction, we observe that any stable matching results from eliminating a closed set of rotations since a rotation cannot be eliminated until it is exposed and cannot be exposed until all of its predecessors are eliminated. And this set of rotations has a unique spanning antichain. □

2.1.4.4 Maximum Cardinality - a better bound

With the use of rotations, we can substantially improve the upper bound on the maximum number of stable matchings, $f(n)$. We will show that $f(n) \in O(\frac{n!}{2^n})$, for all $n \geq 4$.

Let $\rho = \{(m_0, w_0), ..., (m_{n-1}, w_{n-1})\}$ be a rotation. We define its size, $\mathcal{S}(\rho)$, to be n. We also define its rank, $\mathcal{R}(\rho)$ to be the number of stable matchings in which it is exposed.

Lemma 2.20. *Let ρ be a rotation in an instance of size n. Then*

$$\mathcal{R}(\rho) \leq f(n - \mathcal{S}(\rho)).$$

Proof. The proof is immediate since the projection

$$I(m_0, w_0) \cdots (m_{\mathcal{S}(\rho)-1}, w_{\mathcal{S}(\rho)-1})$$

can have at most $f(n - \mathcal{S}(\rho))$ stable matchings. □

Lemma 2.14 suggests a method for finding an exposed rotation in a stable matching M where some man m is matched in M to a woman above his partner in the woman-optimal matching. Then, the man m either belongs to the rotation exposed this way, or the cycle created doesn't contain man m, in which case we say that man m belongs to the tail of the rotation exposed. We modify the definition a bit to suit what follows. We will say that man m belongs to the tail of a rotation exposed in some stable matching, if the above holds, or if man m is partnered with the worst partner he can have in any stable matching. We observe that for a man, m, in a stable matching M, one of the following things holds:

- M is woman optimal
- man m belongs to a rotation exposed in M;
- man m belongs to a tail of a rotation exposed in M.

Lemma 2.21. *A man m belonging to the tail of a rotation ρ can appear to that tail in at most $f(n - \mathcal{S}(\rho) - 1)$ stable matchings.*

Proof. Same as 2.20. □

Now let's assume that the instance I of size n has c rotations and man m belongs to k of them. Then

$$f(n) \leq kf(n-2) + (c-k)f(n-3) + 1$$

because we have maximized to $f(n-2)$ all the appearances of man m exposed to a rotation and to $f(n-3)$ all the appearances of m in a specific tail of a rotation and all the appearances to different tails to $c-k$. Now since $f(n-2) > f(n-3)$ [66], maximizing k makes the quantity on the right bigger. k can be at most $n-1$; also maximizing c (which can be at most $\frac{n(n-1)}{2}$) still makes the quantity on the right bigger. Finally, we get that

$$f(n) \leq (n-1)f(n-2) + \left(\frac{n(n-1)}{2} - (n-1)\right) f(n-3) + 1. \tag{2.1}$$

Using the initial values $f(1) = 1, f(2) = 2, f(3) = 3$ [66] we can produce upper bounds for $f(n)$ for all n. We note that (2.1) produces a proof for Eilers's result that $f(4) = 10$.

Now we can show that $f(n) \leq \frac{n!}{2^{n-3}}$ $\forall n \geq 4$.

- This holds for $n = 4, 5, 6, 7, 8, 9, 10, 11$ (we can establish it explicitly through (2.1)).
- Let's suppose it holds for all $k \leq n-1$.
- Then for $k = n$ we have

$$
\begin{aligned}
f(n) &\leq (n-1)f(n-2) + \frac{(n-1)(n-2)}{2} f(n-3) + 1 \\
&\leq \frac{(n-1)!}{2^{n-5}} + \frac{(n-1)!}{2^{n-5}} + 1 \\
&\leq 3\frac{(n-1)!}{2^{n-5}} \\
&\leq \frac{n!}{2^{n-5}\frac{n}{3}} \\
&\leq \frac{n!}{2^{n-3}}
\end{aligned}
$$

2.1.4.5 Cardinality Problem

Theorem 2.19, is crucial in proving that determining the number of stable matchings for an arbitrary SM instance is #P-complete. The proof is a reduction from a known #P-complete problem, namely, determining the number of antichains in a poset (Provan and Ball [56]). The proof relies on the following theorem.

Theorem 2.22. *Given a poset $(P, \leq)$ with n elements, there exists an instance I of SM, constructible from $(P, \leq)$ in time polynomial in n, such that the stable matchings of I are in one-to-one correspondence with the antichains of $(P, \leq)$.*

We omit proof of these facts; see [37] for a detailed proof.

2.1.4.6 Pair Stability Problem

It can be shown [27] that the converse of Lemma 2.16(2) holds, i.e.

Theorem 2.23. *A pair is stable iff it belongs to M_z or to some rotation.*

Since we can find all the rotations in $O(n^2)$ time using the technique used in Lemma 2.14, we can find all stable pairs in $O(n^2)$ time.

2.1.4.7 Enumeration Problem

Finding all the rotations in $O(n^2)$ time facilitates us in finding all stable matchings efficiently. We have already seen that any stable matching results from a sequence of rotation eliminations. This suggests a method for finding all stable matchings. Begin with M_0 and branch from M_0 to all the stable matchings resulting from eliminating all rotations exposed in M_0. Then repeat the process until there no more exposed rotations. To avoid finding stable matchings more than once, once we have branched to stable matching M_i, resulting from exposed rotation i in matching M, we declare rotations $1, ..., i-1$ as forbidden for matching M_i and all the matchings that result from M_i.That is, we never eliminate rotations $1, ..., i-1$ from M_i or the matchings branched from M_i, and so on. This method finds each stable matching in $O(n^2)$ time and exactly once.Exploiting a variation of the graph $G(\mathcal{M})$ we can achieve finding each stable matching in $O(n)$ time. Needing $O(n^2)$ time to construct M_0 and the aforementioned graph, the total time needed to construct all stable matchings is $O(n^2 + |\mathcal{M}|n)$.

2.1.4.8 Minimum Weight Stable Matching

We observe that the man-optimal stable matching minimizes

$$\sum_{(m,w)\in M} mr(m,w)$$

and maximizes

$$\sum_{(m,w)\in M} wr(w,m)$$

where $mr(m,w)$ and $wr(w,m)$ are just the positions of woman w on m's preference list and of m on w's preference list respectively. Hence the egalitarian stable matching can be thought as treating men and women symmetrically.

We assign weights $c(x,y)$ to all pairs (x,y). For any matching M we define its weight $w(M)$ to be the sum,

$$\sum_{(x,y)\in M} c(x,y)$$

and we want to find a stable matching of minimum possible weight. Let $\rho = \{(m_0, w_0), ..., (m_{r-1}, w_{r-1})\}$ be a rotation; we define its weight $w(\rho)$ as the sum

$$\sum_{i=0}^{r-1} [c(m_i, w_i) - c(m_i, w_{i+1})].$$

That is, we define as the weight of a rotation the difference between a stable matching when the rotation is exposed and when it is eliminated. It is immediate that if S is the closed subset of the rotation poset associated with stable matching M then

$$w(M) = w(M_0) - \sum_{\rho\in S} w(\rho).$$

Thus a minimum weight stable matching is a stable matching generated by a closed subset of the rotation poset of maximum total weight. The egalitarian case is analogous.

Finding a maximum weight closed subset of a weighted poset is a classical problem and many ways of solving this problem are explored in the literature. It has been shown (Picard [55]) that finding the maximum weight closed subset of a poset can be solved by finding a minimum cut in an associated flow network. Exploiting the special structure of the rotation poset (and in particular that of $G(\mathcal{M})$) it can be shown that finding the maximum weight closed subset of $G(\mathcal{M})$ and of the rotation poset can be done in $O(n^3(\log n)^2)$ time (Eirinakis et al. [19]). The egalitarian case is analogous (Gusfield and Irving [27]) and so is another weighted version of SM, Optimal SM (Irving et al. [33]).

2.1.4.9 Equivalent Structures

We state in this section two very interesting results regarding SM establishing that the problem is indeed, very general.

The first result is due to Irving and Leather [37] and establishes that every finite partial order (or poset) is the rotation poset for an SM instance; meaning that the rotation posets contain all partial orders. The second result was first established by Blair in [10] and states that the marriage lattices contain all finite distributive lattices.

2.2 Two-sided generalizations of Stable Marriage

2.2.1 Stable Marriage with Incomplete Lists and/or Ties

As stated in the introduction, we can drop the constraint of strict preference lists thus resulting in indifference among several partners. One variant arises if we assume that a preference list contains all members of the opposite set but possibly some of the entries have the same rank; thus, there are ties in the preference list and we call this problem, as stated in the introduction, SM-T. The other variant is to allow preference lists to be partial orders; we denote this case by SM-P. For SM-T the setting is again n men and n women each with a preference list over all the members of the other sex. However, we modify the stability criterion in order to cope with non-strict preferences. In fact our SM-stability criterion, i.e. Definition 2.1, will be called in this setting super-stability and the corresponding stable matching super-stable. A strongly stable matching, M, will be a matching such that for any pair (x, y) not matched none of the following hold

- $y >_x p_M(x)$ and $x \geq_y p_M(y)$
- $x >_y p_M(y)$ and $y \geq_x p_M(x)$.

A weakly stable matching, M, will be a matching such that for any pair (x, y) not matched the following does not hold

- $y >_x p_M(x)$ and $x >_y p_M(y)$.

Breaking ties arbitrarily and applying the Gale-Sharpley algorithm, one can find a weakly stable matching for the SM problem with ties. Also, even though such an instance may not accept a strongly stable or a super stable matching, polynomial-time algorithms for finding one ($O(n^4)$ and $O(n^2)$ time respectively), if one exists, can be found in [31] (Irving). These results can be generalized to SM-P as well. An improvement of complexity (to $O(n^3)$) for finding a strongly stable matching appears in [50] by Mehlhorn et al.

It is shown in [64], by Spieker, that for SM-P the set of super stable matchings forms a distributive lattice. Even though this structure is absent for the set of weakly stable matchings for SM-T (this result is established by a counter example in [58], by Roth), in the case of strong stability and SM-T the set of stable matchings does form a distributive lattice as shown in [44], by Manlove. Finally, the lattice structure in the case of strong stability does not carry over to the case of SM-P, also shown in [44].

The above results suggest that the corresponding problems, which in SM are polynomial time solvable by exploiting the structure of SM, i.e. the stable pair problem and the optimal and egalitarian stable matching problem, may be polynomial time solvable in the cases where we mentioned that the set of stable matchings forms a distributive lattice. However, the problems of finding an egalitarian weakly stable matching and checking if a pair is weakly stable in an SM-T instance are both NP-hard [46] (Manlove et al.).

We may now turn our attention to SM-IL. In this setting, we have n men and n women and for each person u a strict preference list over some of the people of the other sex but maybe not all; i.e. some of the persons of the other sex may be unacceptable to u. Here we allow matchings that don't exhaust the two sets of men and women. That is, a matching is a partial one-to-one mapping from the set of men to that of the women. A matching, M, is said to be unstable if for an unmatched pair, (x, y), all of the following hold:

- x and y are acceptable to each other;
- x is either unmatched in M,or prefers y to $p_M(x)$;
- y is either unmatched in M,or prefers x to $p_M(y)$.

A matching that is not unstable is stable.

Modifying the Gale-Sharpley algorithm one can show that a stable matching for SM-IL always exists (Gale and Sotomayor [25], Gusfield and Irving [27]). Also the set of stable matchings forms a lattice [25]. An interesting fact is that in all stable matchings the set of people that is unmatched remains unchanged.

We devote the remaining section to the case of both incomplete lists and indifference. In particular the problem that we will discuss (denoted SM-ILT) involves preference lists that are both incomplete and total orders. Extending the notions we used for SM-T we have here as well, super, strong and weak stability. Also a stable matching may be a partial mapping like in SM-IL.

Manlove shows in [45], by , that an extension of the corresponding SM-T algorithms can solve the existence problem for strong and super stability, in an instance of SM-ILT, in $O(n^4)$ and $O(n^2)$ time respectively (again in the case of strong stability the bound is improved in [50]). It is also shown that like in SM-IL, in these problems, the set of persons not matched remains unchanged for all stable matchings. On the other hand, we can find a weakly stable matching for SM-ILT by breaking the ties arbitrarily and solving the resulting SM-IL instance. However, different ways of breaking the ties might produce weakly stable matchings of different sizes (where the size of the stable matching is the number of the men matched). The problem of finding a weakly stable matching of maximum size is NP-hard [46].

The (distributive) lattice structure is present in SM-ILT under super stabilility and is absent under weak stability. We will discuss in a later section that in MM-T under strong stability a (distributive) lattice structure is present. Since SM-ILT is a special case of MM-T (which is the many-to-many generalization of SM) the (distributive) lattice structure is present under strong stability for SM-ILT also.

The lattice structure present in SM-ILT under super-stability allows a treatment of the stable pair problem, of the enumeration problem and of the egalitarian stable matching problem, analogous to that of SM (Scott [62]). We might expect that the same holds for strong stability.

2.2.2 Hospital - Residents Problem

The HR problem involves two sets of possibly unequal cardinality, the hospitals and the residents. The residents can be matched to at most one hospital and each hospital has a "quota", i.e., the maximum number of residents that can be matched to it. We note here that the sum of all quotas is not necessarily equal to the cardinality of the resident set. Each agent (resident or hospital) is associated with a strict preference list involving some, but possibly not all, of the members of the other set. We observe that an instance of HR where the two sets have equal cardinality and each hospital has quota 1, is actually an instance of SM-IL. A matching in the setting we described is a partial mapping from the set of the residents, denoted $\mathcal{R}$ to the set of the hospitals, denoted $\mathcal{H}$, so that none of the hospitals's quotas is exceeded, and for every matched pair each of its members is acceptable to the other. A matching, M, is unstable if there is a pair (r, h) of a resident and a hospital, not matched in M so that

- h and r are acceptable to each other, i.e. h, r appear to each other's preference list;
- either r is unmatched, or r prefers h to his assigned hospital;
- either h has not exhausted its quota or h prefers r to at least one of his assigned partners.

We use similar terminology with SM, i.e. r prefers h to h' if h is above h' on r's list and this is denoted by $h >_r h'$.

A variant of the extended Gale-Sharpley algorithm yields the hospital-oriented and resident-oriented algorithms which in turn yield the hospital-optimal and resident-optimal stable matchings. In the resident-optimal matching each resident is matched to the best stable partner he can have or is unmatched in which case he is unmatched in all stable matchings. In the hospital-optimal matching each hospital either fills all of its q available spaces with its q best stable partners or is assigned a set of less than q residents and exactly this set is assigned to this hospital in all stable matchings. These algorithms run in $O(|\mathcal{R}||\mathcal{H}|)$ time. We state a very interesting theorem regarding HR stable matchings known as the rural hospitals theorem.

Theorem 2.24. *For a given HR instance*

- *each hospital is assigned the same number of residents in all stable matchings;*
- *exactly the same residents are unmatched in all stable matchings;*
- *any hospital that does not exhaust its quota in one stable matching is assigned the same set of residents in all stable matchings.*

A thorough review of HR appears in [27] and the same book is also a good source for references on the subject.

2.2.2.1 A Reduction: HR→SM and Maximum Cardinality

We can easily reduce HR to SM [27], by replacing each hospital $h \in \mathcal{H}$ by the set of hospitals $\{h_1, ..., h_{q_h}\}$, all of which have the same preference list as h, and also, replacing each appearance of hospital h in any resident's, $r \in \mathcal{R}$, list by the sequence $h_1, ..., h_{q_h}$. In the new instance each hospital has quota 1 . Then any stable matching of the new SM instance can easily be transformed into a stable matching for the original HR instance and vice-versa. The constructed SM instance might involve two sets of unequal cardinality and of incomplete lists; these cases can easily be handled with an extension of the Gale-Sharpley algorithm [27]. In particular, in the case of sets of unequal cardinality and complete lists, all agents in the smaller set are always matched and all the agents in the bigger set are partitioned into two sets of agents, those that remain unmatched in all stable matchings and those that are matched in all matchings.

Thus we can derive an upper bound for the maximum number of stable matchings in the HR case.

Proposition 2.25. *Let $I(r,q)$ denote the set of HR instances involving a set of residents, $\mathcal{R}$, with $|\mathcal{R}| = r$, a set of hospitals, $\mathcal{H}$, and a quota function $q : \mathcal{H} \to \mathbb{N}$ satisfying $q = \sum_{h \in \mathcal{H}} q(h)$. Let $f(n)$ be the maximum number of stable matchings for an SM instance of size n and $f(r,q)$ be the maximum number of stable matchings of an HR instance $I \in I(r,q)$. Then*

$$f(r,q) \le f(\min\{r,q\})$$

2.2.2.2 Hospital - Residents with Ties

In the HR problem with ties (or HR-T), like in SM-T, we allow ties in the preference lists. Thus, there appear three notions of stability, i.e., super, strong and weak. Similar results to those for HR are shown for super stability in [36], by Irving et al. In particular, even though an instance of HR-T does not always accept a super stable matching, two algorithms appear in [36] one hospital-oriented and one resident oriented, that construct a super stable matching if one exists. Both algorithms are time-optimal, i.e., they run in $O(|\mathcal{R}||\mathcal{H}|)$ time and, the resulting stable matchings have properties analogous of those for the HR resident-optimal and hospital-optimal matchings. In the super stable hospital-optimal matching, each hospital is either assigned the best q super stable partners it can have (where q is its quota) or fewer than q partners, in which case it is assigned the same set of partners in all super stable matchings. In the super stable resident-optimal matching, every resident that is matched is assigned his best super stable partner and every resident that remains unmatched is unmatched in all super stable matchings. Also an analogue of the rural hospitals theorem (Theorem 2.24) holds for super stability. These results can be extended in the case that the preference lists are arbitrary partial orders (we denote this problem as HR-P).

In the case of strong stability, it is shown in [35], by Irving et al., that a strongly stable matching does not always exist; still, for HR-T, there is an algorithm that finds one when one exists or reports that none exists in $O(a^2)$ time, where a is the total number of mutually acceptable pairs (r, h). This algorithm is resident-oriented; a hospital-oriented algorithm appears in [43], by Malhotra. In contrast to the case of super stability, finding a strongly stable matching in the HR-P case is NP-hard.

Finally, finding a weakly stable matching is always possible by breaking the ties arbitrarily and applying one of the known algorithms for the resulting HR instance.

2.2.2.3 Hospital - Residents with Couples

There is substantially rich literature concerning the extension HR-C of HR, or more generally the case where residents display some sort of preference over their coworkers [58, 57, 17, 12, 39, 40, 41, 48, 49]. Here, we consider the formulation of the problem found in [27] and in [57] by Ronn.

An instance of HR-C comprises one set of hospitals and one set of residents; however, some of the residents might be married. This results in the following set of constraints:

- each hospital has a strict preference list over a subset of the resident set, $\mathcal{R}$;
- each resident is either a single resident or belongs to at most one couple;
- each single resident has a strict preference list over a subset of the hospital set, $\mathcal{H}$;
- each couple of residents has a strict preference list over the elements of the set $\mathcal{H} \times \mathcal{H}$.

In this setting, a matching is unstable if one of the following holds:

- there exists an unmatched pair (r, h) of a single resident and a hospital so that they are mutually acceptable, r is either unmatched or prefers h to his partner, and h has a free place or prefers r to his worst assigned partner;

- there exists a pair (r, h), where resident r is coupled with resident s and (r, s) is matched to (h_r, h_s) with $h_r \neq h$, so that (r, s) prefers (h, h_s) to (h_r, h_s) and h has a free place or prefers r to its worst assigned partner;
- couple (r, s) is matched to (h_r, h_s) and there exist hospitals h'_r, h'_s both having free places or preferring r and s to their assigned worst partners and (r, s) prefers (h'_r, h'_s) to (h_r, h_s).

A matching is stable if not unstable and finding a stable matching is NP-hard [57].

2.2.3 Many - to - Many Problem

The MM problem is a natural generalization of HR. In particular the setting is two sets, $\mathcal{W}, \mathcal{F}$, (of possibly unequal size) of individuals, each of which has a strict preference list over the individuals of the other set that he finds acceptable. Moreover each person has a quota, i.e. the maximum number of persons from the other set that can be matched to him. A matching, M, is a collection of pairs (w, f), $w \in \mathcal{W}$, $f \in \mathcal{F}$ so that

$$|\{f \mid (w, f) \in M\}| \leq q_w \ \forall w \text{ and } |\{w \mid (w, f) \in M\}| \leq q_f \ \forall f$$

where q_x is the quota of the individual x. A matching is stable if for any unmatched pair (w, f), either w and f are unacceptable to each other, or one of w and f, say w, has exhausted its quota and prefers his worst assigned partner to f.

An $O(n^2)$ algorithm, where $n = \max\{|\mathcal{W}|, |\mathcal{F}|\}$, for finding a stable matching appears in [7], by Baïou and Balinski. Actually the algorithm finds the dominant solution under a natural dominance relation and it is also shown that under this dominance relation the set of stable matchings forms a distributive lattice.

An $O(n^6)$ algorithm is presented in [6], by Bansal et al., for solving the equivalent of the egalitarian stable matching problem in the MM setting. Also, equivalent results to those of Section 2.1.4.3 concerning rotations are presented. It is shown in [19], by Eirinakis et al., that we can optimally find all stable pairs in $O(n^2)$ time, exploiting the rotations. Also, analogous results concerning the enumeration problem, i.e. we can generate all stable matchings in $O(n^2 + n|\mathcal{M}|)$ time, and an improvement of the $O(n^6)$ algorithm for the egalitarian stable matching to $O(n^3(\log n)^2)$ are presented in [19]. We note here, that the complexity of these problems remains as in SM.

2.2.3.1 Many - to - Many with Ties

As expected MM has a natural generalization with ties. In this generalization three natural notions of stability arise, weak, strong and super stability. In [43], by Malhotra, a polynomial-time algorithm is presented for finding a strongly stable matching if one exists and it is shown that the set of strongly stable matchings forms a distributive lattice. The strongly stable matching created by the algorithm is $\mathcal{S}$-optimal, where $\mathcal{S}$ is one of the two sets: $\mathcal{W}, \mathcal{F}$, meaning that the members of the set $\mathcal{S}$ are assigned the best set of partners they can have in any strongly stable matching.

2.2.4 Stable Allocation Problem

SAL is a generalization of MM. We still have two sets of agents ,$\mathcal{W}, \mathcal{F}$, the workers and the firms, but instead of seeking to match workers to firms, each worker has a number of hours he can be employed and each firm has a number of hours it can offer for work. Formally, a SAL problem, (Γ, s, d, π), is a graph Γ defined over a grid and arrays of reals $s, d > 0$ and $\pi \geq 0$ as follows. There are two distinct sets of agents, the row agents ($\mathcal{W}$) and the column agents ($\mathcal{F}$), and each agent has a strict preference over the agents of the other set. Each worker $w \in \mathcal{W}$ has $s(w)$ units of work to offer, each firm $f \in \mathcal{F}$ seeks to obtain $d(f)$ units of work and firm f can contract at most $\pi(w, f)$ units of work with worker w. A horizontal directed edge $((f, w), (f, w'))$ in the graph expresses firm's preference for w' over w and the case is similar for vertical edges.

An allocation $x = (x(w, f))$ for (Γ, s, d, π) is a set of real numbers that satisfies the following:

(1) $\sum_{f \in \mathcal{F}} x(w, f) \leq s(w), \ \forall w$

(2) $\sum_{w \in \mathcal{W}} x(w, f) \leq d(f), \ \forall f$

(3) $0 \leq x(w, f) \leq \pi(w, f), \ \forall w \forall f$

In constraint (3) we might also put a strictly positive lower bound on $x(w, f)$.

An allocation is stable if $x(w, f) < \pi(w, f)$ implies that

$$\sum_{w' \geq_f w} x(w', f) = d(f) \ \text{ or } \ \sum_{f' \geq_w f} x(w, f') = s(w).$$

A SAL problem is said to be discrete if s, d, and π are integer valued. We observe that SM is the SAL problem with $s(w) = d(f) = \pi(w, f) = 1$, SM-IL is the SAL problem with $s(w) = d(f) = 1$ and $\pi(w, f) = 0$ or 1, HR is the SAL problem with $s(w) = 1, \ d(f) \geq 1$ and $\pi(w, f) = 0$ or 1, and MM is the SAL problem with $s(w), d(f) \geq 1$ and $\pi(w, f) = 0$ or 1. Of course, all the above problems are discrete.

It is possible to find a row or column optimal stable allocation in $O(mn)$ time where m is the number of edges and n is the number of nodes in the graph ([8] Baïou and Balinski). Moreover, it is shown in [8] that the set of stable allocations forms a distributive lattice. The time bound of finding a stable allocation is improved to $O(m \log n)$ in [16], by Dean and Munshi.

2.2.5 Money Markets

Several generalizations of SM have resulted by examining the problem from the economic point of view. In these variants, we still have two sides of agents that we wish to match, but this time a matching compensates each agent involved in it with money. Using economics-terminology we can incorporate these compensations to the utility function of each agent.

These generalizations can be thought of as a housing market where agents are buyers and sellers of at most one house (one-to-one model [63], by Sharpley and Shubik) or a labor market involving workers and firms (one-to-many model [15, 38], by Crawford, Knoer and Kelso, Crawford respectively; many-to-many model [59, 60], by Roth). These models generalize SM, HR and MM respectively.

The game-theoretic approach involved in these variants suggests that a given assignment is to be stable if it belongs to the core of the game, i.e. if no coalition of agents can improve their positions. Also, the use of monetary compensation gives rise to more elaborate schemes of preferences than simply ordering the opposite set of agents, and relaxations such as an agent not having a quota but an unrestricted number of places to fill in an assignment. In particular the preferences of the firms are rankings over subsets of the worker set according to salary requirements etc. These rankings are achieved through so-called choice functions for each firm that choose from each subset W' of the worker set a subset of W' that the firm would hire if W' and the firm were the only ones present in the market. It has been shown that if a stable assignment is to exist the choice functions must possess the substitutability property which put in simple words suggests that if a firm would like to employ a worker it would still like to employ him if another worker was deleted from the market.

Like in the standard versions, the two sides of the market are always in opposition meaning that if all agents on one side prefer matching M to M' all agents on the other side prefer M' to M.

A generalization of the stable allocation problem where preferences are expressed with choice functions is explored by Alkan and Gale in [5].

In the very interesting work of Fleiner [22], it is shown that some of these generalization results follow directly from the fixed point theorem of Knaster and Tarski.

2.3 Other two sided variants

2.3.1 Exchange Stability

What we have not considered so far is another stability criterion. We now define such a criterion, i.e., exchange stability, in the context of SM. A pair $\{x_0, x_1\}$ of two individuals of the same sex will be called an exchange blocking pair for matching M if $p_M(x_1) >_{x_0} p_M(x_0)$ and $p_M(x_0) >_{x_1} p_M(x_1)$; that is, if x_1 prefers x_0's partner and x_0 prefers x_1's partner. A matching will be called exchange-stable if it admits no exchange-blocking pair.

It is shown in [14], by Cechlárová and Manlove, that finding an exchange-stable matching is NP-hard. Other relaxations of the exchange-stability criterion are also explored in [14], resulting in polynomial time solutions as well as NP-hardness for the variants considered.

2.3.2 One Sided Preferences

The two-sided setting with only the one side having preferences over the other (and the other being indifferent) combined with the notion of rank-optimality was first studied in [29], by Irving, and then in [32], by Irving et al.

In this setting, a rank-maximal matching is a matching between the two sets where the number of individuals matched to their first choice is maximal, and subject to that the number of individuals matched to their second choice is maximal, etc.

In particular, the rank of a matched pair (i, j) is the position of j on i's preference list and it is shown that if C is the maximal rank used on a rank-maximal matching, n is the size of the instance (i.e., of each set) and m is the total size of the preference lists, the problem can be solved in $O(\min\{n + C, C\sqrt{n}\}m)$ time. We note that incomplete lists and ties are allowed.

There are also other optimality criteria related to one sided preference lists; two of them are Pareto-optimality ([1, 4, 61], by Abraham et al., Abdulkadiroglu et al. and Roth et al. respectively), and popularity [2, 3, 47, 51] (by Abraham et al., Manlove et al. and Mestre respectively).

Chapter 3

Stable Roommates

3.1 Stable Roommates Problem

The stable roommates problem is an one-sided generalization of SM, meaning that we drop the two set constraint. In particular, we have now one set of individuals, of even cardinality, and wish to partition it into pairs. Each individual has a strict preference list over all the other members of the set. Of course, we want the partition to be stable i.e. no unmatched pair to prefer each other to their assigned partners. We use the same notation and terminology that we used for SM except for the fact that now a pair of two individuals is a set, $\{x, y\}$, instead of a vector. So a matching, M, is a partition of the individuals into pairs; if y is above z on x's list we say that x prefers y to z and denote it by $y >_x z$; a stable matching is a matching where for any unmatched pair $\{x, y\}$, $p_M(x) >_x y$ or $p_M(y) >_y x$.

3.1.1 Existence Problem

Unlike SM, an SR instance may admit no stable matching. Combining the ideas used in the extended Gale-Sharpley algorithm and those involving rotations, an $O(n^2)$-time algorithm for determining whether a stable matching exists, and if it does to produce such a matching, appears in [30] (Irving). Actually, the notion of a rotation was firstly invented in [30] for the stable roommates problem and then used in [37].

3.1.2 Cardinality and Maximum Cardinality Problems

Let I be an instance of SR of size n. We denote by $\mathcal{R}$, the set of stable matchings of I. Also, we denote by $r(n)$ the maximum number of stable matchings that an instance of SR of size n can have.

Lemma 3.1. *Given an SM instance of size n there is an SR instance of size $2n$ that admits precisely the same stable matchings.*

Proof. We generate the SR instance by appending to the list of every individual all the persons of the same sex in an arbitrary order. Any stable matching of the SM instance remains stable for the SR instance and any matching for the SR instance that pairs two men together must pair two women together as well; the unmatched pair of one of these two men with one of the two women blocks the matching. □

Corollary 3.2. *It is immediate that $r(n) \geq f(\frac{n}{2})$, where $f(n)$ is the maximum number of stable matchings for an SM instance of size n.*

Corollary 3.3. *The cardinality problem is #P-complete for SR also.*

There are many ways of counting the possible number of ways to divide $2n$ objects into unordered pairs. One of those ways is the following. First we choose n objects out of the $2n$ objects. This can be done in $\frac{2n!}{n!n!}$ ways. We want these n objects to belong to different pairs. Now the number of ways we can put the remaining n objects with the already chosen ones so as to form the pairs is $n!$, so that the total number of ways so far is $\frac{2n!}{n!}$. We now observe that for a given partition of the $2n$ items in pairs, there are 2^n ways to follow the previous procedure and obtain this partition. This means that we have counted each possible partition 2^n times, so that the total number of possible partitions is $\frac{2n!}{n!2^n}$.

This proves that the obvious upper bound for $r(n)$ is $\frac{n!}{\frac{n}{2}!2^{\frac{n}{2}}}$. We will prove that

$$r(2n) \leq \frac{2n!}{n!2^n}\frac{f(n)}{n!}.$$

Definition 3.4. Let I be an instance of the Stable Roommates problem involving persons $1,..,2n$. We define the $\{i_1, i_2, ..., i_n\}$ projection of I , using the notation $I(i_1, ..., i_n)$, where $i_k \in \{1, .., 2n\}$, $\forall k$, to be the SM instance involving $\{i_1, ..., i_n\}$ as the men and $\{1, ..., 2n\} \backslash \{i_1, ..., i_n\}$ as the women with preference lists the reduced preference lists from the SR instance, i.e. the preference list for man i_k is the original preference list of I with all the men removed.

Definition 3.5. Let I be an instance of SR. We define the $\{i_1, i_2, ..., i_n\}$ projection of $\mathcal{R}$, using the notation $\mathcal{R}(i_1, ..., i_n)$, where $i_k \in \{1, .., 2n\}$, $\forall k$, to be the subset of $\mathcal{R}$ containing all stable matchings where each person of the set $\{i_1, i_2, ..., i_n\}$ is matched with a person of the set $\{1, ..., 2n\} \backslash \{i_1, ..., i_n\}$.

Proposition 3.6. *Let I be an instance of the SR problem. Then if $S \in \mathcal{R}(i_1, ..., i_n)$, S is a stable matching for $I(i_1, ..., i_n)$.*

Proof. S is indeed a matching between the men and the women of $I(i_1, ..., i_n)$ by definition. If S is blocked in $I(i_1, ..., i_n)$ then S must be blocked in I since the entries of the preference lists in $I(i_1, ..., i_n)$ have the same order in the preference lists of I. □

A projection obtained in the above way can have at most $f(n)$ stable matchings. Now we can choose n men out of $2n$ in $\frac{2n!}{n!n!}$ ways thus covering all stable matchings of I. So the total number of stable matchings for I is so far at most $\frac{2n!}{n!n!}f(n)$. We now observe that for a stable matching of I there are 2^n ways of doing the above process to obtain it. So, we have counted each stable matching of I, 2^n times. This proves that

Proposition 3.7. *The maximum number of stable matchings for an SR instance of order n is*

$$r(n) \leq \frac{n!}{\frac{n}{2}!2^{\frac{n}{2}}} \frac{f(\frac{n}{2})}{\frac{n}{2}!}$$

Using the upper bound for SM established in section 2.1.4.4 we get that

$$r(n) \leq \frac{n!}{\frac{n}{2}!2^{\frac{n}{2}}} \frac{1}{2^{\frac{n}{2}-3}}, \ \forall n = 2k \geq 8$$

3.1.3 Structure and related problems

It has been shown that $\mathcal{R}$ possesses a semilattice[1] structure under a relation that generalizes dominance as it is defined for SM. Exploiting the structure of $\mathcal{R}$ and the algorithm that finds a stable matching , when one exists, similar problems to those defined in Section 2.1.2 can be solved. In particular all stable pairs can be found in $O(n^3 \log n)$ time, all stable matchings can be generated in $O(n^3 \log n + n^2|\mathcal{R}|)$ time and a minimum regret stable matching can be found in $O(n^2)$ time. (A minimum regret stable matching is an analogue of the weighted stable matching defined in Section 2.1.2. Here, the regret of a person is the position his partner occupies on his list, and the regret of a matching is the maximum regret of any person in the matching. So the problem is to find the stable matching with minimum regret.) The minimum regret stable matching is also defined in the bipartite case (Gusfield [26]). All the above results can be found in [27]. Optimal Stable Roommates, an analogue of the Optimal SM, i.e. the weighted version of SM, is shown to be NP-hard by Feder in [21].

[1] A semilattice is like a lattice; the only difference is that it has just one operation under which it is closed: greatest lower bound or least upper bound.

3.2 Generalizations of the Stable Roommates Problem

3.2.1 Stable Roommates with Incomplete Lists and/or Ties

If we just allow unacceptable partners, i.e. incomplete lists, then analogous results to those for SM can be deduced [27]. If however, we allow ties in the preference lists the situation changes dramatically. In particular, we consider the corresponding stability criterion to that of weak stability for SM. An unmatched pair $\{x, y\}$ will block the matching if x strictly prefers y to his partner and y strictly prefers x to his partner.

In the case of SM, this variant can easily be handled by breaking ties, arbitrarily, and applying the Gale-Sharpley algorithm. If we attempt to use this technique here we do not obtain the same result since the resulting instance might admit no stable matching but a different breaking of the ties could produce a solution. It is shown in [57] that the problem of finding a stable matching when ties are allowed in SR is NP-hard.

Irving and Manlove show in [34] that different weakly stable matchings (in the case of incomplete lists and ties) may have different cardinality and that finding a weakly stable matching of maximum cardinality is NP-hard. They also present a linear time algorithm for finding a super-stable matching in the SR-T case and they extend their algorithm to cope with incomplete lists and partially ordered preference lists (SR-P).

3.2.2 3-Person Stable Assignment Problem

There is a natural generalization of SR that asks whether we can partition a set of $3k$ elements into k triples while maintaining stability. Formally, we are given a set containing $3k$ elements and each element (person) has a strict preference list over all possible pairs he can be matched with. A destabilizing triple (x, y, z) of matching M, is an unmatched triple where all of its members prefer the other two (as a pair) to the pair they are assigned in M. This formulation of the problem has been shown to be NP-hard [53],by Ng and Hirschberg. Other formulations of the problem where the preferences are consistent (i.e. a person prefers x to y consistently for all possible pairs) and even when ties are allowed are also shown to be NP-hard in [28], by Huang. These results have been motivated by one of Knuth's posed questions regarding stable matchings [42].

3.2.3 Stable Partnership Problem

A very common formalization of SM and its variants involves graphs. SM can be thought of as the problem where we are given a complete bipartite graph (V, E). Then the incident edges with $v \in V$, denoted $E(v)$ define the acceptable partners of v (when the graph is not complete we have the incomplete lists case) and for each v a linear order ranks the elements of $E(v)$ resulting in v's preferences over its acceptable partners. A stable matching is simply a subset S of E of disjoint edges so that for every $e \in E$ either $e \in S$ or there is an edge $m \in S$ that has a common vertex with e, say v, and v prefers m to e. The same applies to SR only now the graph is not bipartite.

Instead of a linear order on $E(v)$ the rankings of v can be expressed through choice functions for non-bipartite graphs, much like in the case of money markets. In particular the stable partnership problem is defined as follows; we are given a graph (V, E) and for each $v \in V$ a choice function $C_v : 2^{E(v)} \to 2^{E(v)}$ that maps each set X of incident edges with v to a set X' that v chooses from X. We assume that choice functions are substitutable, meaning that if $x \neq y$ and $x \in C_v(X)$ then $x \in C_v(X \backslash \{y\})$. We seek with these inputs to find a stable partnership, i.e. a subset S of E that has the following properties:

- (Individual Rationality) For every vertex v, $C_v(S(v)) = S(v)$, where $S(v)$ are the incident edges with v that belong to S.
- (Stability) There exists no blocking edge $e = \{x, y\} \notin S$ such that $e \in C_x(S(x) \cup \{e\})$ and $e \in C_y(S(y) \cup \{e\})$.

An example of a substitutable choice function is a linear choice function: $E(v)$ is linearly ordered and $C_v(X)$ is the minimal element of X with regard to the linear order. Then, if all choice functions are linear, a stable partnership is a stable matching.

The stable partnership problem is studied by Fleiner in [23] and it is shown that if the choice functions satisfy a monotonicity property also, then the problem is solvable in polynomial time but for random choice functions the problem is NP-hard.

3.2.4 Stable Activities and Stable Multiple Activities Problems

We can easily modify the SR algorithm to treat cases where the underlying set is of odd cardinality and the lists are incomplete. This formulation of the problem is easily expressed with graphs. In particular we consider an SR instance to be a finite simple graph $G = (V, E)$ such that with each vertex $v \in V$ is associated a linear order on the set of edges incident with v, $E(v)$. Then a stable matching is a subset, S, of E, of disjoint edges, such that there is no blocking edge $e \notin S$, i.e. an edge such that each of its vertices either is not covered by S or it prefers e to the edge of S incident with it.

In an instance of the Stable Activities problem we consider one set of persons and several activities between each pair of individuals. Each individual ranks all the activities he can participate in and we want to choose a subset of all activities so that each individual participates in at most one activity and there is no blocking activity, i.e. an activity that both potential participants prefer to the one they are assigned. With graph terminology, we are given a finite multigraph, $G = (V, E)$, and a linear order on $E(v)$ is associated with each vertex, v. We say that a subset, F, of E dominates edge e if there is a vertex, v, of e, and an edge f in F incident with v such that v prefers f to e. We denote by $\mathcal{D}(F)$ all the edges dominated by F. Then a stable matching is a subset, M, of E such that $\mathcal{D}(M) = E \backslash M$. It is straightforward to see that SR is the special case of SAC when the graph is simple.

In an instance of the Stable Multiple Activities problem we are given a finite multigraph, $G = (V, E)$, and a linear order on $E(v)$ is associated with each vertex, $v \in V$. Also, we are given a quota function defined on the vertices, $b : V \to \mathbb{N}$. We will say that a subset F, of E, dominates edge e if there is a vertex, v, of e, and edges $f_1, ..., f_{b(v)} \in F$ incident with v so that v prefers all f_i to e. We denote all edges dominated by F by $\mathcal{D}(F)$. Then a stable matching is a subset, M, of E, such that $\mathcal{D}(M) = E \backslash M$. Again it is straightforward to see that MM is the special case of SMA when the underlying graph is simple and bipartite.

It is shown by Cechlárová and Fleiner [13] that both the above generalizations of SR can be reduced to SR. Thus any algorithm that solves SR can be used to solve these problems as well. In the rest of the section we outline the reduction from the more general case of SMA to SR which is also valid for the bipartite case.

The construction involves two steps; in the first step we transform an instance of SMA to an instance of SMA with the so-called one-to-many property, i.e. each edge having a vertex, v, with $b(v) = 1$. This is achieved if we remove each edge $e = \{u, v\}$, and add the edges

$$\{u, u_0^e\}, \{u_0^e, u_1^e\}, \{u_1^e, v_2^e\}, \{v_2^e, v_0^e\}, \{v_0^e, v\}, \{v_0^e, v_1^e\}, \{v_1^e, u_2^e\}, \{u_2^e, u_0^e\}.$$

That is we put between the vertices u, v, a six-cycle. Any new vertex, x, we add has $b(x) = 1$ while the vertices of the original instance have the same quota. It is obvious that the new instance has the one-to-many property. Also, if we modify the preferences of the vertices (new and old) to suit our needs we can achieve solutions of the two instances being in one-to-one correspondence. In the second step we reduce the newly constructed instance to an SR instance with a construction very much similar to the one in Section 2.2.2.1. That is we make multiple copies of all vertices, x, that have quota $b(x) > 1$. It can be shown that the solutions of the final SR instance are in one-to-one correspondence with the solutions of the original instance. In particular each stable matching of the original instance can be transformed into a stable matching for the SR instance and each stable matching of the SR instance results from such a transformation.

The reduction is valid if the graph is bipartite so the Gale-Sharpley algorithm can be used to solve SMA in its bipartite form, and its special case MM.

3.2.4.1 Maximum Cardinality

A byproduct of the reduction is that it is possible to produce upper bounds for the maximum number of stable matchings in an SMA instance, bipartite or not.

Since the reduction produces , for the most cases, SM or SR instances with incomplete lists, we need to address the SM case with incomplete lists and then it is a simple matter to count the the extra nodes added and produce an upper bound. It is a fact that different SM instances with incomplete lists, even with the same total number of entries in the men's lists (i.e. edges) might admit different bounds. It is easy to see that an instance that has more entries in the lists has potentially more stable matchings since for any instance with incomplete lists there is an instance with more entries having at least as many stable matchings. Let $f(n,m)$ be the maximum number of stable matchings that an instance can have, involving n men in its GS-lists and m entries in the GS-lists of these n men. Let $f(n)$ be the maximum number of stable matchings for an SM instance of size n (with complete lists). Then

$$f(n,m) \leq \frac{m-n}{2} f(n-2) + 1.$$

The above bound is very loose but it can be substantially improved if it is tailored for each particular instance. In particular each instance will have a GS-list with the fewest entries, k, than any other list. Also, for a particular instance we can count the maximum number of entries, m_1, m_2, for all possible 2 and 3 projections of the form $I(i_1, j_1)(i_2, j_2)$, etc. So the number of stable matchings for a particular instance can be bounded by the quantity

$$kf(n-2, m_1) + \left(\frac{m-n}{2} - k\right) f(n-3, m_2) + 1.$$

Of course the bound is not tight but it couldn't have been since counting the number of stable matchings for a particular instance is #P-complete.

Chapter 4

Higher Dimensions

4.1 Three Sided Stable Marriage

Another generalization of SM is generated if we consider more than two sets in the matching scheme. Many formulations exist for the case where three sets are involved but they are all characterized by NP-hardness.
In particular, in the model by Ng and Hirschberg [53], three sets of equal cardinality are involved and each agent ranks all possible pairs of agents from the Cartesian product of the other two sets. A matching is unstable if there exists a destabilizing triple, i.e., an unmatched triple that each of its members prefers the the triple to its assigned one in the matching. The corresponding existence problem in this setting is NP-hard.

Other formulations of the problem where the preferences are consistent (i.e. a person prefers x to y consistently for all possible pairs) and even when ties are allowed are also shown to be NP-hard in [28] by Huang.

In another formulation, where preferences are cyclic (i.e. if sets $A, B, C,$ are involved all $a \in A$ rank the members of B, all $b \in B$ rank the members of C and all $c \in C$ rank the members of A.) the problem is NP-hard when incomplete lists are allowed as shown by Biró and McDermid in [9]. Also shown in this paper is that the problem with cyclic preferences under strong stability is also NP-hard. The preferences in both formulations are assumed strict and a matching is stable if there exists no triple (a, b, c) such that a prefers b to $b' \in B$ with which it is matched, b prefers c to c' with which it is matched and c prefers a to a' with which it is matched. A matching is strongly stable if there exists no triple not in the matching such that a prefers b to b' to which it is matched or $b = b'$ and respectively the same holds for b, c.

The variant of cyclic preferences with complete lists has been shown to always have a solution in the cases that the sets have at most 3 agents (Boros et al. [11]) or at most 4 agents (Eriksson et al. [20]).

4.2 Stable Networks

In this section we describe the model of stable supply chain networks, as it is presented by Ostrovsky in [54], a model that contains all models of Section 2.2. Supply chain networks are a natural generalization of two-sided markets into an arbitrary number of sides in a market. In particular there is a set of individuals (basic suppliers) that can supply only, a set of individuals (final consumers) that can consume only, and several sets of individuals that can supply and consume. Formally, we consider the market as a finite set of nodes, A, with an exogenously given partial ordering, $\prec$, on A, that defines possible trading relationships. That is, $a \prec b$ means that b can potentially sell something to a and then, a is a downstream node for b. By transitivity, there are no loops in the market.

Relationships between nodes are represented by bilateral contracts of the form $c = (s, b, l, p)$ where $s \in A$ is the seller of the unit of a good with serial number l, sold at price p to buyer $b \in A$. Of course $b \prec s$. The set of possible contracts C is finite and is given exogenously.

Nodes have preferences over sets of contracts that involve them as the buyer or the seller. For an agent, $a \in A$, and a set of contracts, X, let $Ch_a(X)$ be a's most preferred subset of X, let $U_a(X)$ be the set of contracts in X in which a is the buyer (i.e. *upstream contracts*) and, let $D_a(X)$ be the set of contracts in X in which a is the seller (i.e. *downstream contracts*). Preferences are strict i.e. $Ch_a(X)$ is a function.

Preferences are *same-side substitutable* if for any two sets of contracts, X, Y, such that $D(X) = D(Y)$ and $U(X) \subseteq U(Y)$ we have that

$$U(X) \backslash U(Ch(X)) \subseteq U(Y) \backslash U(Ch(Y))$$

and if for any two sets of contracts, X, Y, such that $D(X) \subseteq D(Y)$ and $U(X) = U(Y)$ we have that

$$D(X) \backslash D(Ch(X)) \subseteq D(Y) \backslash D(Ch(Y)).$$

The above is a generalization of substitutable choice functions and the two notions are equivalent for two-sided markets.

Preferences are *cross-side complementary* if for any two sets of contracts, X, Y, such that $D(X) = D(Y)$ and $U(X) \subseteq U(Y)$ we have that $D(Ch(X)) \subseteq D(Ch(Y))$ and for any two sets of contracts, X, Y, such that $D(X) \subseteq D(Y)$ and $U(X) = U(Y)$ we have that $U(Ch(X)) \subseteq U(Ch(Y))$. Cross-side complementarity is automatically satisfied in two-sided markets.

A *network* is a set of contracts between nodes of A. Let $\mu(a)$ denote the set of contracts involving a in network μ. Network μ is *individually rational* if for any agent a, $Ch_a(\mu(a)) = \mu(a)$.

A chain is a sequence of contracts, $\{c_1, ..., c_n\}, n \geq 1$, such that for any i, $b_{c_i} = s_{c_{i+1}}$. That is, the buyer of contract c_i is the seller of contract c_{i+1}.

For a network μ a *chain block* is a chain $\{c_1, ..., c_n\}$ that satisfies

- $\forall i \leq n, c_i \notin \mu$
- $c_1 \in Ch_{s_{c_1}}(\mu(s_{c_1}) \cup c_1)$
- $c_n \in Ch_{b_{c_n}}(\mu(b_{c_n}) \cup c_n)$
- $\forall i < n, \{c_i, c_{i+1}\} \subseteq Ch_{b_{c_i}}(\mu(b_{c_i}) \cup c_i \cup c_{i+1})$.

That is, a chain block is a downstream chain of contracts not belonging to μ such that both the buyer and the seller of each contract is willing to add the contract to its contracts in μ.

A network μ is chain stable if it is individually rational and has no chain blocks.

It can be proved that under same-side substitutability and cross-side complementarity, there always exists a chain stable network. The proof relies on objects called pre-networks. A *pre-network* is a set of arrows , R, from nodes in A to other nodes in A. Each arrow $r \in R$ is vector (o_r, d_r, c_r), with origin $o_r \in A$, destination $d_r \in A$ and a contract c_r attached, involving o_r and d_r. If o_r is the seller and d_r is the buyer of c_r, then, the arrow is downstream, else, it is upstream. For a pre-network ν and a node, a, we define

$$\nu(a) = \{c \in \mathrm{C} \mid r = (o_r, a, c) \in \nu\}.$$

For each pre-network ν we define the pre-network

$$T(\nu) = \{r \in R \mid c_r \in Ch_{o_r}(\nu(o_r) \cup c_r)\}.$$

It is proved in [54] that the chain stable networks are in one-to-one correspondence with the fixed points of mapping T. Thus defining a partial ordering on the set of pre-networks and showing that T is monotone under this relation is enough to guarantee the existence of a fixed point of T and of a chain stable network. It is also proved that the set of chain stable networks forms a lattice and that the set of basic suppliers prefers the minimal element of the lattice to any other chain stable network and that the set of final consumers prefers the maximal element of the lattice to any other chain stable network.

For completeness, we follow some of the proofs found in [54]. First let us define a function, F, on pre-networks. For any pre-network, ν, we define a network, $F(\nu)$, as follows.

$$F(\nu) = \{c \in C \mid (s_c, b_c, c) \in \nu \text{ and } (b_c, s_c, c) \in \nu\}$$

Lemma 4.1. *For any fixed point, ν^*, of T i.e., $T(\nu^*) = \nu^*$, the network $\mu^* = F(\nu^*)$, is chain stable.*

Proof. We can assume that there are no contracts in μ^* that differ only in price, since, each agent associated with two contracts that differ only in price, seeks to maximize his utility function, thus, choosing only one.

Let us first show that μ^* is individually rational. For that we need to show that for any agent a, $\mu^*(a) = Ch_a(\mu^*(a))$. To that end, we show that $\mu^*(a) = Ch_a(\nu^*(a))$.

$$\begin{aligned} c \in \mu^*(a) &\iff (a,b,c) \in \nu^* \;\;\&\;\; (b,a,c) \in \nu^* \text{ ,where } b{=}s_c \text{ or } b_c \\ &\iff (a,b,c) \in T(\nu^*) \;\;\&\;\; c \in \nu^*(a) \\ &\iff c \in Ch_a(\nu^*(a) \cup c) \;\;\&\;\; c \in \nu^*(a) \\ &\iff c \in Ch_a(\nu^*(a)) \end{aligned}$$

Now it is immediate that $\mu^*(a) = Ch_a(\nu^*(a)) = Ch_a(Ch_a(\nu^*(a))) = Ch_a(\mu^*(a))$.

Next, we need to show that there are no chain blocks. To that end, we assume that $(c_1, ..., c_n)$ is a chain block of μ^*, and s_i and b_i are the seller and the buyer of contract c_i. Since $(c_1, ..., c_n)$ is a chain block, $c_1 \in Ch_{s_1}(\mu^*(s_1) \cup c_1)$. This means that $c_1 \in Ch_{s_1}(\nu^*(s_1) \cup c_1)$, otherwise, if $c_1 \notin Ch_{s_1}(\nu^*(s_1) \cup c_1)$, from same-side substitutability, we have

$$Ch_{s_1}(\nu^*(s_1) \cup c_1) = Ch_{s_1}(\nu^*(s_1)) = \mu^*(s_1).$$

This, combined with the fact that $\mu^*(s_1) \subseteq \nu^*(s_1)$ means that $c_1 \notin Ch_{s_1}(\mu^*(s_1) \cup c_1)$, a contradiction. Now since, $c_1 \in Ch_{s_1}(\nu^*(s_1) \cup c_1)$, the arrow $r_1 = (s_1, b_1, c_1)$ must be in $T(\nu^*) = \nu^*$. Also, since $(c_1, ..., c_n)$ is a chain block, $\{c_1, c_2\} \subseteq Ch_{s_2}(\mu^*(s_2) \cup c_1 \cup c_2)$. If neither c_1 nor c_2 are in $Ch_{s_2}(\nu^*(s_2) \cup c_1 \cup c_2)$, then by same-side substitutability we have

$$Ch_{s_2}(\nu^*(s_2) \cup c_1 \cup c_2) = Ch_{s_2}(\nu^*(s_2)) = \mu^*(s_2).$$

This, like before, combined with the fact that $\mu^*(s_2) \subseteq \nu^*(s_2)$ produces the following contradiction: $c_1, c_2 \notin Ch_{s_2}(\mu^*(s_2) \cup c_1 \cup c_2)$. If, only $c_2 \notin Ch_{s_2}(\nu^*(s_2) \cup c_1 \cup c_2)$ then then by same-side substitutability we have

$$c_1 \in Ch_{s_2}(\nu^*(s_2) \cup c_1 \cup c_2) = Ch_{s_2}(\nu^*(s_2) \cup c_1)$$

and so, the arrow $r_2 = (s_2, s_1, c_1) \in T(\nu^*) = \nu^*$. Since $s_2 = b_1$ all of the above imply that $c_1 \in \mu^*$, a contradiction since $(c_1, ..., c_n)$ is a chain block. Proceeding inductively there is an arrow $r_i = (s_i, b_i, c_i) \in \nu^*$ for every $i \leq n$ where $s_{i+1} = b_i, \forall i < n$. Doing the reverse process, i.e., starting from b_n and going towards s_n we can show that there is an arrow $r'_n = (b_n, s_n, c_n) \in \nu^*$, which means that $c_n \in \mu^*$, a contradiction. Therefore, $\mu^* = F(\nu^*)$ is chain stable. □

Remark 4.2. *It can be shown that the reverse of Lemma 4.1 holds i.e., for any chain stable network μ, there exists a unique fixed point pre-network ν such that $F(\nu) = \mu$.*

In order to prove that there exists a chain stable network, it now suffices to show that T has a fixed point. To establish that fact, we introduce a partial ordering on the set of pre-networks. Let ν_1, ν_2, be two pre-networks. Then, ν_1 is said to be less than or equal to ν_2 ($\nu_1 \leq \nu_2$) if the set of downstream arrows in ν_1 is a *subset* of the set of downstream arrows in ν_2, while the set of upstream arrows in ν_1 is a *superset* of the set of upstream contracts in ν_2. Let $\nu_{\min}$ be the pre-network that includes all possible upstream arrows and no downstream arrows, and let $\nu_{\max}$ be the pre-network that includes no upstream arrows and all possible downstream arrows. By construction, for any pre-network ν, $\nu_{\min} \leq \nu \leq \nu_{\max}$. Actually, under the relation $\leq$, the set of pre-networks forms a finite lattice, with minimal element, $\nu_{\min}$, and maximal element, $\nu_{\max}$. In particular, if ν_1, ν_2 are two pre-networks, then, the pre-network $\nu_1 \vee \nu_2$ comprises the union of downstream arrows of ν_1 and ν_2, and the intersection of upstream arrows of ν_1 and ν_2. The pre-network $\nu_1 \wedge \nu_2$ is defined analogously.

Lemma 4.3. *For any pair of pre-networks ν_1 and ν_2 such that $\nu_1 \leq \nu_2$, we have $T(\nu_1) \leq T(\nu_2)$.*

Proof. We need to show that all downstream arrows in $T(\nu_1)$ belong to $T(\nu_2)$ and that all upstream arrows in $T(\nu_2)$ belong to $T(\nu_1)$. Let us consider a downstream arrow, r in $T(\nu_1)$. By definition of T we have that the contract attached to r,

$$c_r \in Ch_{o_r}(\nu_1(o_r) \cup c_r).$$

Also, the set of downstream arrows in ν_1 is a subset of the set of downstream arrows in ν_2, and so, the set of contracts attached to downstream arrows pointing to o_r in ν_1 is a subset of the corresponding set in ν_2, i.e., $U_{o_r}(\nu_1(o_r)) \subseteq U_{o_r}(\nu_2(o_r))$. Analogously, we have $D_{o_r}(\nu_1(o_r)) \supseteq D_{o_r}(\nu_2(o_r))$. We have already stated that

$$c_r \in Ch_{o_r}(\nu_1(o_r) \cup c_r) = Ch_{o_r}(U_{o_r}(\nu_1(o_r)) \cup D_{o_r}(\nu_1(o_r)) \cup c_r.$$

Same-side substitutability implies that agent o_r does not accept any contract in $U_{o_r}(\nu_1(o_r)) \cup D_{o_r}(\nu_1(o_r)) \cup c_r$ that he had rejected when choosing from $U_{o_r}(\nu_1(o_r)) \cup D_{o_r}(\nu_2(o_r)) \cup c_r$, which means that,

$$c_r \in Ch_{o_r}(U_{o_r}(\nu_1(o_r)) \cup D_{o_r}(\nu_2(o_r)) \cup c_r).$$

Now by cross-side complementarity we have that

$$c_r \in Ch_{o_r}(U_{o_r}(\nu_2(o_r)) \cup D_{o_r}(\nu_2(o_r)) \cup c_r) = Ch_{o_r}(\nu_2(o_r) \cup c_r)$$

which means that $r \in T(\nu_2)$.
The argument is symmetric for upstream arrows. □

Now, repeatedly applying T to $\nu_{\min}$ we form an increasing sequence of pre-networks:

$$\nu_{\min} \leq T(\nu_{\min}) \leq T^2(\nu_{\min}) \leq \cdots \leq T^n(\nu_{\min}) \leq \cdots$$

which, because the set of pre-networks is finite, must produce a fixed point,

$$\text{i.e., } \exists n_0 : T^{n_0}(\nu_{\min}) = T^{n_0+1}(\nu_{\min}).$$

We denote $T^{n_0}(\nu_{\min})$ by $\nu^*_{\min}$ and, applying F to this fixed point we get a chain stable network $\mu^*_{\min} = F(\nu^*_{\min})$. We therefore have:

Theorem 4.4. *Mapping T has a fixed point, thus, there exists a chain stable network.*

Using as a starting point $\nu_{\max}$ we can construct another fixed point of T, $\nu^*_{\max}$. These two fixed points, are actually the minimal and maximal elements of the lattice of fixed points, guaranteed to exist from Tarski's Fixed Point Theorem on complete lattices (Tarski, [65]). We conclude this chapter by stating a theorem that treats the Dominance problem for chain stable networks:

Theorem 4.5. *Let $\mu^*_{\min} = F(\nu^*_{\min})$, $\mu^*_{\max} = F(\nu^*_{\max})$, and let μ^* be a chain stable network. Then any basic supplier (weakly) prefers $\mu^*_{\min}$ to μ^* and μ^* to $\mu^*_{\max}$, and any final consumer (weakly) prefers $\mu^*_{\max}$ to μ^* and μ^* to $\mu^*_{\min}$.*[1]

[1]Weak preference suggests that an agent is better off when choosing from a larger set.

Chapter 5

Future Work

One of the purposes of this review was to identify potential open problems in the literature.

Reviewing Figures 1.2, 1.3, one can pose several questions that, to the best of my knowledge, haven't been answered yet. Of course most of these questions have already been posed by several authors but for completeness we state them as well.

A first question is whether MM-T and MM-P, under super-stability, can be solved in polynomial time.

Also, the Stable Allocation problem and all problems involving choice functions , to the best of my knowledge, have not yet been examined (in print) under the scope of indifference.

Another question is whether SR-T and SR-P under strong stability can be solved in polynomial time.

Finding if the set of solutions of HR-T and HR-P under super stability possesses a lattice structure is an interesting open question. One can ask the same question for SR-T under super and strong stability and for the stable partnership problem; in these cases, however, we would be looking for a semi-lattice structure.

Finally, all problems where a lattice structure is present are candidates for polynomial-time solutions regarding the corresponding Stable-Pair and Egalitarian problems and also for efficient enumeration of all solutions. For some cases we don't have answers yet as to whether these problems can be efficiently solved, e.g. for MM-T under strong stability.

At this point we conclude with a summary table of known results for some of the problems considered in this thesis. For the problems that are not included in the tables, either I have not found any relevant references, or they can be handled by reductions. Combined with Figures 1.1, 1.2, 1.3, and the implied reductions therein, this table can serve as a guide for the whole document. An entry "P" in the table means that the problem is polynomially solvable; an entry "NP" means that a problem is NP-hard; "No" means that there is no lattice structure, thus the question can't be answered; "#P" means it is #P-complete and, "Unknown" means that I have not found any relevant references.

	SM	SM-T & SM-ILT Weak	SM-T & SM-ILT Super
Dominance	P	No	P
Min-Weight	P	NP	Unknown
Egalitarian	P	NP	P
Minimum Regret	P	NP	P
Cardinality	#P	#P	#P
Enumeration	Efficient	Unknown	Efficient
Pair Stability	P	NP	P

	SM-T & SM-ILT Strong	MM	SAL	SR
Dominance	P	P	P	No
Min-Weight	Unknown	P	P [16]	NP
Egalitarian	Unknown	P	P	NP
Minimum Regret	Unknown	Unknown	Unknown	P
Cardinality	#P	#P	#P	#P
Enumeration	Unknown	Efficient	Unknown	Efficient
Pair Stability	Unknown	P	Unknown	P

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