On a Logical Foundation of the Intersection Types Assignment System
Intersection Logics

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Chapter 1

Introduction

This thesis presents the logical systems *"Intersection Logic"* and *"Intersection Synchronous Logic"*, abbreviated IL and ISL, respectively. The former was proposed by Simona Ronchi Della Rocca and Luca Roversi [RR01, Ro02] and the latter by Elaine Pimentel, Ronchi and Roversi [PR05]. They both aim to establish a logical foundation for the *intersection types assignment system*, a deductive system which assigns formulas built from the intuitionistic implication and the intersection as types to terms of the untyped λ-calculus.

The intersection types assignment system, denoted IT by [RR01, PR05] and D by [Kr93], was introduced in the early '80s by Mario Coppo and Mariangiola Dezani [CD78, CD80] to enhance the typability power of Curry's types assignment system $\lambda_{\rightarrow}$ that assigns formulas built from the implication as types to the untyped λ-calculus. It is very useful as a tool for investigating pure λ-calculus, since it has nice syntactical properties. In particular, we can prove that IT assigns types to all and only the strong normalizing terms [Kr93]. Moreover, if the set of types of IT is extended to include a universal type Ω that can be given to all λ-terms (system D extended to DΩ in [Kr93]), we can prove that an untyped λ-term is typable in the extended system with a non-trivial type if and only if its head reduction is finite [Kr93].

Due to the peculiar nature of the intersection, IT cannot be used as a model for a programming language; however, intersection types have been particularly useful in studying the semantics of various kinds of λ-calculi. This can be done by extending the system with suitable sub-typing relations, so that the types assignment acts as a finitary tool to reason about the interpretation of λ-terms in topological models of λ-calculus, like Scott domains, DI-domains and coherence spaces [Ab91, BC83, HR90, HR92].

Some types assignment systems *à la* Curry correspond to a logic or, even better, have been designed starting from a logic. The bridge relating a logic to a λ-calculus assignment system *à la* Curry is a decoration of the logic's deductions with untyped λ-terms. For example, the implicative fragment of intuitionistic logic decorated with untyped λ-terms that encode the implication delivers Curry's types assignment system $\lambda_{\rightarrow}$, while the implicative and conjunctive fragment of intuitionistic logic (LJ) decorated with untyped λ-terms that encode the implication and the conjunction generates the simple types assignment system $\lambda_{\rightarrow}^{\wedge}$. These decorations embody all the connectives of the logic in question and are called *standard*.

Unlike the systems mentioned above, IT does not originate from a logic; it is a somewhat ad hoc system. The first attempt to give a logical foundation to intersection types was by Betti Venneri [Ve94]. The cloudy relation of LJ to IT was first pointed out by Roger Hindley [Hi84]. To unravel this relation, it suffices to explore the relation of conjunction to intersection. The intersection rules of IT, in which premise and conclusion terms are identical,

$$\frac{\Gamma \vdash M : \sigma \qquad \Gamma \vdash M : \tau}{\Gamma \vdash M : \sigma \cap \tau}\,(\cap I_{IT}) \qquad \frac{\Gamma \vdash M : \sigma \cap \tau}{\Gamma \vdash M : \sigma}\,(\cap E^{l}_{IT})$$

naturally pose the question whether IT-deductions could be obtained through a *non-standard* decoration of LJ-deductions that ignores conjunction and encodes the implication only. This way, the conjunction rules of LJ

$$\frac{\Gamma \vdash \sigma \qquad \Gamma \vdash \tau}{\Gamma \vdash \sigma \wedge \tau}\,(\wedge I_{LJ}) \qquad \frac{\Gamma \vdash \sigma \wedge \tau}{\Gamma \vdash \sigma}\,(\wedge E^{l}_{LJ})$$

would transform to intersection rules of IT. But for this argument to work, the premises $\Gamma \vdash \sigma$ and $\Gamma \vdash \tau$ of a $(\wedge I_{LJ})$ rule would *always* have to be encoded by the same up to α-equivalence λ-term, i.e. they would have to be *isomorphic*. Obviously, this is not always the case. In general, different λ-terms M and N encode the two premises, so the non-standard decoration (ns) cannot proceed. On the other hand, the standard decoration (s) places the pair $\langle M, N\rangle$ before the conjunction[1].

$$\frac{\Gamma \vdash^{\mathrm{ns}} M : \sigma \qquad \Gamma \vdash^{\mathrm{ns}} N : \tau}{\Gamma \vdash^{\mathrm{ns}} \boxed{?} : \sigma \wedge \tau} \qquad \frac{\Gamma \vdash^{\mathrm{s}} M : \sigma \qquad \Gamma \vdash^{\mathrm{s}} N : \tau}{\Gamma \vdash^{\mathrm{s}} \langle M, N\rangle : \sigma \wedge \tau}$$

Nevertheless, *some* LJ-deductions include solely $(\wedge I)$ rules combining isomorphic subdeductions. These can certainly be decorated in the non-standard way to give IT-deductions. Conjunction is converted to intersection. Thus, only a proper subset

$$\frac{\Gamma \vdash^{\mathrm{ns}} M : \sigma \qquad \Gamma \vdash^{\mathrm{ns}} M : \tau}{\Gamma \vdash^{\mathrm{ns}} M : \sigma \wedge \tau} \quad \rightsquigarrow \quad \frac{\Gamma \vdash_{IT} M : \sigma \qquad \Gamma \vdash_{IT} M : \tau}{\Gamma \vdash_{IT} M : \sigma \cap \tau}$$

of LJ corresponds to IT through a decoration with untyped λ-terms. We would like to define a logic expressing this very subset of LJ; introducing IL (and ISL) succeeds in this task.

The standard and non-standard decorations of LJ reveal the *asynchronous* and *synchronous* aspects of intuitionistic conjunction, respectively. The former is *conjunction* as already known, denoted $\wedge$, and the latter is *intersection*, denoted $\cap$. For the logical foundation of IT, it is important to separate between the two and define a logic on the connectives of implication and intersection.

Intersection Logic works with full binary trees called *kits*, whose leaves are formulas generated by the implication and the intersection. It is a natural deduction system which proves judgements in sequent style. Judgements include kits of the same structure (*overlapping* kits). Since IL aims to realize the part of LJ where $(\wedge I)$ is applied on isomorphic premises, the rule introducing the intersection should be such that the "sameness" of premises is explicitly shown. This is achieved by binary trees; the premises become leaves originating from the same parent-node in a kit, so

[1] Even if $M \equiv N$, the standard decoration would still place the pair $\langle M, M\rangle$ before the conjunction.

that the rule introducing the intersection in IL has only one premise. Its conclusion gives a kit where the intersection of the two leaves is a leaf on the parent-node. A non-standard decoration of kits, encoding the implication only, is now free to proceed in *any* IL-deduction.

$$\mathrm{LJ}: \quad \frac{\vdash M : \sigma \qquad \vdash M : \tau}{\vdash M : \sigma \wedge \tau} \qquad\qquad \mathrm{IL}: \quad \frac{\vdash M : [\sigma, \tau]}{\vdash M : \sigma \cap \tau}$$

A similar method is employed in Intersection Synchronous Logic. Its main syntactical structures are sequences of formulas called *atoms*. In [PR05], the formulas are generated by the implication and both synchronous and asynchronous conjunction; however, ISL is restricted to implication and intersection to provide a logical foundation for IT. It is a natural deduction system proving multisets called *molecules*, whose members are atoms. Isomorphic LJ-premises become atoms of the same molecule, so that the rule introducing the intersection has again a single premise. Its conclusion gives a molecule where the two atoms have merged in one that contains the intersection of the formulas. A non-standard decoration of molecules can run through *any* ISL-deduction.

$$\mathrm{LJ}: \quad \frac{\vdash M : \sigma \qquad \vdash M : \tau}{\vdash M : \sigma \wedge \tau} \qquad\qquad \mathrm{ISL}: \quad \frac{M : [(\quad ; \sigma), (\quad ; \tau)]}{M : [(\quad ; \sigma \cap \tau)]}$$

Conclusively, as far as LJ is concerned, IT corresponds to this proper subset obtained by imposing the metatheoretical condition of isomorphic premises on the use of conjunction. In the logical systems IL and ISL, there is no longer need for such a condition, since $(\cap I)$ rules involve a single premise. In other words, these two systems capture the synchronous aspect of conjunction only and for this reason can each be considered as a logic behind IT. The relation between them is reminiscent of the relation between sequent calculus and natural deduction in predicate logic.

In Chapter 2, we present the implicative and conjunctive fragment of intuitionistic logic (LJ) and state its properties (strong normalization theorem for LJ). We then introduce the simple types assignment system $\lambda^{\wedge}_{\rightarrow}$ and the types assignment system LJr—which is $\lambda^{\wedge}_{\rightarrow}$ supplied with intersection rules—and use them to distinguish between synchronous and asynchronous conjunction. By restricting LJr to implication and intersection, we derive the intersection types assignment system IT and explore by examples its typability power compared to Curry's types assignment system $\lambda_{\rightarrow}$. We also investigate the relation of LJ to each of the λ-calculus assignment systems $\lambda^{\wedge}_{\rightarrow}$, LJr and IT.

In Chapter 3, we start by defining the basic structures of Intersection Logic (kits) and the tools for manipulating them (overlapping of kits, substitution of subkits by new kits, pruning, implication of overlapping kits etc.). We then present the deductive system "pre Intersection Logic" (pIL), whose judgements are in sequent style and comprise of overlapping kits. Equivalence classes of pIL lead to the definition of IL. We continue by showing the transition from a pIL-deduction to a set of LJ-deductions that share some structural properties, namely they are all decoratable non-standardly by the same λ-term, which is also the term that decorates non-standardly the pIL-deduction. These LJ-deductions are called *LJ-projections* and we describe a *projection algorithm* for constructing them, given the pIL-deduction. We use the transition from pIL to LJ to prove the strong normalization property of IL and argue on the relation of IL to the subset of LJ that can be decorated non-standardly. On the latter, we offer new contributions by:

1. proving the transition from a set of LJ-deductions that are all decoratable non-standardly by the same λ-term to a pIL-deduction that is decoratable non-standardly by this very λ-term (section 3.6, theorem 3.6.4),
2. describing a *simulation algorithm* for constructing the pIL-deduction, given the LJ-deductions (section 3.6, example 3.6.6) and
3. proving a one-to-one correspondence between P-normal, proper IL-deductions and LJ-deductions decoratable non-standardly (section 3.6, theorem 3.6.11).

We end by giving the relation of IL to IT through a non-standard decoration of IL, from which we derive a proof that any term typable in IT is strongly normalizable. This relation of IL to IT establishes the appropriatness of IL as a logical foundation for IT.

In Chapter 4, we present ISL including the connectives of implication, intersection and conjunction. We define its main building blocks (atoms, molecules) and exhibit its deductive rules, which derive molecules. We then restrict ISL to implication and intersection and explore its relation to the part of LJ decoratable non-standardly. In particular, we show the transition from an ISL-deduction to a set of LJ-deductions that are all decoratable non-standardly by the same λ-term—which also decorates non-standardly the ISL-deduction—and contribute a proof on the transition from a set of LJ-deductions all decoratable non-standardly by the same λ-term to an ISL-deduction decoratable non-standardly by this very λ-term. We continue to work with ISL restricted and give its relation to IT through a non-standard decoration of its deductions. We finally prove the strong normalization property of ISL (considered with all three connectives) by reduction to the strong normalization of LJ. The results of this chapter are to a great extent similar to the ones shown in chapter 3 for IL.

In Chapter 5, we make a new contribution by *proving the equivalence of IL and ISL*. We first enrich IL with conjunction and then show the transition from a pIL-deduction to an ISL-deduction and vice versa. These transitions reduce to eliminating the binary structure from a kit-judgement to form a molecule on one hand and representing a molecule by a sequence of overlapping kits on the other.

Throughout the thesis, we focus on the analogy of IL and ISL, trying to highlight the similarities of methods despite the differences of structures.

Chapter 2

Intuitionistic Logic and Types Assignment Systems

2.1 Implicative and Conjunctive Fragment of Intuitionistic Logic (LJ)

We start by recalling the natural deduction of the *implicative and conjunctive fragment of Intuitionistic Logic (LJ)*. This logical system proves judgements in sequent style.

Definition 2.1.1 (LJ) *(i) The set $\mathcal{F}_{LJ}$ of* formulas of LJ *is generated by the grammar: $\sigma ::= \alpha \mid \sigma \to \sigma \mid \sigma \wedge \sigma$, where α belongs to a denumerable set of propositional variables. The implication is right associative, while the conjunction is left associative and the latter takes precedence over the former. Lowercase greek letters α, β, γ will denote propositional variables, while σ, τ, ρ will denote any formula.*

(ii) A LJ-context *is a finite multiset $\{\sigma_1, \ldots, \sigma_n\}$ of formulas of LJ. We will denote LJ-contexts by Γ, Δ.*

(iii) LJ proves statements of the form $\Gamma \vdash_{LJ} \sigma$, where Γ is a LJ-context and σ is a formula. Its rules are shown in Figure 2.1.

(iv) Writing $\Pi : \Gamma \vdash_{LJ} \sigma$ means that the LJ-deduction Π concludes by proving $\Gamma \vdash_{LJ} \sigma$.

$$\frac{\sigma \in \Gamma}{\Gamma \vdash_{LJ} \sigma}\,(A_{LJ}) \qquad \frac{\Gamma \vdash_{LJ} \sigma \qquad \Gamma \vdash_{LJ} \tau}{\Gamma \vdash_{LJ} \sigma \wedge \tau}\,(\wedge I_{LJ})$$

$$\frac{\Gamma \vdash_{LJ} \sigma \wedge \tau}{\Gamma \vdash_{LJ} \sigma}\,(\wedge E^l_{LJ}) \qquad \frac{\Gamma \vdash_{LJ} \sigma \wedge \tau}{\Gamma \vdash_{LJ} \tau}\,(\wedge E^r_{LJ})$$

$$\frac{\Gamma \cup \{\sigma\} \vdash_{LJ} \tau}{\Gamma \vdash_{LJ} \sigma \to \tau}\,(\to I_{LJ}) \qquad \frac{\Gamma \vdash_{LJ} \sigma \to \tau \qquad \Gamma \vdash_{LJ} \sigma}{\Gamma \vdash_{LJ} \tau}\,(\to E_{LJ})$$

Figure 2.1: The rules of LJ.

Example 2.1.2 Let σ denote the formula $((\alpha \to \alpha) \to \alpha \to \alpha) \wedge (\alpha \to \alpha)$. Deductions Π_1 and Π_2 combine under $(\to E)$ to give Π.

$$\dfrac{\dfrac{\dfrac{\{\sigma\} \vdash_{LJ} \sigma}{\{\sigma\} \vdash_{LJ} (\alpha \to \alpha) \to \alpha \to \alpha}\,(\wedge E^l) \qquad \dfrac{\{\sigma\} \vdash_{LJ} \sigma}{\{\sigma\} \vdash_{LJ} \alpha \to \alpha}\,(\wedge E^r)}{\{\sigma\} \vdash_{LJ} \alpha \to \alpha}\,(\to E)}{\Pi_1 : \vdash_{LJ} \sigma \to \alpha \to \alpha}\,(\to I)$$

$$\dfrac{\dfrac{\{\alpha \to \alpha\} \vdash_{LJ} \alpha \to \alpha}{\vdash_{LJ} (\alpha \to \alpha) \to \alpha \to \alpha}\,(\to I) \qquad \dfrac{\{\alpha\} \vdash_{LJ} \alpha}{\vdash_{LJ} \alpha \to \alpha}\,(\to I)}{\Pi_2 : \vdash_{LJ} ((\alpha \to \alpha) \to \alpha \to \alpha) \wedge (\alpha \to \alpha) \equiv \sigma}\,(\wedge I)$$

$$\dfrac{\Pi_1 : \vdash_{LJ} \sigma \to \alpha \to \alpha \qquad \Pi_2 : \vdash_{LJ} \sigma}{\Pi : \vdash_{LJ} \alpha \to \alpha}\,(\to E)$$

An equivalent version of LJ can be obtained by considering contexts as *sequences* of formulas instead of multisets, changing the axiom and adding rules for context weakening and exchange. The rules of this version of LJ are shown in Figure 2.2.

$$\dfrac{}{\sigma \vdash_{LJ} \sigma}\,(A_{LJ}) \qquad \dfrac{\Gamma \vdash_{LJ} \sigma}{\Gamma, \tau \vdash_{LJ} \sigma}\,(W_{LJ}) \qquad \dfrac{\Gamma, \sigma, \tau, \Delta \vdash_{LJ} \rho}{\Gamma, \tau, \sigma, \Delta \vdash_{LJ} \rho}\,(X_{LJ})$$

$$\dfrac{\Gamma \vdash_{LJ} \sigma \qquad \Gamma \vdash_{LJ} \tau}{\Gamma \vdash_{LJ} \sigma \wedge \tau}\,(\wedge I_{LJ}) \qquad \dfrac{\Gamma \vdash_{LJ} \sigma_l \wedge \sigma_r}{\Gamma \vdash_{LJ} \sigma_s}\,(\wedge E^s_{LJ}, s \in \{l, r\})$$

$$\dfrac{\Gamma, \sigma \vdash_{LJ} \tau}{\Gamma \vdash_{LJ} \sigma \to \tau}\,(\to I_{LJ}) \qquad \dfrac{\Gamma \vdash_{LJ} \sigma \to \tau \qquad \Gamma \vdash_{LJ} \sigma}{\Gamma \vdash_{LJ} \tau}\,(\to E_{LJ})$$

Figure 2.2: The rules of LJ, when contexts are sequences.

It is worth noting that the system, as presented in Definition 2.1.1, has the weakening property.

Proposition 2.1.3 (Weakening property for LJ) *If* $\Pi : \Gamma \vdash_{LJ} \sigma$*, then, for every formula* τ*, there exists* $\Pi_\tau : \Gamma \cup \{\tau\} \vdash_{LJ} \sigma$.

Proof: By easy induction on Π. $\dashv$

We continue to define implicative and conjunctive redexes of a LJ-deduction and to show how to eliminate them. We aim to state the strong normalization property of LJ.

Definition 2.1.4 *Let Π be a LJ-deduction and $s \in \{l, r\}$.*

(i) A $\to$-redex *of Π is a sequence $(\to I_{LJ}), (\to E_{LJ})$ in Π of consecutive rules introducing and eliminating the implication.*

(ii) A $\wedge$-redex *of Π is a sequence $(\wedge I_{LJ}), (\wedge E^s_{LJ})$ in Π of consecutive rules introducing and eliminating the conjunction.*

$$\dfrac{\dfrac{\Gamma \cup \{\tau\} \vdash_{LJ} \sigma}{\Gamma \vdash_{LJ} \tau \to \sigma}(\to I) \qquad \Gamma \vdash_{LJ} \tau}{\Gamma \vdash_{LJ} \sigma}(\to E) \qquad\qquad \dfrac{\dfrac{\Gamma \vdash_{LJ} \sigma_l \qquad \Gamma \vdash_{LJ} \sigma_r}{\Gamma \vdash_{LJ} \sigma_l \wedge \sigma_r}(\wedge I)}{\Gamma \vdash_{LJ} \sigma_s}(\wedge E^s)$$

(iii) We say that Π is normal, *if it is free of redexes.*

Definition 2.1.5 *Let $\{\tau_1, \ldots, \tau_n\} \vdash_{LJ} \sigma$ be a statement in position x of a LJ-deduction that consists of k steps $(0 \leqslant x \leqslant k)$. The context-formula τ_i is said to be* open, *if it doesn't move to the right of $\vdash_{LJ}$ by a $(\to I)$ rule in steps $x+1, \ldots, k$.*

The following lemma is used for the elimination of $\to$-redexes from LJ-deductions.

Lemma 2.1.6 (Substitution lemma) *Let $\Pi_0 : \Gamma \cup \{\tau\} \vdash_{LJ} \sigma$, $\Pi_1 : \Gamma \vdash_{LJ} \tau$ be LJ-deductions and $S(\Pi_1, \Pi_0)$ be the deductive structure obtained from Π_0 by substituting all axioms $\Gamma' \cup \{\tau\} \vdash_{LJ} \tau$ $(\Gamma \subseteq \Gamma')$ with τ open by $\Pi'_1 : \Gamma' \vdash_{LJ} \tau$. Then, $S(\Pi_1, \Pi_0) : \Gamma \vdash_{LJ} \sigma$.*

Proof: Use double induction, see [Pr65]. The method is also given in [Gi89] for λ-calculus normalization. $\dashv$

The next definition describes single normalization steps.

Definition 2.1.7 *Let Π be a LJ-deduction and $s \in \{l, r\}$.*

(i) A $\to$-rewriting step on Π is a normalization step that eliminates a $\to$-redex of the deduction.

$$\dfrac{\dfrac{\Pi_0 : \Gamma \cup \{\tau\} \vdash_{LJ} \sigma}{\Gamma \vdash_{LJ} \tau \to \sigma}(\to I) \qquad \Pi_1 : \Gamma \vdash_{LJ} \tau}{\Gamma \vdash_{LJ} \sigma}(\to E) \quad \hookrightarrow_{\to} \quad S(\Pi_1, \Pi_0) : \Gamma \vdash_{LJ} \sigma$$

(ii) A $\wedge$-rewriting step on Π is a normalization step that eliminates a $\wedge$-redex of the deduction.

$$\dfrac{\dfrac{\Pi_l : \Gamma \vdash_{LJ} \sigma_l \qquad \Pi_r : \Gamma \vdash_{LJ} \sigma_r}{\Gamma \vdash_{LJ} \sigma_l \wedge \sigma_r}(\wedge I)}{\Gamma \vdash_{LJ} \sigma_s}(\wedge E^s) \quad \hookrightarrow_{\wedge} \quad \Pi_s : \Gamma \vdash_{LJ} \sigma_s$$

Theorem 2.1.8 *LJ is strongly mormalizable, i.e. every LJ-deduction is strongly normalizable.*

Proof: See [Pr65, Gi89]. $\dashv$

We close this section by mentioning the sub-formula property for LJ.

Definition 2.1.9 *Let σ be a LJ-formula. Then:*

(i) σ is a sub-formula of σ and

(ii) if $\tau \diamond \rho$ is a sub-formula of σ, where $\diamond \in \{\to, \wedge\}$, then τ and ρ are sub-formulas of σ.

Theorem 2.1.10 *If $\Pi : \Gamma \vdash_{LJ} \sigma$ is a normal LJ-deduction, every formula appearing in Π is a sub-formula of one of the formulas occuring in the judgement $\Gamma \vdash_{LJ} \sigma$.*

Proof: By induction on Π. $\dashv$

2.2 The simple types assignment system $\lambda^{\wedge}_{\rightarrow}$

By decorating LJ with untyped λ-terms that encode the introduction and elimination of logical connectives, we derive the simple types assignment system $\lambda^{\wedge}_{\rightarrow}$, a deductive system which attributes formulas—built from the implication and the conjunction—as types to the untyped λ-calculus with pairs. Let us call this kind of decoration of LJ *standard* and denote it by $\mathsf{d}^{\wedge}_{\rightarrow}$. In this section, we present $\lambda^{\wedge}_{\rightarrow}$ and uncoil its correspondence with LJ.

Definition 2.2.1 ($\lambda^{\wedge}_{\rightarrow}$) *(i) The terms of the untyped λ-calculus with pairs Λ_p are defined by the grammar: $M ::= x \mid \lambda x.M \mid (M)M \mid \langle M, M\rangle \mid \pi_1(M) \mid \pi_2(M)$, where x belongs to a countable set of variables. Instead of $\pi_1(M), \pi_2(M)$, we can equivalently use $\pi_l(M), \pi_r(M)$, respectively. Application*[1] *is left associative. Letters M, N will range over terms in Λ_p.*

(ii) The types of $\lambda^{\wedge}_{\rightarrow}$ or simple types *are generated by the grammar:*

$$\sigma ::= \alpha \mid \sigma \rightarrow \sigma \mid \sigma \wedge \sigma$$

i.e. they coincide with LJ-formulas.

*(iii) A $\lambda^{\wedge}_{\rightarrow}$-*context*, denoted by Γ or Δ, is a finite set of simple types assignments to distinct variables*[2]*. If Γ is the $\lambda^{\wedge}_{\rightarrow}$-context $\{x_1 : \sigma_1, \dots, x_n : \sigma_n\}$, we define $dom(\Gamma)$ to be the set $\{x_1, \dots, x_n\}$.*

(iv) The system $\lambda^{\wedge}_{\rightarrow}$ derives judgements of the form $\Gamma \vdash_{\lambda^{\wedge}_{\rightarrow}} M : \sigma$, read "$M$ is of type σ in the context Γ" or "M may be given type σ in the context Γ". Its rules are exhibited in Figure 2.3. The expression $\Pi : \Gamma \vdash_{\lambda^{\wedge}_{\rightarrow}} M : \sigma$ acquires the expected meaning.

$$\frac{x : \sigma \in \Gamma}{\Gamma \vdash_{\lambda^{\wedge}_{\rightarrow}} x : \sigma}\ (A) \qquad \frac{\Gamma \vdash_{\lambda^{\wedge}_{\rightarrow}} M : \sigma \qquad \Gamma \vdash_{\lambda^{\wedge}_{\rightarrow}} N : \tau}{\Gamma \vdash_{\lambda^{\wedge}_{\rightarrow}} \langle M, N\rangle : \sigma \wedge \tau}\ (\wedge I)$$

$$\frac{\Gamma \vdash_{\lambda^{\wedge}_{\rightarrow}} M : \sigma \wedge \tau}{\Gamma \vdash_{\lambda^{\wedge}_{\rightarrow}} \pi_1(M) : \sigma}\ (\wedge E^l) \qquad \frac{\Gamma \vdash_{\lambda^{\wedge}_{\rightarrow}} M : \sigma \wedge \tau}{\Gamma \vdash_{\lambda^{\wedge}_{\rightarrow}} \pi_2(M) : \tau}\ (\wedge E^r)$$

$$\frac{\Gamma \cup \{x : \sigma\} \vdash_{\lambda^{\wedge}_{\rightarrow}} M : \tau}{\Gamma \vdash_{\lambda^{\wedge}_{\rightarrow}} \lambda x.M : \sigma \rightarrow \tau}\ (\rightarrow I) \qquad \frac{\Gamma \vdash_{\lambda^{\wedge}_{\rightarrow}} M : \sigma \rightarrow \tau \qquad \Gamma \vdash_{\lambda^{\wedge}_{\rightarrow}} N : \sigma}{\Gamma \vdash_{\lambda^{\wedge}_{\rightarrow}} MN : \tau}\ (\rightarrow E)$$

Figure 2.3: The rules of $\lambda^{\wedge}_{\rightarrow}$.

[1]We follow the Krivine notation on application.

[2]Note that we can assign a simple type σ to more than one variables in a $\lambda^{\wedge}_{\rightarrow}$-context.

If we decorate the sequence version of LJ by $\mathsf{d}^{\wedge}_{\to}$, we obtain a sequence version of $\lambda^{\wedge}_{\to}$, whose rules are shown in Figure 2.4. Contexts are now *sequences* of variable assignments, the variables still being distinct. If Γ is the context $x_1 : \sigma_1, \ldots, x_n : \sigma_n$, then $dom(\Gamma) = \{x_1, \ldots, x_n\}$. In the rule (W), $BV(M)$ denotes the set of bound variables of the term M.

$$\frac{}{x : \sigma \vdash_{\lambda^{\wedge}_{\to}} x : \sigma}\,(A)$$

$$\frac{\Gamma \vdash_{\lambda^{\wedge}_{\to}} M : \sigma,\ x \notin dom(\Gamma) \cup BV(M)}{\Gamma, x : \tau \vdash_{\lambda^{\wedge}_{\to}} M : \sigma}\,(W)$$

$$\frac{\Gamma_1, x : \sigma_1, y : \sigma_2, \Gamma_2 \vdash_{\lambda^{\wedge}_{\to}} M : \sigma}{\Gamma_1, y : \sigma_2, x : \sigma_1, \Gamma_2 \vdash_{\lambda^{\wedge}_{\to}} M : \sigma}\,(X)$$

$$\frac{\Gamma \vdash_{\lambda^{\wedge}_{\to}} M : \sigma \qquad \Gamma \vdash_{\lambda^{\wedge}_{\to}} N : \tau}{\Gamma \vdash_{\lambda^{\wedge}_{\to}} \langle M, N\rangle : \sigma \wedge \tau}\,(\wedge I) \qquad \frac{\Gamma \vdash_{\lambda^{\wedge}_{\to}} M : \sigma_l \wedge \sigma_r}{\Gamma \vdash_{\lambda^{\wedge}_{\to}} \pi_s(M) : \sigma_s}\,(\wedge E^s, s \in \{l, r\})$$

$$\frac{\Gamma, x : \sigma \vdash_{\lambda^{\wedge}_{\to}} M : \tau}{\Gamma \vdash_{\lambda^{\wedge}_{\to}} \lambda x.M : \sigma \to \tau}\,(\to I) \qquad \frac{\Gamma \vdash_{\lambda^{\wedge}_{\to}} M : \sigma \to \tau \qquad \Gamma \vdash_{\lambda^{\wedge}_{\to}} N : \sigma}{\Gamma \vdash_{\lambda^{\wedge}_{\to}} (M)N : \tau}\,(\to E)$$

Figure 2.4: The rules of $\lambda^{\wedge}_{\to}$, when contexts are sequences.

Conclusively, we can say that LJ is the logic behind the simple types assignment system $\lambda^{\wedge}_{\to}$ or, in other words, that LJ offers a logical foundation for $\lambda^{\wedge}_{\to}$ through $\mathsf{d}^{\wedge}_{\to}$.

2.3 Synchronous and asynchronous conjunction

In the simple types assignment system $\lambda^{\wedge}_{\to}$, the rule

$$\frac{\Gamma \vdash M : \sigma \qquad \Gamma \vdash N : \tau}{\Gamma \vdash \langle M, N\rangle : \sigma \wedge \tau}\,(\wedge I)$$

concludes that the pair $\langle M, N\rangle$ is of type $\sigma \wedge \tau$ from the premises "M of type σ" and "N of type τ", no matter if M and N are identical[3] or not. Thus, the conclusion term $\langle M, N\rangle$ captures in its syntax the introduction of conjunction between the premise types σ and τ. A similar remark holds for $(\wedge E^s)$: the conclusion term mirrors the elimination of conjunction from the premise type. This sort of typing reflects the general aspect of conjunction: the *asynchronous conjunction* or *conjunction*, denoted $\wedge$.

Suppose, though, that, when introducing the conjunction, we distinguish the case where premise terms are identical, say denoted by M, and choose to assign the conjunction of the premise types to this term M.

[3] We identify λ-terms modulo α-conversion.

$$\frac{\Gamma \vdash M : \sigma \qquad \Gamma \vdash M : \tau}{\Gamma \vdash M : \sigma \wedge \tau} (\wedge I)$$

Similarly, when eliminating the conjunction, suppose we choose to assign the premise and conclusion types to the same term. This way, the conclusion term gets syntactically disconnected from its type and a special aspect of conjunction is revealed: the *synchronous conjunction* or *intersection*, denoted $\cap$. In what follows, we consider $\cap$ to be a logical connective, standing for conjunction with synchronous behaviour.

2.3.1 The types assignment system LJr

Adding rules to $\lambda^{\wedge}_{\rightarrow}$ for the introduction and elimination of intersection, we get a more elaborate types assignment system for terms in Λ_p, denoted LJr in [PR05].

Definition 2.3.1 (LJr) *(i) The set $\mathcal{F}_{LJr}$ of types of LJr is defined by the grammar: $\sigma ::= \alpha \mid \sigma \to \sigma \mid \sigma \wedge \sigma \mid \sigma \cap \sigma$, where α belongs to a countable set of propositional variables, implication is right associative and both conjunction and intersection are left associative. The connectives $\wedge$ and $\cap$ are equivalent with respect to order of application, but they both precede $\to$.*

(ii) A LJr-context*, denoted by Γ or Δ, is a finite set of LJr-types assignments to distinct variables. If $\Gamma = \{x_1 : \sigma_1, \ldots, x_n : \sigma_n\}$, then $dom(\Gamma) = \{x_1, \ldots, x_n\}$.*

(iii) The system LJr derives statements of the form $\Gamma \vdash_{LJr} M : \sigma$, where Γ is a LJr-context, $M \in \Lambda_p$ and σ is a LJr-type. Its rules are gathered in Figure 2.5. The expression $\Pi : \Gamma \vdash_{LJr} M : \sigma$ is interpreted as usual.

$$\frac{x : \sigma \in \Gamma}{\Gamma \vdash_{LJr} x : \sigma} (A)$$

$$\frac{\Gamma \vdash_{LJr} M : \sigma \qquad \Gamma \vdash_{LJr} N : \tau}{\Gamma \vdash_{LJr} \langle M, N \rangle : \sigma \wedge \tau} (\wedge I) \qquad \frac{\Gamma \vdash_{LJr} M : \sigma_l \wedge \sigma_r}{\Gamma \vdash_{LJr} \pi_s(M) : \sigma_s} (\wedge E^s, s \in \{l, r\})$$

$$\frac{\Gamma \vdash_{LJr} M : \sigma \qquad \Gamma \vdash_{LJr} M : \tau}{\Gamma \vdash_{LJr} M : \sigma \cap \tau} (\cap I) \qquad \frac{\Gamma \vdash_{LJr} M : \sigma_l \cap \sigma_r}{\Gamma \vdash_{LJr} M : \sigma_s} (\cap E^s,\ s \in \{l, r\})$$

$$\frac{\Gamma \cup \{x : \sigma\} \vdash_{LJr} M : \tau}{\Gamma \vdash_{LJr} \lambda x.M : \sigma \to \tau} (\to I) \qquad \frac{\Gamma \vdash_{LJr} M : \sigma \to \tau \qquad \Gamma \vdash_{LJr} N : \sigma}{\Gamma \vdash_{LJr} MN : \tau} (\to E)$$

Figure 2.5: The rules of LJr.

As in the case of $\lambda^{\wedge}_{\rightarrow}$, a sequence version of LJr can be formed.

2.4 Relation of LJ to LJr

As already mentioned in section 2.2, if we decorate *any* LJ-deduction by $\mathsf{d}^{\wedge}_{\rightarrow}$, we obtain a $\lambda^{\wedge}_{\rightarrow}$-deduction, which is also a LJr-deduction, since $\lambda^{\wedge}_{\rightarrow}$ is a subsystem of LJr. In this section, we follow [RR01] in defining a *non-standard* decoration of LJ, denoted

$\mathsf{d}_{\rightarrow}$ in this thesis, which encodes the implicative rules only. We show that *some* LJ-deductions can be decorated by $\mathsf{d}_{\rightarrow}$ and have $\wedge$ altered to $\cap$ to give LJr-deductions. On the other hand, any LJr-deduction gives uniquely a LJ-deduction, when fired by an erasing function E that converts LJr-contexts to LJ-contexts, LJr-types to LJ-formulas and erases the typable terms. The picture is this: some LJ-deductions can be decorated by both $\mathsf{d}^{\wedge}_{\rightarrow}$ and $\mathsf{d}_{\rightarrow}$ to give two distinct LJr-deductions, which will both return the initial LJ-deduction, when fired by E. Which LJ-deductions are these? The ones that can be decorated by $\mathsf{d}_{\rightarrow}$, since all can be decorated by $\mathsf{d}^{\wedge}_{\rightarrow}$. And which are these? We examine this crucial question in what follows.

Definition 2.4.1 ($\mathsf{d}_{\rightarrow}$: non-standard decoration of LJ) *(i) Consider a LJ-context* $\Delta \equiv \{\sigma_1, \ldots, \sigma_n\}$. *A* decoration Δ^* of Δ *is a set* $\{x_1 : \sigma_1, \ldots, x_n : \sigma_n\}$, *where the* x_i*'s are distinct* λ*-variables. Then,* $dom(\Delta^*)$ *is the sequence* $x_1, \ldots, x_n$.

(ii) Every $\Pi : \Delta \vdash_{LJ} \sigma$ *can be associated through an inductive algorithm to a decorated deduction*

$$\Pi^* : \Delta^* \vdash^*_{LJ} T_{dom(\Delta^*)}(\Pi) : \sigma$$

where Δ^* *is a decoration of* Δ, $\vdash^*_{LJ}$ *denotes the decoration of* $\vdash_{LJ}$ *and* $T_{dom(\Delta^*)}(\Pi)$ *is in* Λ.

- $$\frac{\sigma \in \Delta}{\Pi : \Delta \vdash_{LJ} \sigma}\,(A) \quad \Rightarrow \quad \frac{x : \sigma \in \Delta^*}{\Pi^* : \Delta^* \vdash^*_{LJ} x : \sigma}\,(A^*)$$

  and $T_{dom(\Delta^*)}(\Pi) \equiv x$.

- $$\frac{\Pi_1 : \Delta \vdash_{LJ} \sigma \qquad \Pi_2 : \Delta \vdash_{LJ} \tau}{\Pi : \Delta \vdash_{LJ} \sigma \wedge \tau}\,(\wedge I) \quad \Rightarrow$$

  $$\frac{\Pi_1^* : \Delta^* \vdash^*_{LJ} T_{dom(\Delta^*)}(\Pi_1) : \sigma \qquad \Pi_2^* : \Delta^* \vdash^*_{LJ} T_{dom(\Delta^*)}(\Pi_2) : \tau}{\Pi^* : \Delta^* \vdash^*_{LJ} T_{dom(\Delta^*)}(\Pi) : \sigma \wedge \tau}\,(\wedge I^*)$$

  where $T_{dom(\Delta^*)}(\Pi) =_\alpha T_{dom(\Delta^*)}(\Pi_1)$, *if* $T_{dom(\Delta^*)}(\Pi_1) =_\alpha T_{dom(\Delta^*)}(\Pi_2)$ *and is undefined otherwise.*

- $$\frac{\Pi_1 : \Delta \vdash_{LJ} \sigma_l \wedge \sigma_r}{\Pi : \Delta \vdash_{LJ} \sigma_s}\,(\wedge E^s) \quad \Rightarrow$$

  $$\frac{\Pi_1^* : \Delta^* \vdash^*_{LJ} T_{dom(\Delta^*)}(\Pi_1) : \sigma_l \wedge \sigma_r}{\Pi^* : \Delta^* \vdash^*_{LJ} T_{dom(\Delta^*)}(\Pi) : \sigma_s}\,(\wedge E^s)^*$$

  where $T_{dom(\Delta^*)}(\Pi) \equiv T_{dom(\Delta^*)}(\Pi_1)$.

- $$\frac{\Pi_1 : \Delta \cup \{\sigma\} \vdash_{LJ} \tau}{\Pi : \Delta \vdash_{LJ} \sigma \rightarrow \tau}\,(\rightarrow I) \quad \Rightarrow$$

  $$\frac{\Pi_1^* : \Delta^* \cup \{x : \sigma\} \vdash^*_{LJ} T_{dom(\Delta^*),x}(\Pi_1) : \tau}{\Pi^* : \Delta^* \vdash^*_{LJ} T_{dom(\Delta^*)}(\Pi) : \sigma \rightarrow \tau}\,(\rightarrow I^*)$$

where x is not in $dom(\Delta^*)$ and $T_{dom(\Delta^*)}(\Pi) \equiv \lambda x.T_{dom(\Delta^*),x}(\Pi_1)$.

- $\dfrac{\Pi_1 : \Delta \vdash_{LJ} \sigma \rightarrow \tau \qquad \Pi_2 : \Delta \vdash_{LJ} \sigma}{\Pi : \Delta \vdash_{LJ} \tau} (\rightarrow E) \;\Rightarrow$

$$\dfrac{\Pi_1^* : \Delta^* \vdash_{LJ}^* T_{dom(\Delta^*)}(\Pi_1) : \sigma \rightarrow \tau \qquad \Pi_2^* : \Delta^* \vdash_{LJ}^* T_{dom(\Delta^*)}(\Pi_2) : \sigma}{\Pi^* : \Delta^* \vdash_{LJ}^* T_{dom(\Delta^*)}(\Pi) : \tau} (\rightarrow E^*)$$

where $T_{dom(\Delta^*)}(\Pi) \equiv T_{dom(\Delta^*)}(\Pi_1)T_{dom(\Delta^*)}(\Pi_2)$.

(iii) If $\Pi : \Delta \vdash_{LJ} \sigma$ is decoratable by $\mathsf{d}_{\rightarrow}$, then

$$U(\Pi) \stackrel{\text{def}}{=} \{T_{dom(\Delta^*)}(\Pi) \mid dom(\Delta^*) \textit{ is a sequence of } |\Delta| \textit{ distinct variables}\}$$

The set $U(\Pi)$ is called the form *of Π.*

Remark 2.4.2 (i) The set $FV(T_{dom(\Delta^*)}(\Pi))$ of free variables of $T_{dom(\Delta^*)}(\Pi)$ is a subset of $\{dom(\Delta^*)\}$. (ii) The set $BV(T_{dom(\Delta^*)}(\Pi))$ of bound variables of $T_{dom(\Delta^*)}(\Pi)$ is disjoint from $\{dom(\Delta^*)\}$. (iii) If $BV(T_{dom(\Delta^*)}(\Pi))$ is non-empty and x, y belong to it, then $x \not\equiv y$. (iv) For every sequence $dom(\Delta^*)$, $T_{dom(\Delta^*)}(\Pi)$ is actually *a set of α-equivalent terms*, since every possible choice of bound variables should be considered; hence, it is actually

$$U(\Pi) = \bigcup_{dom(\Delta^*)} T_{dom(\Delta^*)}(\Pi)$$

(v) If $M \not\equiv N$ and $M, N \in U(\Pi)$, then M and N have the same term-structure, since they both trace the implicative rules of Π, but there is at least one variable position (free or bound) on which they differ.

Definition 2.4.3 (Erasing function e) *(i) Let $e : \mathcal{F}_{LJr} \rightarrow \mathcal{F}_{LJ}$ be defined as: $e(\alpha) = \alpha$, $e(\sigma \rightarrow \tau) = e(\sigma) \rightarrow e(\tau)$ and $e(\sigma \wedge \tau) = e(\sigma \cap \tau) = e(\sigma) \wedge e(\tau)$.*

(ii) The function e can be extended to contexts to convert LJr-contexts to LJ-contexts in the obvious way: $e(\{x_1 : \sigma_1, \ldots, x_n : \sigma_n\}) = \{e(\sigma_1), \ldots, e(\sigma_n)\}$.

A theorem relating LJ and LJr is now in order.

Theorem 2.4.4 *Let E be a function from LJr-deductions to LJ-deductions that erases all term information and collapses $\cap$ to $\wedge$.*

(i) If $\Pi : \Gamma \vdash_{LJr} M : \sigma$, then $E(\Pi) : e(\Gamma) \vdash_{LJ} e(\sigma)$.

(ii) If $\Pi : \Gamma \vdash_{LJ} \sigma$, then there is a LJr-deduction $\Pi' : \Gamma' \vdash_{LJr} M : \sigma'$, such that $E(\Pi') \equiv \Pi$, $e(\Gamma') \equiv \Gamma$ and $e(\sigma') \equiv \sigma$.

Proof: (i) By induction on Π.

Inductive step: The case of $(\cap I)$ is shown.

- $\dfrac{\Pi_1 : \Gamma \vdash_{LJr} M : \sigma \qquad \Pi_2 : \Gamma \vdash_{LJr} M : \tau}{\Pi : \Gamma \vdash_{LJr} M : \sigma \cap \tau} (\cap I)$

By the Ind. Hyp., we have $E(\Pi_1) : e(\Gamma) \vdash_{LJ} e(\sigma)$ and $E(\Pi_2) : e(\Gamma) \vdash_{LJ} e(\tau)$. Applying $(\wedge I_{LJ})$ on $E(\Pi_1)$ and $E(\Pi_2)$, we get $E(\Pi) : e(\Gamma) \vdash_{LJ} e(\sigma) \wedge e(\tau) \equiv e(\sigma \cap \tau)$.

(ii) Decorate Π by $\mathsf{d}^{\wedge}_{\rightarrow}$ to take Π'. $\dashv$

It is obvious that if Π is decoratable by $\mathsf{d}_{\rightarrow}$, we can find two distinct LJr-deductions with the property stated in Theorem 2.4.4(ii). One formed by decorating Π by $\mathsf{d}^{\wedge}_{\rightarrow}$ and another one by decorating it by $\mathsf{d}_{\rightarrow}$ and changing $\wedge$ to $\cap$. This point is illustrated in the following example.

Example 2.4.5 We present two LJr-deductions Π_1 and Π_2. Let

$$\sigma \equiv ((\alpha \to \alpha) \to \alpha \to \alpha) \wedge (\alpha \to \alpha), \quad \tau \equiv ((\alpha \to \alpha) \to \alpha \to \alpha) \cap (\alpha \to \alpha)$$

Deduction Π_1 consists of subdeductions Π_1' and Π_1'', while Π_2 includes Π_2' and Π_2''.

$$\dfrac{\dfrac{\dfrac{\{x:\sigma\} \vdash_{LJr} x:\sigma}{\{x:\sigma\} \vdash_{LJr} \pi_1(x):(\alpha \to \alpha) \to \alpha \to \alpha}\,(\wedge E^l) \qquad \dfrac{\{x:\sigma\} \vdash_{LJr} x:\sigma}{\{x:\sigma\} \vdash_{LJr} \pi_2(x):\alpha \to \alpha}\,(\wedge E^r)}{\{x:\sigma\} \vdash_{LJr} \pi_1(x)\pi_2(x):\alpha \to \alpha}\,(\to E)}{\Pi_1' : \vdash_{LJr} \lambda x.\pi_1(x)\pi_2(x):\sigma \to \alpha \to \alpha}\,(\to I)$$

$$\dfrac{\dfrac{\{x:\alpha \to \alpha\} \vdash_{LJr} x:\alpha \to \alpha}{\vdash_{LJr} \lambda x.x:(\alpha \to \alpha) \to \alpha \to \alpha}\,(\to I) \qquad \dfrac{\{x:\alpha\} \vdash_{LJr} x:\alpha}{\vdash_{LJr} \lambda x.x:\alpha \to \alpha}\,(\to I)}{\Pi_1'' : \vdash_{LJr} \langle \lambda x.x, \lambda x.x \rangle : \sigma}\,(\wedge I)$$

$$\dfrac{\Pi_1' : \vdash_{LJr} \lambda x.\pi_1(x)\pi_2(x):\sigma \to \alpha \to \alpha \qquad \Pi_1'' : \vdash_{LJr} \langle \lambda x.x, \lambda x.x \rangle : \sigma}{\Pi_1 : \vdash_{LJr} (\lambda x.\pi_1(x)\pi_2(x))\langle \lambda x.x, \lambda x.x \rangle : \alpha \to \alpha}\,(\to E)$$

$$\dfrac{\dfrac{\dfrac{\{x:\tau\} \vdash_{LJr} x:\tau}{\{x:\tau\} \vdash_{LJr} x:(\alpha \to \alpha) \to \alpha \to \alpha}\,(\cap E^l) \qquad \dfrac{\{x:\tau\} \vdash_{LJr} x:\tau}{\{x:\tau\} \vdash_{LJr} x:\alpha \to \alpha}\,(\cap E^r)}{\{x:\tau\} \vdash_{LJr} xx:\alpha \to \alpha}\,(\to E)}{\Pi_2' : \vdash_{LJr} \lambda x.xx:\tau \to \alpha \to \alpha}\,(\to I)$$

$$\dfrac{\dfrac{\{x:\alpha \to \alpha\} \vdash_{LJr} x:\alpha \to \alpha}{\vdash_{LJr} \lambda x.x:(\alpha \to \alpha) \to \alpha \to \alpha}\,(\to I) \qquad \dfrac{\{x:\alpha\} \vdash_{LJr} x:\alpha}{\vdash_{LJr} \lambda x.x:\alpha \to \alpha}\,(\to I)}{\Pi_2'' : \vdash_{LJr} \lambda x.x : \tau}\,(\cap I)$$

$$\dfrac{\Pi_2' : \vdash_{LJr} \lambda x.xx:\tau \to \alpha \to \alpha \qquad \Pi_2'' : \vdash_{LJr} \lambda x.x:\tau}{\Pi_2 : \vdash_{LJr} (\lambda x.xx)\lambda x.x:\alpha \to \alpha}\,(\to E)$$

The erasing function E applied on both Π_1 and Π_2 returns the LJ-deduction Π of example 2.1.2. In fact, decorating Π by $\mathsf{d}^{\wedge}_{\rightarrow}$ gives Π_1, while decorating it by $\mathsf{d}_{\rightarrow}$ and having $\wedge$ converted to $\cap$ gives Π_2. So, Π belongs to the class of LJ-deductions that can be decorated by $\mathsf{d}_{\rightarrow}$; let us call it $D_{\overrightarrow{LJ}}$. For a LJ-deduction to belong to $D_{\overrightarrow{LJ}}$, the following must hold: wherever $\wedge$ is introduced, it must be done between two subdeductions which belong to $D_{\overrightarrow{LJ}}$ and are encoded by the same λ-term, i.e. by subdeductions *isomorphic with respect to $d_{\rightarrow}$*. We clarify this point with yet another example of a LJ-deduction in $D_{\overrightarrow{LJ}}$.

Example 2.4.6 Let $\sigma \equiv (\alpha \to \beta \wedge \gamma) \wedge (\beta \to \alpha \wedge \gamma) \wedge (\gamma \to \alpha)$, with α, β and γ propositional variables. The LJ-deductions

$$\Pi_1 : \{\sigma\} \vdash_{LJ} \alpha \to \beta,\ \Pi_2 : \{\sigma\} \vdash_{LJ} \alpha \to \gamma,\ \Pi_3 : \{\sigma\} \vdash_{LJ} \beta \to \alpha,\ \Pi_4 : \{\sigma\} \vdash_{LJ} \gamma \to \alpha$$

combine to create $\Pi_4^{(12)3} \in D_{LJ}^{\to}$ in the following way:

$$\dfrac{\dfrac{\dfrac{\dfrac{\{\alpha,\sigma\} \vdash_{LJ} \sigma}{\{\alpha,\sigma\} \vdash_{LJ} \alpha \to \beta \wedge \gamma}{\scriptstyle(\wedge E^l)\times 2} \qquad \{\alpha,\sigma\} \vdash_{LJ} \alpha}{\{\alpha,\sigma\} \vdash_{LJ} \beta \wedge \gamma}{\scriptstyle(\to E)}}{\{\alpha,\sigma\} \vdash_{LJ} \beta}{\scriptstyle(\wedge E^l)}}{\Pi_1 : \{\sigma\} \vdash_{LJ} \alpha \to \beta}{\scriptstyle(\to I)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\{\alpha,\sigma\} \vdash_{LJ} \sigma}{\{\alpha,\sigma\} \vdash_{LJ} \alpha \to \beta \wedge \gamma}{\scriptstyle(\wedge E^l)\times 2} \qquad \{\alpha,\sigma\} \vdash_{LJ} \alpha}{\{\alpha,\sigma\} \vdash_{LJ} \beta \wedge \gamma}{\scriptstyle(\to E)}}{\{\alpha,\sigma\} \vdash_{LJ} \gamma}{\scriptstyle(\wedge E^r)}}{\Pi_2 : \{\sigma\} \vdash_{LJ} \alpha \to \gamma}{\scriptstyle(\to I)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\{\beta,\sigma\} \vdash_{LJ} \sigma}{\{\beta,\sigma\} \vdash_{LJ} \beta \to \alpha \wedge \gamma}{\scriptstyle(\wedge E^l),(\wedge E^r)} \qquad \{\beta,\sigma\} \vdash_{LJ} \beta}{\{\beta,\sigma\} \vdash_{LJ} \alpha \wedge \gamma}{\scriptstyle(\to E)}}{\{\beta,\sigma\} \vdash_{LJ} \alpha}{\scriptstyle(\wedge E^l)}}{\Pi_3 : \{\sigma\} \vdash_{LJ} \beta \to \alpha}{\scriptstyle(\to I)}$$

$$\dfrac{\dfrac{\dfrac{\{\gamma,\sigma\} \vdash_{LJ} \sigma}{\{\gamma,\sigma\} \vdash_{LJ} \gamma \to \alpha}{\scriptstyle(\wedge E^r)} \qquad \{\gamma,\sigma\} \vdash_{LJ} \gamma}{\{\gamma,\sigma\} \vdash_{LJ} \alpha}{\scriptstyle(\to E)}}{\Pi_4 : \{\sigma\} \vdash_{LJ} \gamma \to \alpha}{\scriptstyle(\to I)}$$

$$\dfrac{\dfrac{\Pi_1 : \{\sigma\} \vdash_{LJ} \alpha \to \beta \qquad \Pi_2 : \{\sigma\} \vdash_{LJ} \alpha \to \gamma}{\Pi_2^1 : \{\sigma\} \vdash_{LJ} (\alpha \to \beta) \wedge (\alpha \to \gamma)}{\scriptstyle(\wedge I)} \qquad \Pi_3 : \{\sigma\} \vdash_{LJ} \beta \to \alpha}{\Pi_3^{12} : \{\sigma\} \vdash_{LJ} (\alpha \to \beta) \wedge (\alpha \to \gamma) \wedge (\beta \to \alpha)}{\scriptstyle(\wedge I)}$$

$$\dfrac{\Pi_3^{12} : \{\sigma\} \vdash_{LJ} (\alpha \to \beta) \wedge (\alpha \to \gamma) \wedge (\beta \to \alpha) \qquad \Pi_4 : \{\sigma\} \vdash_{LJ} \gamma \to \alpha}{\Pi_4^{(12)3} : \{\sigma\} \vdash_{LJ} (\alpha \to \beta) \wedge (\alpha \to \gamma) \wedge (\beta \to \alpha) \wedge (\gamma \to \alpha)}{\scriptstyle(\wedge I)}$$

We shall briefly explain how $\Pi_4^{(12)3}$ is decorated by $\mathsf{d}_{\rightarrow}$, showing the isomorphism with respect to $\mathsf{d}_{\rightarrow}$ of $(\Pi_1, \Pi_2), (\Pi_2^1, \Pi_3)$ and (Π_3^{12}, Π_4). Suppose we decorate $\{\sigma\}$ of the judgement proved by $\Pi_4^{(12)3}$ by x. The decoration moves contextwise upwards and then to the right in axiom level. Thus, we get the decorated axioms $\{y : \alpha, x : \sigma\} \vdash^*_{LJ} x : \sigma$, $\{y : \alpha, x : \sigma\} \vdash^*_{LJ} y : \alpha$ of Π_1 and Π_2, $\{y : \beta, x : \sigma\} \vdash^*_{LJ} x : \sigma$ and $\{y : \beta, x : \sigma\} \vdash^*_{LJ} y : \beta$ of Π_3 and $\{y : \gamma, x : \sigma\} \vdash^*_{LJ} x : \sigma$, $\{y : \gamma, x : \sigma\} \vdash^*_{LJ} y : \gamma$ of Π_4. Applying the non-standard decoration rules of definition 2.4.1 to Π_1, Π_2, Π_3 and Π_4, we get the decorated deductions

$$\Pi_1^* : \{x : \sigma\} \vdash^*_{LJ} \lambda y.xy : \alpha \rightarrow \beta, \quad \Pi_2^* : \{x : \sigma\} \vdash^*_{LJ} \lambda y.xy : \alpha \rightarrow \gamma$$

$$\Pi_3^* : \{x : \sigma\} \vdash^*_{LJ} \lambda y.xy : \beta \rightarrow \alpha, \quad \Pi_4^* : \{x : \sigma\} \vdash^*_{LJ} \lambda y.xy : \gamma \rightarrow \alpha$$

Deductions Π_1^* and Π_2^* give $(\Pi_2^1)^* : \{x : \sigma\} \vdash^*_{LJ} \lambda y.xy : (\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \gamma)$ and then $(\Pi_2^1)^*$ and Π_3^* give $(\Pi_3^{12})^* : \{x : \sigma\} \vdash^*_{LJ} \lambda y.xy : (\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \gamma) \wedge (\beta \rightarrow \alpha)$. Finally, $(\Pi_3^{12})^*$ and Π_4^* give

$$(\Pi_4^{(12)3})^* : \{x : \sigma\} \vdash^*_{LJ} \lambda y.xy : (\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \gamma) \wedge (\beta \rightarrow \alpha) \wedge (\gamma \rightarrow \alpha)$$

2.5 The intersection types assignment system IT

So far, we have presented the logical system LJ and the types assignment systems $\lambda^{\wedge}_{\rightarrow}$ and LJr. LJ corresponds to $\lambda^{\wedge}_{\rightarrow}$ in the following manner: any LJ-deduction can be decorated by $\mathsf{d}^{\wedge}_{\rightarrow}$ to produce a $\lambda^{\wedge}_{\rightarrow}$-deduction which, when fired by the erasing function E, gives back the LJ-deduction and any $\lambda^{\wedge}_{\rightarrow}$-deduction can decompose through E to a LJ-deduction which, when decorated by $\mathsf{d}^{\wedge}_{\rightarrow}$, returns the $\lambda^{\wedge}_{\rightarrow}$-deduction modulo sequences of distinct variables decorating contexts. In other words, for every LJ-deduction there is a unique—modulo sequences of variables—$\lambda^{\wedge}_{\rightarrow}$-deduction which collapses to it, if fired by E and for every $\lambda^{\wedge}_{\rightarrow}$-deduction there is a unique LJ-deduction which produces it, if decorated by $\mathsf{d}^{\wedge}_{\rightarrow}$. Since the decoration involved encodes all the connectives of the logic, the correspondence described follows the manner of the Curry-Howard isomorphism[4].

The intersection types assignment system IT is the subsystem of LJr where only synchronous conjunction is used. Which logical system, if any, corresponds to IT and which is the mode of correspondence in this case? These are the main issues this thesis attempts to examine. In this section, we present IT and state its main properties.

Definition 2.5.1 (IT) *(i) Terms of the untyped λ-calculus Λ are defined by the grammar: $M ::= x \mid \lambda x.M \mid MM$. We have that $\Lambda \subsetneq \Lambda_p$.*

(ii) The set $\mathcal{F}_{IT}$ of types of IT or intersection types *is generated by the grammar: $\sigma ::= \alpha \mid \sigma \rightarrow \sigma \mid \sigma \cap \sigma$. We have that $\mathcal{F}_{IT} \subsetneq \mathcal{F}_{LJr}$.*

(iii) An IT-context *is a finite set $\{x_1 : \sigma_1, \ldots, x_n : \sigma_n\}$ of intersection types assignments to distinct variables. IT-contexts will be denoted by Γ, Δ and context domains are defined as usual.*

[4]According to the Curry-Howard isomorphism, a logical proof is related to a typed λ-term M^σ that captures in its syntax the structure of the proof. This is done by decorating the proof with typed λ-terms encoding all logical connectives and thus obtaining a deduction of a Church assignment system concluding by assigning σ to M^σ. The assignment system $\lambda^{\wedge}_{\rightarrow}$ is a Curry system, i.e. it involves typable λ-terms, not typed ones. For this reason, we cite a *manner* of Curry-Howard isomorphism instead of Curry-Howard isomorphism itself between LJ and $\lambda^{\wedge}_{\rightarrow}$.

$$\frac{x:\sigma \in \Gamma}{\Gamma \vdash_{IT} x:\sigma}\,(A_{IT}) \qquad \frac{\Gamma \vdash_{IT} M:\sigma \qquad \Gamma \vdash_{IT} M:\tau}{\Gamma \vdash_{IT} M:\sigma \cap \tau}\,(\cap I_{IT})$$

$$\frac{\Gamma \vdash_{IT} M:\sigma \cap \tau}{\Gamma \vdash_{IT} M:\sigma}\,(\cap E^{l}_{IT}) \qquad \frac{\Gamma \vdash_{IT} M:\sigma \cap \tau}{\Gamma \vdash_{IT} M:\tau}\,(\cap E^{r}_{IT})$$

$$\frac{\Gamma \cup \{x:\sigma\} \vdash_{IT} M:\tau}{\Gamma \vdash_{IT} \lambda x.M:\sigma \to \tau}\,(\to I_{IT}) \qquad \frac{\Gamma \vdash_{IT} M:\sigma \to \tau \qquad \Gamma \vdash_{IT} N:\sigma}{\Gamma \vdash_{IT} MN:\tau}\,(\to E_{IT})$$

Figure 2.6: The rules of IT.

(iv) The system IT proves statements of the form $\Gamma \vdash_{IT} M:\sigma$*, where* Γ *is an IT-context,* $M \in \Lambda$ *and* σ *is an intersection type. Its rules are shown in Figure 2.6. The expression* $\Pi : \Gamma \vdash_{IT} M:\sigma$ *carries the usual meaning.*

Note that in a typing deduction of IT the λ-term to the right of $\vdash_{IT}$ undergoes syntactical change by the implicative rules only.

Example 2.5.2 Let $\sigma \equiv \rho \to \alpha$ and $\tau \equiv \rho \to \beta$. Deduction Π includes Π_2 and Π_4 as subdeductions, which in turn include Π_1 and Π_3, respectively.

$$\frac{\dfrac{\{x:\sigma, y:\sigma\cap\tau, z:\rho\} \vdash_{IT} x:\sigma}{\{y:\sigma\cap\tau, z:\rho\} \vdash_{IT} \lambda x.x:\sigma \to \sigma}\,(\to I_{IT}) \qquad \dfrac{\{y:\sigma\cap\tau, z:\rho\} \vdash_{IT} y:\sigma\cap\tau}{\{y:\sigma\cap\tau, z:\rho\} \vdash_{IT} y:\sigma}\,(\cap E^{l}_{IT})}{\Pi_1 : \{y:\sigma\cap\tau, z:\rho\} \vdash_{IT} (\lambda x.x)y:\sigma}\,(\to E_{IT})$$

$$\frac{\Pi_1 : \{y:\sigma\cap\tau, z:\rho\} \vdash_{IT} (\lambda x.x)y:\sigma \qquad \{y:\sigma\cap\tau, z:\rho\} \vdash_{IT} z:\rho}{\Pi_2 : \{y:\sigma\cap\tau, z:\rho\} \vdash_{IT} (\lambda x.x)yz:\alpha}\,(\to E_{IT})$$

$$\frac{\dfrac{\{x:\tau, y:\sigma\cap\tau, z:\rho\} \vdash_{IT} x:\tau}{\{y:\sigma\cap\tau, z:\rho\} \vdash_{IT} \lambda x.x:\tau \to \tau}\,(\to I_{IT}) \qquad \dfrac{\{y:\sigma\cap\tau, z:\rho\} \vdash_{IT} y:\sigma\cap\tau}{\{y:\sigma\cap\tau, z:\rho\} \vdash_{IT} y:\tau}\,(\cap E^{r}_{IT})}{\Pi_3 : \{y:\sigma\cap\tau, z:\rho\} \vdash_{IT} (\lambda x.x)y:\tau}\,(\to E_{IT})$$

$$\frac{\Pi_3 : \{y:\sigma\cap\tau, z:\rho\} \vdash_{IT} (\lambda x.x)y:\tau \qquad \{y:\sigma\cap\tau, z:\rho\} \vdash_{IT} z:\rho}{\Pi_4 : \{y:\sigma\cap\tau, z:\rho\} \vdash_{IT} (\lambda x.x)yz:\beta}\,(\to E_{IT})$$

$$\frac{\Pi_2 : \{y:\sigma\cap\tau, z:\rho\} \vdash_{IT} (\lambda x.x)yz:\alpha \qquad \Pi_4 : \{y:\sigma\cap\tau, z:\rho\} \vdash_{IT} (\lambda x.x)yz:\beta}{\Pi : \{y:\sigma\cap\tau, z:\rho\} \vdash_{IT} (\lambda x.x)yz:\alpha\cap\beta}\,(\cap I_{IT})$$

Proposition 2.5.3 (Weakening property for IT) *If* $\Pi : \Gamma \vdash_{IT} M:\sigma$*, then, for every variable* $x \notin dom(\Gamma) \cup BV(M)$ *and every* $\tau \in \mathcal{F}_{IT}$*, there exists*

$$\Pi_\tau : \Gamma \cup \{x:\tau\} \vdash_{IT} M:\sigma$$

Proof: By induction on Π. $\dashv$

An equivalent version of IT can be obtained by considering contexts as *sequences* of variable assignments, changing the axiom and adding rules for context weakening and exchange. If Γ is the context $x_1 : \sigma_1, \ldots, x_n : \sigma_n$, then $dom(\Gamma) = \{x_1, \ldots, x_n\}$. The rules for this version are shown in Figure 2.7.

$$\frac{}{x : \sigma \vdash_{ITs} x : \sigma}\,(A) \qquad \frac{\Gamma \vdash_{ITs} M : \sigma,\ x \notin dom(\Gamma) \cup BV(M)}{\Gamma, x : \tau \vdash_{ITs} M : \sigma}\,(W)$$

$$\frac{\Gamma_1, x : \sigma_1, y : \sigma_2, \Gamma_2 \vdash_{ITs} M : \sigma}{\Gamma_1, y : \sigma_2, x : \sigma_1, \Gamma_2 \vdash_{ITs} M : \sigma}\,(X) \qquad \frac{\Gamma \vdash_{ITs} M : \sigma \qquad \Gamma \vdash_{ITs} M : \tau}{\Gamma \vdash_{ITs} M : \sigma \cap \tau}\,(\cap I)$$

$$\frac{\Gamma \vdash_{ITs} M : \sigma \cap \tau}{\Gamma \vdash_{ITs} M : \sigma}\,(\cap E^l) \qquad \frac{\Gamma \vdash_{ITs} M : \sigma \cap \tau}{\Gamma \vdash_{ITs} M : \tau}\,(\cap E^r)$$

$$\frac{\Gamma, x : \sigma \vdash_{ITs} M : \tau}{\Gamma \vdash_{ITs} \lambda x.M : \sigma \to \tau}\,(\to I) \qquad \frac{\Gamma \vdash_{ITs} M : \sigma \to \tau \qquad \Gamma \vdash_{ITs} N : \sigma}{\Gamma \vdash_{ITs} MN : \tau}\,(\to E)$$

Figure 2.7: The rules of IT, when contexts are sequences.

For the rest of this section, consider IT as presented in Figure 2.6. Putting aside the intersection rules, we retrieve Curry's types assignment system $\lambda_\to$ for terms in Λ. For both IT and $\lambda_\to$, the following theorem holds.

Theorem 2.5.4 *Let $\star \in \{IT, \lambda_\to\}$, $M \in \Lambda$ and $Fv(M)$ denote the set of free variables of M. Then:*
(i) If $\Gamma \vdash_\star M : \sigma$, then $Fv(M) \subseteq dom(\Gamma)$.
(ii) If $\Gamma \vdash_\star M : \sigma$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash_\star M : \sigma$.
(iii) If $\Gamma \vdash_\star M : \sigma$ and $\Gamma' \subseteq \Gamma$ is the set of those assignments in Γ which concern variables occuring free in M, then $\Gamma' \vdash_\star M : \sigma$.

Proof: By induction on $\Gamma \vdash_\star M : \sigma$ for all cases. $\dashv$

IT possesses greater typability power than $\lambda_\to$, which was the main reason for its creation. The introduction of intersection types allows the typing in IT of terms which are not typable in $\lambda_\to$. In particular, IT assigns types to all and only the strongly normalizable terms. As not every term is strongly normalizable, it is obvious that there exist terms not typable in IT.

Example 2.5.5 The term $\lambda x.xx$ is typable in IT, but not in $\lambda_\to$.
Proof: Suppose $\lambda x.xx$ is typable in $\lambda_\to$, i.e. $\Gamma \vdash_{\lambda_\to} \lambda x.xx : \rho$, for some Γ and ρ. Then, by 2.5.5(iii), we have that $\vdash_{\lambda_\to} \lambda x.xx : \rho$. By the Generation lemma for $\lambda_\to$ (see [Ba92], p. 40), there exist types σ_1 and σ_2, such that $\{x : \sigma_1\} \vdash_{\lambda_\to} xx : \sigma_2$ and $\rho \equiv \sigma_1 \to \sigma_2$. By the Generation lemma again, $\{x : \sigma_1\} \vdash_{\lambda_\to} xx : \sigma_2$ implies that there is a type τ, such that $\{x : \sigma_1\} \vdash_{\lambda_\to} x : \tau \to \sigma_2$ and $\{x : \sigma_1\} \vdash_{\lambda_\to} x : \tau$. But then, the Generation lemma gives that $x : \tau \to \sigma_2 \in \{x : \sigma_1\}$, so that $\sigma_1 \equiv \tau \to \sigma_2$ and $x : \tau \in \{x : \sigma_1\}$, so that $\sigma_1 \equiv \tau$, which is absurd. On the other hand, $\lambda x.xx$ is typable in IT, as the following deduction shows. Context-braces are omitted.

$$\dfrac{\dfrac{x : (\sigma \to \tau) \cap \sigma \vdash_{IT} x : (\sigma \to \tau) \cap \sigma}{x : (\sigma \to \tau) \cap \sigma \vdash_{IT} x : \sigma \to \tau}\,(\cap E^l) \qquad \dfrac{x : (\sigma \to \tau) \cap \sigma \vdash_{IT} x : (\sigma \to \tau) \cap \sigma}{x : (\sigma \to \tau) \cap \sigma \vdash_{IT} x : \sigma}\,(\cap E^r)}{\dfrac{x : (\sigma \to \tau) \cap \sigma \vdash_{IT} xx : \tau}{\vdash_{IT} \lambda x.xx : (\sigma \to \tau) \cap \sigma \to \tau}\,(\to I)}\,(\to E)$$

Definition 2.5.6 *(i) A type σ in $\mathcal{F}_{IT}$ is* prime*, if it is not an intersection. So, a prime type is either a type variable or an implication.*

(ii) If σ and τ are prime, then σ and τ are prime factors *of $\sigma \cap \tau$. A type σ is a prime factor of a type $\rho_1 \cap \rho_2$, if it is either a prime factor of ρ_1 or a prime factor of ρ_2.*

By inference, the main building blocks of any type σ in $\mathcal{F}_{IT}$ are prime factors and the connective $\cap$.

Example 2.5.7 The term $(\lambda x.xx)\lambda x.xx$ is not typable in IT.
Proof: Suppose $(\lambda x.xx)\lambda x.xx$ is typable in IT, i.e. $\Gamma \vdash_{IT} (\lambda x.xx)\lambda x.xx : \sigma$, for some Γ and σ. Then, by 2.5.5(iii), we have that $\vdash_{IT} (\lambda x.xx)\lambda x.xx : \sigma$. If σ is an intersection, then $\vdash_{IT} (\lambda x.xx)\lambda x.xx : \rho$, for every prime factor ρ of σ. If σ is not an intersection, it is prime. In any case, we have that $\vdash_{IT} (\lambda x.xx)\lambda x.xx : \rho$, where ρ is a prime type. By the Generation lemma for IT (see [Kr93], p. 50), we have that $\vdash_{IT} \lambda x.xx : \tau' \to \rho'$ and $\vdash_{IT} \lambda x.xx : \tau'$, where ρ is a prime factor of ρ'. Since $\tau' \to \rho'$ is prime, the typing of example 2.5.6 reveals that τ' is an intersection of the form $(\upsilon_1 \to \upsilon_2) \cap \upsilon_1$. But then, it should be $\vdash_{IT} \lambda x.xx : \upsilon_1 \to \upsilon_2$ and $\vdash_{IT} \lambda x.xx : \upsilon_1$. Since $\upsilon_1 \to \upsilon_2$ is prime, the typing of example 2.5.6 reveals again that υ_1 is an intersection of the form $(\upsilon_3 \to \upsilon_4) \cap \upsilon_3$. It is $\vdash_{IT} \lambda x.xx : \upsilon_3 \to \upsilon_4$ and $\vdash_{IT} \lambda x.xx : \upsilon_3$, type υ_3 should be an intersection and the argument is infinitely reproducted. This is a contradiction, since τ' is a finite chain of sub-types and connectives.

The extension of Curry's types assignment system with intersection types enlarged the class of typable terms and resulted to a very useful tool for investigating pure λ-calculus. Indeed, IT has such nice syntactical properties that a very important characterization of the class of typable terms can be proved.

Theorem 2.5.8 (Strong normalization theorem) *Let $M \in \Lambda$. Then*

$$M \textit{ is typable in IT} \;\Leftrightarrow\; M \textit{ is strongly normalizable}$$

Proof: See [Kr93]. $\dashv$

We saw in example 2.5.6 that the normal term $\lambda x.xx$ is typable in IT, while the non-normalizable $(\lambda x.xx)\lambda x.xx$ is not. We also note that, in example 2.4.5, the LJr-deduction Π_2 uses only implication and intersection, thus being actually an IT-deduction. It shows a typing of the strongly normalizable term $(\lambda x.xx)\lambda x.x$. On the other hand, the normalizable, but not strongly normalizable term

$$(\lambda x.y)(\lambda x.xx)\lambda x.xx$$

is not typable in IT. For if it were, then $(\lambda x.xx)\lambda x.xx$ would be typable—a contradiction.

2.6 Relation of LJ to IT

Not all LJ-deductions are decoratable by $\mathsf{d}_{\rightarrow}$. However, the part of LJ decoratable by $\mathsf{d}_{\rightarrow}$ corresponds to the intersection types assignment system IT via $\mathsf{d}_{\rightarrow}$, as the whole of LJ corresponds to the simple types assignment system $\lambda^{\wedge}_{\rightarrow}$ via $\mathsf{d}^{\wedge}_{\rightarrow}$. In particular, any LJ-deduction in $D^{\rightarrow}_{LJ}$ can be decorated[5] by $\mathsf{d}_{\rightarrow}$ to produce an IT-deduction which, when fired by E, returns the LJ-deduction and any IT-deduction can collapse through E to a LJ-deduction in $D^{\rightarrow}_{LJ}$ which, when decorated by $\mathsf{d}_{\rightarrow}$, gives back the IT-deduction modulo sequences of distinct variables decorating contexts. This is to say that for every LJ-deduction in $D^{\rightarrow}_{LJ}$ there is a unique—modulo sequences of variables—IT-deduction which decomposes to it through E and for every IT-deduction there is a unique LJ-deduction in $D^{\rightarrow}_{LJ}$ which generates it through $\mathsf{d}_{\rightarrow}$. In this section, we formalize the relation just described.

Definition 2.6.1 *Let f be a function from $\mathcal{F}_{LJ}$ to $\mathcal{F}_{IT}$ defined as: $f(\alpha) = \alpha$, $f(\sigma \rightarrow \tau) = f(\sigma) \rightarrow f(\tau)$, $f(\sigma \wedge \tau) = f(\sigma) \cap f(\tau)$. Then $(e \restriction \mathcal{F}_{IT}) \circ f = id_{\mathcal{F}_{LJ}}$ and $f \circ (e \restriction \mathcal{F}_{IT}) = id_{\mathcal{F}_{IT}}$.*

Theorem 2.6.2 (From $\mathbf{D^{\rightarrow}_{LJ}}$ to IT) *If $\Pi : \{\sigma_1, \ldots, \sigma_m\} \vdash_{LJ} \tau$ is in $D^{\rightarrow}_{LJ}$, then $\{x_1 : f(\sigma_1), \ldots, x_m : f(\sigma_m)\} \vdash_{IT} T_{x_1,\ldots,x_m}(\Pi) : f(\tau)$, for every sequence $x_1, \ldots, x_m$ of distinct variables .*

Proof: By induction on Π. Suppose $x_1, \ldots, x_m$ is fixed, but arbitrary.

Base: If $\Pi : \{\sigma_1, \ldots, \sigma_m\} \vdash_{LJ} \sigma_1$ is an axiom, then

$$\{x_1 : f(\sigma_1), \ldots, x_m : f(\sigma_m)\} \vdash_{IT} T_{x_1,\ldots,x_m}(\Pi) \equiv x_1 : f(\sigma_1)$$

is an axiom, as well.

Inductive step: We go through the LJ-rules one by one.

- $$\frac{\Pi_1 : \{\sigma_1, \ldots, \sigma_m\} \vdash_{LJ} \tau_1 \qquad \Pi_2 : \{\sigma_1, \ldots, \sigma_m\} \vdash_{LJ} \tau_2}{\Pi : \{\sigma_1, \ldots, \sigma_m\} \vdash_{LJ} \tau_1 \wedge \tau_2 \equiv \tau}\ (\wedge I)$$

Since Π is in $D^{\rightarrow}_{LJ}$, both Π_1 and Π_2 are in $D^{\rightarrow}_{LJ}$ and $T_{x_1,\ldots,x_m}(\Pi_1) \equiv T_{x_1,\ldots,x_m}(\Pi_2) \equiv T_{x_1,\ldots,x_m}(\Pi) \equiv M$. By the Inductive Hypothesis (IH), we have that

$$\{x_1 : f(\sigma_1), \ldots, x_m : f(\sigma_m)\} \vdash_{IT} M : f(\tau_1)$$

$$\{x_1 : f(\sigma_1), \ldots, x_m : f(\sigma_m)\} \vdash_{IT} M : f(\tau_2)$$

By $(\cap I_{IT})$, we get $\{x_1 : f(\sigma_1), \ldots, x_m : f(\sigma_m)\} \vdash_{IT} M : f(\tau_1) \cap f(\tau_2) \equiv f(\tau_1 \wedge \tau_2) \equiv f(\tau)$.

- $$\frac{\Pi' : \{\sigma_1, \ldots, \sigma_m\} \vdash_{LJ} \tau \wedge \tau'}{\Pi : \{\sigma_1, \ldots, \sigma_m\} \vdash_{LJ} \tau}\ (\wedge E^l)$$

Since Π is in $D^{\rightarrow}_{LJ}$, Π' is in $D^{\rightarrow}_{LJ}$ and $T_{x_1,\ldots,x_m}(\Pi) \equiv T_{x_1,\ldots,x_m}(\Pi') \equiv M$. By the IH, we have that $\{x_1 : f(\sigma_1), \ldots, x_m : f(\sigma_m)\} \vdash_{IT} M : f(\tau \wedge \tau') \equiv f(\tau) \cap f(\tau')$. Applying $(\cap E^l_{IT})$, we get $\{x_1 : f(\sigma_1), \ldots, x_m : f(\sigma_m)\} \vdash_{IT} M : f(\tau)$.

The case of $(\wedge E^r)$ is similar.

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[5] The decoration by $\mathsf{d}_{\rightarrow}$ goes along with the conversion of conjunction to intersection.

- $$\frac{\Pi' : \{\sigma_1, \ldots, \sigma_m, \tau_1\} \vdash_{LJ} \tau_2}{\Pi : \{\sigma_1, \ldots, \sigma_m\} \vdash_{LJ} \tau_1 \to \tau_2 \equiv \tau} (\to I)$$

Since Π is in $D^{\to}_{LJ}$, Π' is in $D^{\to}_{LJ}$ and $T_{x_1,\ldots,x_m}(\Pi) \equiv \lambda x.T_{x_1,\ldots,x_m,x}(\Pi')$, for some x distinct from the x_i ($1 \leqslant i \leqslant m$). By the IH, we have that

$$\{x_1 : f(\sigma_1), \ldots, x_m : f(\sigma_m), x : f(\tau_1)\} \vdash_{IT} T_{x_1,\ldots,x_m,x}(\Pi') : f(\tau_2)$$

Applying $(\to I_{IT})$, we get

$$\{x_1 : f(\sigma_1), \ldots, x_m : f(\sigma_m)\} \vdash_{IT} \lambda x.T_{x_1,\ldots,x_m,x}(\Pi') : f(\tau_1) \to f(\tau_2) \equiv f(\tau_1 \to \tau_2) \equiv f(\tau)$$

- $$\frac{\Pi_1 : \{\sigma_1, \ldots, \sigma_m\} \vdash_{LJ} \rho \to \tau \qquad \Pi_2 : \{\sigma_1, \ldots, \sigma_m\} \vdash_{LJ} \rho}{\Pi : \{\sigma_1, \ldots, \sigma_m\} \vdash_{LJ} \tau} (\to E)$$

Since Π is in $D^{\to}_{LJ}$, both Π_1 and Π_2 are in $D^{\to}_{LJ}$ and

$$T_{x_1,\ldots,x_m}(\Pi) \equiv T_{x_1,\ldots,x_m}(\Pi_1)T_{x_1,\ldots,x_m}(\Pi_2)$$

By the IH, it is

$$\{x_1 : f(\sigma_1), \ldots, x_m : f(\sigma_m)\} \vdash_{IT} T_{x_1,\ldots,x_m}(\Pi_1) : f(\rho \to \tau) \equiv f(\rho) \to f(\tau)$$

$$\{x_1 : f(\sigma_1), \ldots, x_m : f(\sigma_m)\} \vdash_{IT} T_{x_1,\ldots,x_m}(\Pi_2) : f(\rho)$$

Applying $(\to E_{IT})$, we get $\{x_1 : f(\sigma_1), \ldots, x_m : f(\sigma_m)\} \vdash_{IT} T_{x_1,\ldots,x_m}(\Pi) : f(\tau)$. $\dashv$

Theorem 2.6.3 (From IT to $\mathbf{D^{\to}_{LJ}}$) *Suppose that $x_1, \ldots, x_m$ is a fixed, but arbitrary sequence of distinct variables. If $\Pi : \{x_1 : \sigma_1, \ldots, x_m : \sigma_m\} \vdash_{IT} M : \tau$, then $E(\Pi) : \{e(\sigma_1), \ldots, e(\sigma_m)\} \vdash_{LJ} e(\tau)$ is in $D^{\to}_{LJ}$ and $T_{x_1,\ldots,x_m}(E(\Pi)) \equiv M$.*

Proof: By induction on Π.

Base: If $\Pi : \{x_1 : \sigma_1, \ldots, x_m : \sigma_m\} \vdash_{IT} x_1 : \sigma_1$ is an axiom, then

$$E(\Pi) : \{e(\sigma_1), \ldots, e(\sigma_m)\} \vdash_{LJ} e(\sigma_1)$$

is an axiom—hence in $D^{\to}_{LJ}$—and $T_{x_1,\ldots,x_m}(E(\Pi)) \equiv x_1$.

Inductive step: We examine all IT-rules.

- $$\frac{\Pi_1 : \{x_1 : \sigma_1, \ldots, x_m : \sigma_m\} \vdash_{IT} M : \tau_1 \qquad \Pi_2 : \{x_1 : \sigma_1, \ldots, x_m : \sigma_m\} \vdash_{IT} M : \tau_2}{\Pi : \{x_1 : \sigma_1, \ldots, x_m : \sigma_m\} \vdash_{IT} M : \tau_1 \cap \tau_2 \equiv \tau} (\cap I)$$

By the IH, we have that

$$E(\Pi_1) : \{e(\sigma_1), \ldots, e(\sigma_m)\} \vdash_{LJ} e(\tau_1)$$

$$E(\Pi_2) : \{e(\sigma_1), \ldots, e(\sigma_m)\} \vdash_{LJ} e(\tau_2)$$

are in $D^{\to}_{LJ}$ and $T_{x_1,\ldots,x_m}(E(\Pi_1)) \equiv M \equiv T_{x_1,\ldots,x_m}(E(\Pi_2))$ (1). By $(\wedge I_{LJ})$, we get

$$E(\Pi) : \{e(\sigma_1), \ldots, e(\sigma_m)\} \vdash_{LJ} e(\tau_1) \wedge e(\tau_2) \equiv e(\tau_1 \cap \tau_2) \equiv e(\tau)$$

which is in $D^{\to}_{LJ}$, since $E(\Pi_1)$ and $E(\Pi_2)$ are and (1) holds. It is $T_{x_1,\ldots,x_m}(E(\Pi)) \equiv M$.

- $$\frac{\Pi' : \{x_1 : \sigma_1, \ldots, x_m : \sigma_m\} \vdash_{IT} M : \tau \cap \rho}{\Pi : \{x_1 : \sigma_1, \ldots, x_m : \sigma_m\} \vdash_{IT} M : \tau} \ (\cap E^l)$$

By the IH, we have that $E(\Pi') : \{e(\sigma_1), \ldots, e(\sigma_m)\} \vdash_{LJ} e(\tau \cap \rho) \equiv e(\tau) \wedge e(\rho)$ is in $D^{\rightarrow}_{LJ}$ and $T_{x_1,\ldots,x_m}(E(\Pi')) \equiv M$. By $(\wedge E^l_{LJ})$, we get $E(\Pi) : \{e(\sigma_1), \ldots, e(\sigma_m)\} \vdash_{LJ} e(\tau)$ in $D^{\rightarrow}_{LJ}$ with $T_{x_1,\ldots,x_m}(E(\Pi)) \equiv M$.

The case of $(\cap E^r)$ is similar.

- $$\frac{\Pi' : \{x_1 : \sigma_1, \ldots, x_m : \sigma_m, x : \sigma\} \vdash_{IT} N : \sigma'}{\Pi : \{x_1 : \sigma_1, \ldots, x_m : \sigma_m\} \vdash_{IT} \lambda x.N : \sigma \rightarrow \sigma'} \ (\rightarrow I)$$

By the IH, we have that $E(\Pi') : \{e(\sigma_1), \ldots, e(\sigma_m), e(\sigma)\} \vdash_{LJ} e(\sigma')$ is in $D^{\rightarrow}_{LJ}$ and $T_{x_1,\ldots,x_m,x}(E(\Pi')) \equiv N$. Applying $(\rightarrow I_{LJ})$, we get

$$E(\Pi) : \{e(\sigma_1), \ldots, e(\sigma_m)\} \vdash_{LJ} e(\sigma) \rightarrow e(\sigma') \equiv e(\sigma \rightarrow \sigma')$$

in $D^{\rightarrow}_{LJ}$ with $T_{x_1,\ldots,x_m}(E(\Pi)) \equiv \lambda x.T_{x_1,\ldots,x_m,x}(E(\Pi')) \equiv \lambda x.N$.

- $$\frac{\Pi_1 : \{x_1 : \sigma_1, \ldots, x_m : \sigma_m\} \vdash_{IT} N_1 : \rho \rightarrow \tau \quad \Pi_2 : \{x_1 : \sigma_1, \ldots, x_m : \sigma_m\} \vdash_{IT} N_2 : \rho}{\Pi : \{x_1 : \sigma_1, \ldots, x_m : \sigma_m\} \vdash_{IT} N_1 N_2 : \tau} \ (\rightarrow E)$$

By the IH, we have that

$$E(\Pi_1) : \{e(\sigma_1), \ldots, e(\sigma_m)\} \vdash_{LJ} e(\rho \rightarrow \tau) \equiv e(\rho) \rightarrow e(\tau)$$

$$E(\Pi_2) : \{e(\sigma_1), \ldots, e(\sigma_m)\} \vdash_{LJ} e(\rho)$$

are in $D^{\rightarrow}_{LJ}$ and $T_{x_1,\ldots,x_m}(E(\Pi_1)) \equiv N_1$, $T_{x_1,\ldots,x_m}(E(\Pi_2)) \equiv N_2$. By $(\rightarrow E_{LJ})$, we get

$$E(\Pi) : \{e(\sigma_1), \ldots, e(\sigma_m)\} \vdash_{LJ} e(\tau)$$

in $D^{\rightarrow}_{LJ}$ with $T_{x_1,\ldots,x_m}(E(\Pi)) \equiv T_{x_1,\ldots,x_m}(E(\Pi_1))T_{x_1,\ldots,x_m}(E(\Pi_2)) \equiv N_1 N_2$. $\dashv$

The logical systems “Intersection Logic” and “Intersecrion Synchromous Logic” introduced in chapters 3 and 4, respectively, aim to realize the part of LJ decoratable by $\mathsf{d}_{\rightarrow}$ in a logic of its own, so that IT relates to a logic—and not to *part* of a logic—through a decoration encoding the implication only.

Chapter 3

Intersection Logic (IL)

3.1 Preliminaries

We start with some preliminary notions. We define the main structure of Intersection Logic—full binary trees, called *kits*—along with its characteristic parts—*leaves, paths, subtrees*—as well as the tools for manipulating it—*overlapping* of kits, substitution of subtrees by new kits, *pruning* of leaves. We give some basic properties of kits resulting from the definitions.

Definition 3.1.1 *(i) A* kit *is a full binary tree in the language generated by the grammar: $K ::= \sigma \mid [K, K]$, where the leaves σ, also called* atoms*, are generated by the grammar: $\sigma ::= \alpha \mid \sigma \to \sigma \mid \sigma \cap \sigma$, with α belonging to a denumerable set of propositional variables. Kits are denoted by H, K, L and leaves by lowercase greek letters.*

(ii) Two kits H, K overlap*, denoted $H \simeq K$, if they have exactly the same tree structure but may differ on the names of their leaves. For example,*

$$[\sigma_1, [[\sigma_2, \sigma_3], \sigma_4]] \simeq [\tau, [[\sigma_2, \rho], \sigma_4]]$$

(iii) If $H \simeq K$, then $H \to K$ denotes a kit that overlaps with H, K and is inductively defined as follows: for $H \equiv \sigma$ and $K \equiv \tau$, $H \to K$ is defined to be $\sigma \to \tau$, while, for $H \equiv [H_1, H_2]$ and $K \equiv [K_1, K_2]$, $H \to K$ is defined to be $[H_1 \to K_1,\ H_2 \to K_2]$.

(iv) A path *is a finite string built over the alphabet $\{l, r\}$. Paths are denoted by p, q and ϵ denotes the empty path. The* subtree *of a kit H at path p, denoted H^p, is inductively defined as follows: $H^\epsilon = H$, $[H_1, H_2]^{lp} = H_1^p$, $[H_1, H_2]^{rp} = H_2^p$. For $p \not\equiv \epsilon$, σ^p is undefined. A path p is* defined *in H if and only if H^p is defined, so, for p defined in H, H^p is actually the subtree of H rooted at the end of p in H. If p is defined in H, p is* terminal *in H if and only if H^p is a leaf. The set of terminal paths of a kit H is denoted by $P_T(H)$. Two paths p and q of H are* different*, if they split on a node of H.*

(v) For any path p defined in H, $H[p := K]$ denotes the kit resulting from the substitution of K for H^p in H and, if $p_1, \ldots, p_n \in P_T(H)$, $H[p_i := K_i \mid 1 \leq i \leq n]$ denotes the kit resulting from the substitution of K_i for H^{p_i} in H, for each i in the set $\{1, \ldots, n\}$.

(vi) Let ps be a path defined in H, where $s \in \{l, r\}$. The pruning *of H at path ps is defined as $H\backslash^{ps} \stackrel{\text{def}}{=} H[p := H^{ps}]$. For example, if*

$$H \equiv [\sigma_1, [[[[\sigma_5, \sigma_6], \sigma_3], [\sigma_4, [\sigma_7, \sigma_8]]], \sigma_2]]$$

$p \equiv rl$ and $s \equiv r$, then $H\backslash^{ps} \equiv [\sigma_1, [[\sigma_4, [\sigma_7, \sigma_8]], \sigma_2]]$.

It is obvious from the definition that, for any kits H and K, $H \simeq K \Leftrightarrow P_T(H) = P_T(K)$. Also, if $H \simeq K$, then for every $p \in P_T(H)$, $(H \to K)^p \equiv H^p \to K^p$.

Proposition 3.1.2 *(i) For any kits H and K and any path p, if $H \simeq K$, $p \not\equiv \epsilon$ and p is defined in H, then $H\backslash^p \to K\backslash^p \equiv (H \to K)\backslash^p$.*

(ii) For any kit H and non-empty paths p, q, if p and q are both defined in H and also p is defined in $H\backslash^q$ and q is defined in $H\backslash^p$, we have that $(H\backslash^p)\backslash^q \equiv (H\backslash^q)\backslash^p$.

Proof: (i) By induction on the structure of H and K.

Base: For single-node kits $H \equiv \sigma$ and $K \equiv \tau$, the only path defined is the empty path ϵ, so (i) holds trivially.

Inductive step: Suppose $H \equiv [H_1, H_2]$ and $K \equiv [K_1, K_2]$ are overlapping and $p \not\equiv \epsilon$ is defined in H. Then $H_1 \simeq K_1$, $H_2 \simeq K_2$, $p \equiv sq$, for $s \in \{l, r\}$, and q is defined in H_1, if $s \equiv l$, while q is defined in H_2, if $s \equiv r$. We consider the cases:

1. $q \equiv \epsilon$: If $s \equiv l$, we have:

$$\begin{array}{rcl} H\backslash^p \to K\backslash^p & \equiv & [H_1, H_2]\backslash^l \to [K_1, K_2]\backslash^l \\ & \equiv & H_1 \to K_1 \\ & \equiv & [H_1 \to K_1,\ H_2 \to K_2]\backslash^l \\ & \equiv & ([H_1, H_2] \to [K_1, K_2])\backslash^l \\ & \equiv & (H \to K)\backslash^p \end{array}$$

If $s \equiv r$, we work similarly.

2. $q \not\equiv \epsilon$: If $s \equiv l$, we have:

$$\begin{array}{rcll} H\backslash^p \to K\backslash^p & \equiv & [H_1, H_2]\backslash^{lq} \to [K_1, K_2]\backslash^{lq} & \\ & \equiv & [H_1\backslash^q, H_2] \to [K_1\backslash^q, K_2] & \\ & \equiv & [H_1\backslash^q \to K_1\backslash^q,\ H_2 \to K_2] & \\ & \equiv & [(H_1 \to K_1)\backslash^q,\ H_2 \to K_2] & \text{(ind. hyp.)} \\ & \equiv & [H_1 \to K_1,\ H_2 \to K_2]\backslash^{lq} & \\ & \equiv & ([H_1, H_2] \to [K_1, K_2])\backslash^{lq} & \\ & \equiv & (H \to K)\backslash^p & \end{array}$$

If $s \equiv r$, we work similarly.

(ii) Since $q \not\equiv \epsilon$, suppose, without loss of generality, that $q \equiv q_1 l$. If $q_1 \equiv \epsilon$, it is not always the case that p is defined in $H\backslash^q$: if $q_1 \equiv \epsilon$, then $q \equiv l$ and, if, in addition, H^q is a leaf, then $H\backslash^q$ is a single-node kit, namely the leaf H^q; so then, the non-empty path p is not defined in $H\backslash^q$, a contradiction. Thus, $q_1 \not\equiv \epsilon$. Suppose, wlog, that $q_1 \equiv q_1' r$, so that $q \equiv q_1' rl$. We can't have $q \subseteq p$, because, if this is so, it is not always the case that p is defined in $H\backslash^q$. Similarly, we cant't have $p \subseteq q$. Also, p can't have the form $q_1 r p'$, because, if this is so, it is again not always the case that p is defined in $H\backslash^q$. Combining the above, we conclude that $p \equiv p_1' s p_1''$, with $s \in \{l, r\}$, for some $p_1' \subseteq q_1'$. If $p_1' \equiv q_1'$, $s \equiv l$, while, if $p_1' \subsetneq q_1'$, say $q_1' \equiv p_1' l q_1''$, then $s \equiv r$. Also, $p_1'' \not\equiv \epsilon$, because, if $p_1'' \equiv \epsilon$, it is not always the case that q is defined in $H\backslash^p$. Suppose, wlog, that $p_1'' \equiv p_1'''l$. Consider the general case, where $q \equiv p_1' l q_1'' r l$ and $p \equiv p_1' r p_1''' l$. Then, using the fact that, if $\hat{p}$ and $\hat{q}$ are defined in H and different, we have

$$H[\hat{p} := K_1][\hat{q} := K_2] \equiv H[\hat{q} := K_2][\hat{p} := K_1]$$

we get that

$$\begin{aligned}
(H\backslash^p)\backslash^q &\equiv H[p_1' r p_1''' := H^p][p_1' l q_1'' r := H^q] \\
&\equiv H[p_1' l q_1'' r := H^q][p_1' r p_1''' := H^p] \quad (p_1' r p_1''', p_1' l q_1'' r \text{ different}) \\
&\equiv (H\backslash^q)\backslash^p
\end{aligned}$$

$\dashv$

Remark 3.1.3 Note that for two non-empty paths p and q, defined in a kit H, to be such that one is defined in the pruning of H at the other, they must split on a node of H, which has at least two descendants in each of them; it is not enough for p and q to be different.

3.2 Pre Intersection Logic (pIL)

In this section, we give the definition of the deductive system "pre Intersection Logic", denoted pIL, on which we shall define Intersection Logic (see section 3.3). The key feature of pIL is that its judgements exclusively contain overlapping kits. We also present a decoration of pIL-deductions with untyped λ-terms that encode the implication only.

Definition 3.2.1 (pIL) *The deductive system pIL, that we call* pre Intersection Logic*, derives judgements of the form* $\Gamma \vdash_{pIL} K$*, where the* pIL-context Γ *is a multiset of kits and* K *is a kit. Its rules are shown in Figure 3.1. We denote pIL-contexts by* Γ, Δ*. Writing* $\Pi : \Gamma \vdash_{pIL} K$ *means that the pIL-deduction* Π *concludes by proving* $\Gamma \vdash_{pIL} K$*.*

Remark 3.2.2 (i) In the rule (P_{pIL}), the notation $\Gamma\backslash^{ps}$ stands for the distribution of the pruning to the elements of Γ. (ii) In $(\cap I_{pIL})$, we say that the introduction of intersection *concerns path* p or *is applied on* p; similarly, in $(\cap E^s_{pIL})$, $s \in \{l, r\}$, the elimination of intersection concerns or is applied on p. (iii) The rules $(\rightarrow I)$ and $(\rightarrow E)$ are *global* rules, in the sense that they affect all the leaves of the kits to the right of $\vdash_{pIL}$, while $(\cap I)$ and $(\cap E)$ are *local* rules, since they concern particular paths of the kits.

Definition 3.2.3 *(i) A judgement* $\{H_1, \ldots, H_n\} \vdash_{pIL} K$ *is* proper *if and only if* $H_1, \ldots, H_n, K$ *are single-node kits, i.e. leaves.*

(ii) A deduction $\Pi : \{H_1, \ldots, H_n\} \vdash_{pIL} K$ *is* proper*, if its conclusive judgement* $\{H_1, \ldots, H_n\} \vdash_{pIL} K$ *is proper.*

The judgements of pIL enjoy an invariant, stated in the following lemma.

Lemma 3.2.4 *If* $\Pi : \{H_1, \ldots, H_n\} \vdash_{pIL} K$*, then the kits* $H_1, \ldots, H_n$ *and* K *are overlapping.*

Proof: By induction on Π. We remark thar the binary relation $\simeq \subseteq (\mathcal{K} \times \mathcal{K})$ on the set of all kits $\mathcal{K}$ is reflexive, symmetric and transitive, i.e. it is an equivalence relation on $\mathcal{K}$. $\dashv$

$$\frac{\forall H \in \Gamma : H \simeq K,\ K \in \Gamma}{\Gamma \vdash_{pIL} K}\ (A_{pIL})$$

$$\frac{\Gamma \vdash_{pIL} K,\ s \in \{l, r\},\ ps \text{ defined in } K}{\Gamma \backslash^{ps} \vdash_{pIL} K \backslash^{ps}}\ (P_{pIL})$$

$$\frac{\Gamma \cup \{H\} \vdash_{pIL} K}{\Gamma \vdash_{pIL} H \to K}\ (\to I_{pIL}) \qquad \frac{\Gamma \vdash_{pIL} H \to K \quad \Gamma \vdash_{pIL} H}{\Gamma \vdash_{pIL} K}\ (\to E_{pIL})$$

$$\frac{\{H_1[p := [\sigma_1, \sigma_1]], \ldots, H_n[p := [\sigma_n, \sigma_n]]\} \vdash_{pIL} K[p := [\sigma, \tau]]}{\{H_1[p := \sigma_1], \ldots, H_n[p := \sigma_n]\} \vdash_{pIL} K[p := \sigma \cap \tau]}\ (\cap I_{pIL})$$

$$\frac{\Gamma \vdash_{pIL} K[p := \sigma \cap \tau]}{\Gamma \vdash_{pIL} K[p := \sigma]}\ (\cap E^l_{pIL}) \qquad \frac{\Gamma \vdash_{pIL} K[p := \sigma \cap \tau]}{\Gamma \vdash_{pIL} K[p := \tau]}\ (\cap E^r_{pIL})$$

Figure 3.1: The rules of pIL.

An equivalent version of pIL can be formed, deriving judgements $\Gamma \vdash_{pIL} K$, where Γ is a *sequence* of kits and K is a kit. Its rules are demonstrated in Figure 3.2. In this thesis, though, only the multiset version of pIL will concern us.

Before defining Intersection Logic, we follow [RR01] in introducing a non-standard decoration of pIL-deductions, denoted $\mathsf{pd}_{\to}$ in this thesis. As the notation witnesses, it encodes the implicative rules only, not the whole structure of the deduction. We will use it to argue about deductions in IL and to establish a correspondence between IL and IT.

Definition 3.2.5 ($\mathsf{pd}_{\to}$: non-standard decoration of pIL) *(i) Consider a pIL-context* $\Delta \equiv \{H_1, \ldots, H_n\}$*. A* decoration Δ^* of Δ *is a set* $\{x_1 : H_1, \ldots, x_n : H_n\}$*, where the* x_i*'s are distinct* λ*-variables. Then,* $dom(\Delta^*)$ *is the sequence* $x_1, \ldots, x_n$*.*

(ii) Every $\Pi : \Delta \vdash_{pIL} K$ *can be associated through an inductive algorithm to a decorated deduction*

$$\Pi^* : \Delta^* \vdash^*_{pIL} T_{dom(\Delta^*)}(\Pi) : K$$

where Δ^* *is a decoration of* Δ*,* $\vdash^*_{pIL}$ *denotes the decoration of* $\vdash_{pIL}$ *and* $T_{dom(\Delta^*)}(\Pi)$ *is in* Λ*.*

- $$\frac{K \in \Delta}{\Pi : \Delta \vdash_{pIL} K}\ (A_{pIL}) \quad \Rightarrow \quad \frac{x : K \in \Delta^*}{\Pi^* : \Delta^* \vdash^*_{pIL} T_{dom(\Delta^*)}(\Pi) : K}\ (A^*_{pIL})$$

  where $T_{dom(\Delta^*)}(\Pi) \equiv x$*.*

- $$\frac{\Pi_1 : \Delta_1 \vdash_{pIL} K}{\Pi : \Delta_1 \backslash^{ps} \equiv \Delta \vdash_{pIL} K \backslash^{ps}}\ (P_{pIL}) \quad \Rightarrow \quad \frac{\Pi^*_1 : \Delta^*_1 \vdash^*_{pIL} T_{dom(\Delta^*_1)}(\Pi_1) : K}{\Pi^* : \Delta^* \vdash^*_{pIL} T_{dom(\Delta^*)}(\Pi) : K \backslash^{ps}}\ (P^*_{pIL})$$

  where $s \in \{l, r\}$*,* ps *is defined in* K*,* $dom(\Delta^*_1) \equiv dom(\Delta^*)$ *and* $T_{dom(\Delta^*)}(\Pi) \equiv T_{dom(\Delta^*_1)}(\Pi_1)$*.*

$$\frac{}{K \vdash_{pIL} K}\,(A_{pIL}) \qquad \frac{\Gamma, H_1, H_2, \Delta \vdash_{pIL} K}{\Gamma, H_2, H_1, \Delta \vdash_{pIL} K}\,(X_{pIL})$$

$$\frac{H_1, \ldots, H_n \vdash_{pIL} K,\ \ H' \simeq K}{H_1, \ldots, H_n, H' \vdash_{pIL} K}\,(W_{pIL})$$

$$\frac{\Gamma \vdash_{pIL} K,\ \ s \in \{l, r\},\ \ ps \text{ defined in } K}{\Gamma\backslash^{ps} \vdash_{pIL} K\backslash^{ps}}\,(P_{pIL})$$

$$\frac{\Gamma, H \vdash_{pIL} K}{\Gamma \vdash_{pIL} H \to K}\,(\to I_{pIL}) \qquad \frac{\Gamma \vdash_{pIL} H \to K \qquad \Gamma \vdash_{pIL} H}{\Gamma \vdash_{pIL} K}\,(\to E_{pIL})$$

$$\frac{H_1[p := [\sigma_1, \sigma_1]], \ldots, H_n[p := [\sigma_n, \sigma_n]] \vdash_{pIL} K[p := [\sigma, \tau]]}{H_1[p := \sigma_1], \ldots, H_n[p := \sigma_n] \vdash_{pIL} K[p := \sigma \cap \tau]}\,(\cap I_{pIL})$$

$$\frac{\Gamma \vdash_{pIL} K[p := \sigma \cap \tau]}{\Gamma \vdash_{pIL} K[p := \sigma]}\,(\cap E^l_{pIL}) \qquad \frac{\Gamma \vdash_{pIL} K[p := \sigma \cap \tau]}{\Gamma \vdash_{pIL} K[p := \tau]}\,(\cap E^r_{pIL})$$

Figure 3.2: The rules of pIL, when contexts are sequences.

- $$\dfrac{\Pi_1 : \Delta \cup \{H\} \vdash_{pIL} K}{\Pi : \Delta \vdash_{pIL} H \rightarrow K}\,(\rightarrow I_{pIL}) \quad \Rightarrow$$

$$\dfrac{\Pi_1^* : \Delta^* \cup \{x : H\} \vdash^*_{pIL} T_{dom(\Delta^*),x}(\Pi_1) : K}{\Pi^* : \Delta^* \vdash^*_{pIL} T_{dom(\Delta^*)}(\Pi) : H \rightarrow K}\,(\rightarrow I^*_{pIL})$$

where $x \notin dom(\Delta^*)$ *and* $T_{dom(\Delta^*)}(\Pi) \equiv \lambda x.T_{dom(\Delta^*),x}(\Pi_1)$.

- $$\dfrac{\Pi_1 : \Delta \vdash_{pIL} H \rightarrow K \qquad \Pi_2 : \Delta \vdash_{pIL} H}{\Pi : \Delta \vdash_{pIL} K}\,(\rightarrow E_{pIL}) \quad \Rightarrow$$

$$\dfrac{\Pi_1^* : \Delta^* \vdash^*_{pIL} T_{dom(\Delta^*)}(\Pi_1) : H \rightarrow K \qquad \Pi_2^* : \Delta^* \vdash^*_{pIL} T_{dom(\Delta^*)}(\Pi_2) : H}{\Pi^* : \Delta^* \vdash^*_{pIL} T_{dom(\Delta^*)}(\Pi) : K}\,(\rightarrow E^*_{pIL})$$

where $T_{dom(\Delta^*)}(\Pi) \equiv T_{dom(\Delta^*)}(\Pi_1)T_{dom(\Delta^*)}(\Pi_2)$.

- $$\dfrac{\Pi_1 : \Delta_1 \vdash_{pIL} K[p := [\sigma, \tau]]}{\Pi : \Delta \vdash_{pIL} K[p := \sigma \cap \tau]}\,(\cap I_{pIL}) \quad \Rightarrow$$

$$\dfrac{\Pi_1^* : \Delta_1^* \vdash^*_{pIL} T_{dom(\Delta_1^*)}(\Pi_1) : K[p := [\sigma, \tau]]}{\Pi^* : \Delta^* \vdash^*_{pIL} T_{dom(\Delta^*)}(\Pi) : K[p := \sigma \cap \tau]}\,(\cap I^*_{pIL})$$

where $dom(\Delta_1^*) \equiv dom(\Delta^*)$ *and* $T_{dom(\Delta^*)}(\Pi) \equiv T_{dom(\Delta_1^*)}(\Pi_1)$.

- $$\dfrac{\Pi_1 : \Delta \vdash_{pIL} K[p := \sigma_l \cap \sigma_r]}{\Pi : \Delta \vdash_{pIL} K[p := \sigma_s]}\,(\cap E^s_{pIL}) \quad \Rightarrow$$

$$\dfrac{\Pi_1^* : \Delta^* \vdash^*_{pIL} T_{dom(\Delta^*)}(\Pi_1) : K[p := \sigma_l \cap \sigma_r]}{\Pi^* : \Delta^* \vdash^*_{pIL} T_{dom(\Delta^*)}(\Pi) : K[p := \sigma_s]}\,(\cap E^s_{pIL})^*$$

where $s \in \{l, r\}$ *and* $T_{dom(\Delta^*)}(\Pi) \equiv T_{dom(\Delta^*)}(\Pi_1)$.

(iii) If $\Pi : \Delta \vdash_{pIL} K$, *we define the* form *of* Π, *denoted* $U(\Pi)$, *to be the set* $\{T_{dom(\Delta^*)}(\Pi) \mid dom(\Delta^*)$ *is a sequence of* $|\Delta|$ *distinct variables*$\}$.

Remark 3.2.6 All points of remark 2.4.2 made for the non-standard decoration of LJ hold for the non-standard decoration of pIL, as well.

3.3 Definition of IL

A deduction in Intersection Logic, proving $\{H_1, \ldots, H_n\} \vdash_{IL} K$, is defined as an equivalence class of deductions in pIL, all proving $\{H_1, \ldots, H_n\} \vdash_{pIL} K$. The equivalence relation between derivations of pIL is defined to eliminate unnecessary differentiations resulting from differences in the order of application of consecutive local rules concerning different paths. For convenience though, a deduction in IL is identified with a deduction in pIL belonging to the specified equivalence class.

This section is devoted to the definition and further explanation of the equivalence relation in question.

Definition 3.3.1 (Intersection Logic) *(i) We define a binary relation, denoted $\sim$, on the set D_{pIL} of all pIL-deductions, as follows: for all Π, Π' in D_{pIL}, $\Pi \sim \Pi'$ if and only if Π' results from Π by interchanging two consecutive local rules concerning different paths. The interchange cases*[1] *are shown in Figure 3.3. Somewhat abusing the notation, we use the symbol $\sim$ between deduction-parts, which are responsible for the relation $\sim$ between the corresponding pIL-deductions.*

(ii) As the three displayed schemas make clear, the interchange of two consecutive local rules applied on different paths leaves the resulting judgement unchanged. So, if $\Pi \sim \Pi'$, they both prove the same judgement.

(iii) We can then define an equivalence relation, denoted $\approx$, on D_{pIL} as the reflexive and transitive closure of the relation $\sim$, defined in (i). Note that $\sim$ is, by definition, symmetric. It is easy to see that, if $\Pi \approx \Pi'$, they still both prove the same judgement.

(iv) The set D_{IL} of all IL-deductions is defined to be D_{pIL} quotiented by $\approx$. It is $D_{IL} = [\,D_{pIL}/\!\approx] = \{\,[\,\Pi/\!\approx] \mid \Pi \in D_{pIL}\,\}$, where $[\,\Pi/\!\approx] = \{\,\Pi' \in D_{pIL} \mid \Pi \approx \Pi'\,\}$. An equivalence class in D_{IL}, whose deductions prove $\Gamma \vdash_{pIL} K$, is denoted $\Gamma \vdash_{IL} K$ or $\pi : \Gamma \vdash_{IL} K$. So, $\pi \equiv [\,\Pi/\!\approx]$, for some $\Pi \in D_{pIL}$. If $\Pi' \approx \Pi$, we write $\Pi' \in \pi$. For practical reasons though, we usually identify π with a $\Pi' \in \pi$.

Remark 3.3.2 (i) In the first interchange case, we have that $(\Gamma\backslash^{pl})\backslash^{ql} \equiv (\Gamma\backslash^{ql})\backslash^{pl}$, meaning that $(H\backslash^{pl})\backslash^{ql} \equiv (H\backslash^{ql})\backslash^{pl}$, for every $H \in \Gamma$. This is because p and q are different, allowing for pl and ql to be such that the hypotheses of proposition 3.1.2(ii) are satisfied. (ii) In the second and third cases, s and s' belong to $\{l, r\}$. (iii) Since local rules are not registered by $\mathsf{pd}_{\rightarrow}$ and equivalent pIL-deductions differ solely in the order of application of local rules, if $\Pi_1 \approx \Pi_2 : \{H_1, \ldots, H_n\} \vdash_{pIL} K$, then $T_{x_1,\ldots,x_n}(\Pi_1) = T_{x_1,\ldots,x_n}(\Pi_2)$, for every $x_1, \ldots, x_n$. That's why we can identify an IL-deduction $\pi \equiv [\,\Pi/\!\approx] : \{H_1, \ldots, H_n\} \vdash_{IL} K$ with any member Π' of the equivalence class and have that $T_{x_1,\ldots,x_n}(\pi) = T_{x_1,\ldots,x_n}(\Pi')$, for every $x_1, \ldots, x_n$. Consequently, if $\Pi_1 \approx \Pi_2$, then $U(\Pi_1) = U(\Pi_2)$ and, if $\Pi' \in \pi$, then $U(\pi) = U(\Pi')$. (iv) There are finitely many pIL-deductions in an equivalence class $[\,\Pi/\!\approx]$.

Example 3.3.3 Let $\sigma \equiv \alpha \cap \beta \cap \gamma$. Deductions Π_1, Π_2 and Π_3, shown in Figure 3.4 on page 31, are equivalent. If we number the rules of Π_1 from 1 to 11, we swap pairs $(2,3), (5,6)$ and $(7,8)$ of Π_1 in three simple steps to produce Π_2—in which the rules appear in the order $1,3,2,4,6,5,8,7,9,10,11$—and then we swap pairs $(1,3), (2,4)$ and $(7,9)$ of Π_2 in three further steps to attain Π_3, in which the rules have been mixed in the order $3,1,4,2,6,5,8,9,7,10,11$. Context-braces are omitted.

[1] In [RR01], the case of consecutive intersection introduction rules on different paths is not mentioned. We consider its inclusion to the definition necessary.

$$\dfrac{\dfrac{\Gamma \vdash_{pIL} K[p := [\sigma_l, \sigma_r]]\,[q := [\tau_l, \tau_r]]}{\Gamma\backslash^{pl} \vdash_{pIL} K[p := \sigma_l \cap \sigma_r]\,[q := [\tau_l, \tau_r]]}\,(\cap I_{pIL})}{(\Gamma\backslash^{pl})\backslash^{ql} \vdash_{pIL} K[p := \sigma_l \cap \sigma_r]\,[q := \tau_l \cap \tau_r]}\,(\cap I_{pIL}) \quad \sim$$

$$\dfrac{\dfrac{\Gamma \vdash_{pIL} K[p := [\sigma_l, \sigma_r]]\,[q := [\tau_l, \tau_r]]}{\Gamma\backslash^{ql} \vdash_{pIL} K[p := [\sigma_l, \sigma_r]]\,[q := \tau_l \cap \tau_r]}\,(\cap I_{pIL})}{(\Gamma\backslash^{ql})\backslash^{pl} \vdash_{pIL} K[p := \sigma_l \cap \sigma_r]\,[q := \tau_l \cap \tau_r]}\,(\cap I_{pIL})$$

$$\dfrac{\dfrac{\Gamma \vdash_{pIL} K[p := [\sigma_l, \sigma_r]]\,[q := \tau_l \cap \tau_r]}{\Gamma\backslash^{pl} \vdash_{pIL} K[p := \sigma_l \cap \sigma_r]\,[q := \tau_l \cap \tau_r]}\,(\cap I_{pIL})}{\Gamma\backslash^{pl} \vdash_{pIL} K[p := \sigma_l \cap \sigma_r]\,[q := \tau_s]}\,(\cap E^{s}_{pIL}) \quad \sim$$

$$\dfrac{\dfrac{\Gamma \vdash_{pIL} K[p := [\sigma_l, \sigma_r]]\,[q := \tau_l \cap \tau_r]}{\Gamma \vdash_{pIL} K[p := [\sigma_l, \sigma_r]]\,[q := \tau_s]}\,(\cap E^{s}_{pIL})}{\Gamma\backslash^{pl} \vdash_{pIL} K[p := \sigma_l \cap \sigma_r]\,[q := \tau_s]}\,(\cap I_{pIL})$$

$$\dfrac{\dfrac{\Gamma \vdash_{pIL} K[p := \sigma_l \cap \sigma_r]\,[q := \tau_l \cap \tau_r]}{\Gamma \vdash_{pIL} K[p := \sigma_s]\,[q := \tau_l \cap \tau_r]}\,(\cap E^{s}_{pIL})}{\Gamma \vdash_{pIL} K[p := \sigma_s]\,[q := \tau_{s'}]}\,(\cap E^{s'}_{pIL}) \quad \sim$$

$$\dfrac{\dfrac{\Gamma \vdash_{pIL} K[p := \sigma_l \cap \sigma_r]\,[q := \tau_l \cap \tau_r]}{\Gamma \vdash_{pIL} K[p := \sigma_l \cap \sigma_r]\,[q := \tau_{s'}]}\,(\cap E^{s'}_{pIL})}{\Gamma \vdash_{pIL} K[p := \sigma_s]\,[q := \tau_{s'}]}\,(\cap E^{s}_{pIL})$$

Figure 3.3: Related deduction-parts. Paths p and q are different.

$$
\begin{array}{cl}
[[[\tau,\tau],[\tau,\tau]],\gamma]\,,\ [[[\sigma,\sigma],[\sigma,\sigma]],\alpha\cap\beta] \vdash_{pIL} [[[\sigma,\sigma],[\sigma,\sigma]],\alpha\cap\beta] & \\ \hline
[[[\tau,\tau],[\tau,\tau]],\gamma]\,,\ [[[\sigma,\sigma],[\sigma,\sigma]],\alpha\cap\beta] \vdash_{pIL} [[[\alpha\cap\beta,\sigma],[\sigma,\sigma]],\alpha\cap\beta] & (\cap E^l)\text{ on } lll \\ \hline
[[[\tau,\tau],[\tau,\tau]],\gamma]\,,\ [[[\sigma,\sigma],[\sigma,\sigma]],\alpha\cap\beta] \vdash_{pIL} [[[\beta,\sigma],[\sigma,\sigma]],\alpha\cap\beta] & (\cap E^r)\text{ on } lll \\ \hline
[[[\tau,\tau],[\tau,\tau]],\gamma]\,,\ [[[\sigma,\sigma],[\sigma,\sigma]],\alpha\cap\beta] \vdash_{pIL} [[[\beta,\gamma],[\sigma,\sigma]],\alpha\cap\beta] & (\cap E^r)\text{ on } llr \\ \hline
[[[\tau,\tau],[\tau,\tau]],\gamma]\,,\ [[[\sigma,\sigma],[\sigma,\sigma]],\alpha\cap\beta] \vdash_{pIL} [[[\beta,\gamma],[\alpha\cap\beta,\sigma]],\alpha\cap\beta] & (\cap E^l)\text{ on } lrl \\ \hline
[[[\tau,\tau],[\tau,\tau]],\gamma]\,,\ [[[\sigma,\sigma],[\sigma,\sigma]],\alpha\cap\beta] \vdash_{pIL} [[[\beta,\gamma],[\alpha,\sigma]],\alpha\cap\beta] & (\cap E^l)\text{ on } lrl \\ \hline
[[[\tau,\tau],[\tau,\tau]],\gamma]\,,\ [[[\sigma,\sigma],[\sigma,\sigma]],\alpha\cap\beta] \vdash_{pIL} [[[\beta,\gamma],[\alpha,\gamma]],\alpha\cap\beta] & (\cap E^r)\text{ on } lrr \\ \hline
[[\tau,[\tau,\tau]],\gamma]\,,\ [[\sigma,[\sigma,\sigma]],\alpha\cap\beta] \vdash_{pIL} [[\beta\cap\gamma,[\alpha,\gamma]],\alpha\cap\beta] & (\cap I)\text{ on } ll \\ \hline
[[\tau,\tau],\gamma]\,,\ [[\sigma,\sigma],\alpha\cap\beta] \vdash_{pIL} [[\beta\cap\gamma,\alpha\cap\gamma],\alpha\cap\beta] & (\cap I)\text{ on } lr \\ \hline
[[\tau,\tau],\gamma]\,,\ [[\sigma,\sigma],\alpha\cap\beta] \vdash_{pIL} [[\beta\cap\gamma,\alpha\cap\gamma],\alpha] & (\cap E^l)\text{ on } r \\ \hline
[[\sigma,\sigma],\alpha\cap\beta] \vdash_{pIL} [[\tau\to\beta\cap\gamma\,,\ \tau\to\alpha\cap\gamma],\gamma\to\alpha] & (\to I) \\ \hline
\Pi_1 : [\sigma,\alpha\cap\beta] \vdash_{pIL} [(\tau\to\beta\cap\gamma)\cap(\tau\to\alpha\cap\gamma),\gamma\to\alpha] & (\cap I)\text{ on } l
\end{array}
$$

$$
\begin{array}{cl}
[[[\tau,\tau],[\tau,\tau]],\gamma]\,,\ [[[\sigma,\sigma],[\sigma,\sigma]],\alpha\cap\beta] \vdash_{pIL} [[[\sigma,\sigma],[\sigma,\sigma]],\alpha\cap\beta] & \\ \hline
[[[\tau,\tau],[\tau,\tau]],\gamma]\,,\ [[[\sigma,\sigma],[\sigma,\sigma]],\alpha\cap\beta] \vdash_{pIL} [[[\alpha\cap\beta,\sigma],[\sigma,\sigma]],\alpha\cap\beta] & (\cap E^l)\text{ on } lll \\ \hline
[[[\tau,\tau],[\tau,\tau]],\gamma]\,,\ [[[\sigma,\sigma],[\sigma,\sigma]],\alpha\cap\beta] \vdash_{pIL} [[[\alpha\cap\beta,\gamma],[\sigma,\sigma]],\alpha\cap\beta] & (\cap E^r)\text{ on } llr \\ \hline
[[[\tau,\tau],[\tau,\tau]],\gamma]\,,\ [[[\sigma,\sigma],[\sigma,\sigma]],\alpha\cap\beta] \vdash_{pIL} [[[\beta,\gamma],[\sigma,\sigma]],\alpha\cap\beta] & (\cap E^r)\text{ on } lll \\ \hline
[[[\tau,\tau],[\tau,\tau]],\gamma]\,,\ [[[\sigma,\sigma],[\sigma,\sigma]],\alpha\cap\beta] \vdash_{pIL} [[[\beta,\gamma],[\alpha\cap\beta,\sigma]],\alpha\cap\beta] & (\cap E^l)\text{ on } lrl \\ \hline
[[[\tau,\tau],[\tau,\tau]],\gamma]\,,\ [[[\sigma,\sigma],[\sigma,\sigma]],\alpha\cap\beta] \vdash_{pIL} [[[\beta,\gamma],[\alpha\cap\beta,\gamma]],\alpha\cap\beta] & (\cap E^r)\text{ on } lrr \\ \hline
[[[\tau,\tau],[\tau,\tau]],\gamma]\,,\ [[[\sigma,\sigma],[\sigma,\sigma]],\alpha\cap\beta] \vdash_{pIL} [[[\beta,\gamma],[\alpha,\gamma]],\alpha\cap\beta] & (\cap E^l)\text{ on } lrl \\ \hline
[[[\tau,\tau],\tau],\gamma]\,,\ [[[\sigma,\sigma],\sigma],\alpha\cap\beta] \vdash_{pIL} [[[\beta,\gamma],\alpha\cap\gamma],\alpha\cap\beta] & (\cap I)\text{ on } lr \\ \hline
[[\tau,\tau],\gamma]\,,\ [[\sigma,\sigma],\alpha\cap\beta] \vdash_{pIL} [[\beta\cap\gamma,\alpha\cap\gamma],\alpha\cap\beta] & (\cap I)\text{ on } ll \\ \hline
[[\tau,\tau],\gamma]\,,\ [[\sigma,\sigma],\alpha\cap\beta] \vdash_{pIL} [[\beta\cap\gamma,\alpha\cap\gamma],\alpha] & (\cap E^l)\text{ on } r \\ \hline
[[\sigma,\sigma],\alpha\cap\beta] \vdash_{pIL} [[\tau\to\beta\cap\gamma\,,\ \tau\to\alpha\cap\gamma],\gamma\to\alpha] & (\to I) \\ \hline
\Pi_2 : [\sigma,\alpha\cap\beta] \vdash_{pIL} [(\tau\to\beta\cap\gamma)\cap(\tau\to\alpha\cap\gamma),\gamma\to\alpha] & (\cap I)\text{ on } l
\end{array}
$$

$$
\begin{array}{cl}
[[[\tau,\tau],[\tau,\tau]],\gamma]\,,\ [[[\sigma,\sigma],[\sigma,\sigma]],\alpha\cap\beta] \vdash_{pIL} [[[\sigma,\sigma],[\sigma,\sigma]],\alpha\cap\beta] & \\ \hline
[[[\tau,\tau],[\tau,\tau]],\gamma]\,,\ [[[\sigma,\sigma],[\sigma,\sigma]],\alpha\cap\beta] \vdash_{pIL} [[[\sigma,\gamma],[\sigma,\sigma]],\alpha\cap\beta] & (\cap E^r)\text{ on } llr \\ \hline
[[[\tau,\tau],[\tau,\tau]],\gamma]\,,\ [[[\sigma,\sigma],[\sigma,\sigma]],\alpha\cap\beta] \vdash_{pIL} [[[\alpha\cap\beta,\gamma],[\sigma,\sigma]],\alpha\cap\beta] & (\cap E^l)\text{ on } lll \\ \hline
[[[\tau,\tau],[\tau,\tau]],\gamma]\,,\ [[[\sigma,\sigma],[\sigma,\sigma]],\alpha\cap\beta] \vdash_{pIL} [[[\alpha\cap\beta,\gamma],[\alpha\cap\beta,\sigma]],\alpha\cap\beta] & (\cap E^l)\text{ on } lrl \\ \hline
[[[\tau,\tau],[\tau,\tau]],\gamma]\,,\ [[[\sigma,\sigma],[\sigma,\sigma]],\alpha\cap\beta] \vdash_{pIL} [[[\beta,\gamma],[\alpha\cap\beta,\sigma]],\alpha\cap\beta] & (\cap E^r)\text{ on } lll \\ \hline
[[[\tau,\tau],[\tau,\tau]],\gamma]\,,\ [[[\sigma,\sigma],[\sigma,\sigma]],\alpha\cap\beta] \vdash_{pIL} [[[\beta,\gamma],[\alpha\cap\beta,\gamma]],\alpha\cap\beta] & (\cap E^r)\text{ on } lrr \\ \hline
[[[\tau,\tau],[\tau,\tau]],\gamma]\,,\ [[[\sigma,\sigma],[\sigma,\sigma]],\alpha\cap\beta] \vdash_{pIL} [[[\beta,\gamma],[\alpha,\gamma]],\alpha\cap\beta] & (\cap E^l)\text{ on } lrl \\ \hline
[[[\tau,\tau],\tau],\gamma]\,,\ [[[\sigma,\sigma],\sigma],\alpha\cap\beta] \vdash_{pIL} [[[\beta,\gamma],\alpha\cap\gamma],\alpha\cap\beta] & (\cap I)\text{ on } lr \\ \hline
[[[\tau,\tau],\tau],\gamma]\,,\ [[[\sigma,\sigma],\sigma],\alpha\cap\beta] \vdash_{pIL} [[[\beta,\gamma],\alpha\cap\gamma],\alpha] & (\cap E^l)\text{ on } r \\ \hline
[[\tau,\tau],\gamma]\,,\ [[\sigma,\sigma],\alpha\cap\beta] \vdash_{pIL} [[\beta\cap\gamma,\alpha\cap\gamma],\alpha] & (\cap I)\text{ on } ll \\ \hline
[[\sigma,\sigma],\alpha\cap\beta] \vdash_{pIL} [[\tau\to\beta\cap\gamma\,,\ \tau\to\alpha\cap\gamma],\gamma\to\alpha] & (\to I) \\ \hline
\Pi_3 : [\sigma,\alpha\cap\beta] \vdash_{pIL} [(\tau\to\beta\cap\gamma)\cap(\tau\to\alpha\cap\gamma),\gamma\to\alpha] & (\cap I)\text{ on } l
\end{array}
$$

Figure 3.4: Deductions Π_1, Π_2 and Π_3 of example 3.3.3.

It is $T_x(\Pi_1) = T_x(\Pi_2) = T_x(\Pi_3) = \{\lambda y.x \mid y(\not\equiv x) \text{ a variable}\}$, for every x. If $\pi \equiv [\Pi_1/\!\approx] : [\sigma, \alpha\cap\beta] \vdash_{IL} [(\tau\to\beta\cap\gamma)\cap(\tau\to\alpha\cap\gamma), \gamma\to\alpha]$, we can identify π with Π_i $(i \in \{1,2,3\})$. In any case, $T_x(\pi) = \{\lambda y.x \mid y(\not\equiv x) \text{ a variable}\}$, for every x.

3.4 From pIL to LJ

In this section, we examine the transition from pIL to LJ. We show that a derivation $\Pi : \{H_1,\dots,H_n\} \vdash_{pIL} K$ in pIL groups a set, denoted $LJ(\Pi)$, of derivations in LJ, which are all decoratable by $\mathsf{d}_\to$ and have the same form[2], namely the form of Π *(projection theorem)*. Each derivation $\Pi' : \{\sigma_1,\dots,\sigma_n\} \vdash_{LJ} \tau$ in $LJ(\Pi)$ can be obtained from Π by considering a certain terminal path of the kits in Π's conclusion, taking the leaves to which it leads in the kits, changing the intersection to conjunction and thus reducing the kits $H_1,\dots,H_n,K$ to LJ-formulas $\sigma_1,\dots,\sigma_n,\tau$. We also investigate the conditions that a LJ-deduction must satisfy to be decoratable by $\mathsf{d}_\to$ and explain why LJ-deductions originating from a pIL-deduction, in the manner just described, satisfy these conditions.

Theorem 3.4.1 (Projection theorem) *Let $\Pi : \{H_1,\dots,H_n\} \vdash_{pIL} K$. For all paths p in $P_T(K)$, we have that $\Pi^p : \{e(H_1^p),\dots,e(H_n^p)\} \vdash_{LJ} e(K^p)$ is a LJ-deduction in $D_{LJ}^{\to}$, such that $T_{x_1,\dots,x_n}(\Pi^p) = T_{x_1,\dots,x_n}(\Pi)$, for every sequence $x_1,\dots,x_n$ of distinct variables.*

Proof: By induction on Π.

Base: For $\Pi : \{H_1,\dots,H_{n-1},K\} \vdash_{pIL} K$ an axiom of pIL and $p \in P_T(K)$, we have that $\Pi^p : \{e(H_1^p),\dots,e(H_{n-1}^p),e(K^p)\} \vdash_{LJ} e(K^p)$ is an axiom of LJ—hence in $D_{LJ}^{\to}$—with $T_{x_1,\dots,x_n}(\Pi^p) = \{x_n\} = T_{x_1,\dots,x_n}(\Pi)$, for every $x_1,\dots,x_n$.

Inductive step: We thoroughly examine all pIL-rules.

- $$\frac{\Pi_1 : \{H_1,\dots,H_n\} \vdash_{pIL} K}{\Pi : \{H_1\backslash^{ps},\dots,H_n\backslash^{ps}\} \vdash_{pIL} K\backslash^{ps}}\,(P_{pIL})$$

Suppose $q \in P_T(K\backslash^{ps})$. Then there exists $\bar{q} \in P_T(K)$, such that $(H_i\backslash^{ps})^q \equiv (H_i)^{\bar{q}}$ $(1\leqslant i\leqslant n)$ and $(K\backslash^{ps})^q \equiv K^{\bar{q}}$. By the IH, $\Pi_1^{\bar{q}} : \{e(H_1^{\bar{q}}),\dots,e(H_n^{\bar{q}})\} \vdash_{LJ} e(K^{\bar{q}})$ is in $D_{LJ}^{\to}$ and $T_{x_1,\dots,x_n}(\Pi_1^{\bar{q}}) = T_{x_1,\dots,x_n}(\Pi_1)$, for every $x_1,\dots,x_n$. But $e(H_i^{\bar{q}}) \equiv e((H_i\backslash^{ps})^q)$ and $e(K^{\bar{q}}) \equiv e((K\backslash^{ps})^q)$, so $\Pi_1^{\bar{q}} \equiv \Pi^q$ and $T_{x_1,\dots,x_n}(\Pi_1^{\bar{q}}) = T_{x_1,\dots,x_n}(\Pi^q)$, for every $x_1,\dots,x_n$. Also, $T_{x_1,\dots,x_n}(\Pi_1) = T_{x_1,\dots,x_n}(\Pi)$, for every $x_1,\dots,x_n$, so $T_{x_1,\dots,x_n}(\Pi^q) = T_{x_1,\dots,x_n}(\Pi)$, for every $x_1,\dots,x_n$.

- $$\frac{\Pi_1 : \{H_1,\dots,H_n,K_1\} \vdash_{pIL} K_2}{\Pi : \{H_1,\dots,H_n\} \vdash_{pIL} K_1 \to K_2}\,(\to I_{pIL})$$

Suppose $p \in P_T(K_1\to K_2)$. Then $p \in P_T(K_2)$, so, by the IH,

$$\Pi_1^p : \{e(H_1^p),\dots,e(H_n^p),e(K_1^p)\} \vdash_{LJ} e(K_2^p)$$

is in $D_{LJ}^{\to}$ and $T_{x_1,\dots,x_n,x}(\Pi_1^p) = T_{x_1,\dots,x_n,x}(\Pi_1)$, for every $x_1,\dots,x_n,x$. Applying $(\to I_{LJ})$ on Π_1^p, we get

$$\Pi^p : \{e(H_1^p),\dots,e(H_n^p)\} \vdash_{LJ} e(K_1^p)\to e(K_2^p) \equiv e(K_1^p\to K_2^p) \equiv e((K_1\to K_2)^p)$$

We also have that Π^p is in $D_{LJ}^{\to}$, since Π_1^p is in $D_{LJ}^{\to}$ and Π^p results from Π_1^p by $(\to I)$.

[2] We actually prove a stronger result, see theorem 3.4.1 and remark 3.4.2 for details.

It remains to show that $T_{x_1,\dots,x_n}(\Pi^p) = T_{x_1,\dots,x_n}(\Pi)$, for every $x_1,\dots,x_n$. It is

$$\begin{array}{rcl}
T_{x_1,\dots,x_n,x}(\Pi_1^p) & = & T_{x_1,\dots,x_n,x}(\Pi_1),\ \text{for every } x_1,\dots,x_n,x \implies \\
\lambda x.T_{x_1,\dots,x_n,x}(\Pi_1^p) & = & \lambda x.T_{x_1,\dots,x_n,x}(\Pi_1)^3,\ \text{for every } x_1,\dots,x_n \\
 & & \text{and every } x \text{ distinct from } x_i\ (1 \leqslant i \leqslant n) \implies \\
\bigcup_x \lambda x.T_{x_1,\dots,x_n,x}(\Pi_1^p) & = & \bigcup_x \lambda x.T_{x_1,\dots,x_n,x}(\Pi_1),\ \text{for every } x_1,\dots,x_n \iff \\
T_{x_1,\dots,x_n}(\Pi^p) & = & T_{x_1,\dots,x_n}(\Pi),\ \text{for every } x_1,\dots,x_n
\end{array}$$

- $$\frac{\Pi_1 : \{H_1,\dots,H_n\} \vdash_{pIL} H \to K \qquad \Pi_2 : \{H_1,\dots,H_n\} \vdash_{pIL} H}{\Pi : \{H_1,\dots,H_n\} \vdash_{pIL} K}\ (\to E_{pIL})$$

Suppose $p \in P_T(K)$. Then $p \in P_T(H \to K) = P_T(H)$, so, by the IH,

$$\Pi_1^p : \{\, e(H_1^p),\dots,e(H_n^p)\,\} \vdash_{LJ} e((H \to K)^p) \equiv e(H^p) \to e(K^p)$$

$$\Pi_2^p : \{\, e(H_1^p),\dots,e(H_n^p)\,\} \vdash_{LJ} e(H^p)$$

Applying $(\to E_{LJ})$ on Π_1^p, Π_2^p, we get $\Pi^p : \{\, e(H_1^p),\dots,e(H_n^p)\,\} \vdash_{LJ} e(K^p)$. We have that Π^p is in $D_{LJ}^{\to}$, since Π_1^p and Π_2^p are in $D_{LJ}^{\to}$, by the IH, and Π^p follows from Π_1^p and Π_2^p by $(\to E)$. It remains to show that $T_{x_1,\dots,x_n}(\Pi^p) = T_{x_1,\dots,x_n}(\Pi)$, for every $x_1,\dots,x_n$. By the IH, we have $T_{x_1,\dots,x_n}(\Pi_1^p) = T_{x_1,\dots,x_n}(\Pi_1)$ and $T_{x_1,\dots,x_n}(\Pi_2^p) = T_{x_1,\dots,x_n}(\Pi_2)$, for every $x_1,\dots,x_n$. So

$$\begin{array}{rcl}
T_{x_1,\dots,x_n}(\Pi_1^p)\,T_{x_1,\dots,x_n}(\Pi_2^p) & = & T_{x_1,\dots,x_n}(\Pi_1)\,T_{x_1,\dots,x_n}(\Pi_2)^4,\ \text{for every } x_1,\dots,x_n \implies \\
T_{x_1,\dots,x_n}(\Pi^p) & = & T_{x_1,\dots,x_n}(\Pi),\ \text{for every } x_1,\dots,x_n
\end{array}$$

- $$\frac{\Pi_1 : \{H_1[p := [\sigma_1,\sigma_1]],\dots,H_n[p := [\sigma_n,\sigma_n]]\} \vdash_{pIL} K[p := [\sigma,\tau]]}{\Pi : \{H_1[p := \sigma_1],\dots,H_n[p := \sigma_n]\} \vdash_{pIL} K[p := \sigma \cap \tau]}\ (\cap I_{pIL})$$

Suppose $q \in P_T(K[p := \sigma \cap \tau])$. We distinguish two cases. (1) If $q \not\equiv p$, then $q \in P_T(K[p := [\sigma,\tau]])$, so, by the IH,

$$\Pi_1^q : \{\, e(H_1[p := [\sigma_1,\sigma_1]]^q),\dots,e(H_n[p := [\sigma_n,\sigma_n]]^q)\,\} \vdash_{LJ} e(K[p := [\sigma,\tau]]^q)$$

is in $D_{LJ}^{\to}$ and $T_{x_1,\dots,x_n}(\Pi_1^q) = T_{x_1,\dots,x_n}(\Pi_1)$, for every $x_1,\dots,x_n$. But $H_i[p := [\sigma_i,\sigma_i]]^q \equiv H_i[p := \sigma_i]^q$ $(1 \leqslant i \leqslant n)$ and $K[p := [\sigma,\tau]]^q \equiv K[p := \sigma \cap \tau]^q$, so

$$\Pi_1^q \equiv \Pi^q : \{\, e(H_1[p := \sigma_1]^q),\dots,e(H_n[p := \sigma_n]^q)\,\} \vdash_{LJ} e(K[p := \sigma \cap \tau]^q)$$

---

[3] If Π' belongs to $D_{LJ}^{\to} \cup D_{pIL}$ and gives Π under $(\to I)$, it is

$$T_{x_1,\dots,x_n}(\Pi) = \bigcup_x \lambda x.T_{x_1,\dots,x_n,x}(\Pi') = \bigcup_x \{\, \lambda x.M \mid M \in T_{x_1,\dots,x_n,x}(\Pi')\}$$

[4] If Π' and Π'' both belong to $D_{LJ}^{\to}$ or to D_{pIL} and give Π under $(\to E)$, it is

$$\begin{array}{rcl}
T_{x_1,\dots,x_n}(\Pi) & = & T_{x_1,\dots,x_n}(\Pi')\,T_{x_1,\dots,x_n}(\Pi'') \\
 & = & \{MN \mid M \in T_{x_1,\dots,x_n}(\Pi'),\ N \in T_{x_1,\dots,x_n}(\Pi''),\ BV(M) \cap BV(N) = \emptyset\}
\end{array}$$

Hence, Π^q is in $D_{LJ}^{\rightarrow}$ and $T_{x_1,\ldots,x_n}(\Pi^q) = T_{x_1,\ldots,x_n}(\Pi_1^q) = T_{x_1,\ldots,x_n}(\Pi_1) = T_{x_1,\ldots,x_n}(\Pi)$, for every $x_1, \ldots, x_n$. (2) If $q \equiv p$, then pl and pr are in $P_T(K[p := [\sigma, \tau]])$, so, by the IH,

$$\Pi_1^{pl} : \{ e(\sigma_1), \ldots, e(\sigma_n) \} \vdash_{LJ} e(\sigma) \qquad \Pi_1^{pr} : \{ e(\sigma_1), \ldots, e(\sigma_n) \} \vdash_{LJ} e(\tau)$$

Applying $(\wedge I_{LJ})$ on Π_1^{pl}, Π_1^{pr}, we get

$$\Pi^p : \{ e(\sigma_1), \ldots, e(\sigma_n) \} \vdash_{LJ} e(\sigma) \wedge e(\tau) \equiv e(\sigma \cap \tau)$$

which is in $D_{LJ}^{\rightarrow}$, since, by the IH, Π_1^{pl} and Π_1^{pr} are in $D_{LJ}^{\rightarrow}$ and $T_{x_1,\ldots,x_n}(\Pi_1^{pl}) = T_{x_1,\ldots,x_n}(\Pi_1) = T_{x_1,\ldots,x_n}(\Pi_1^{pr})$, for every $x_1, \ldots, x_n$. Also, $T_{x_1,\ldots,x_n}(\Pi^p) = T_{x_1,\ldots,x_n}(\Pi_1^{pl}) = T_{x_1,\ldots,x_n}(\Pi_1) = T_{x_1,\ldots,x_n}(\Pi)$, for every $x_1, \ldots, x_n$.

- $$\frac{\Pi_1 : \{H_1, \ldots, H_n\} \vdash_{pIL} K[p := \sigma \cap \tau]}{\Pi : \{H_1, \ldots, H_n\} \vdash_{pIL} K[p := \sigma]} \; (\cap E_{pIL}^l)$$

Suppose $q \in P_T(K[p := \sigma])$. Then $q \in P_T(K[p := \sigma \cap \tau])$. We distinguish two cases. (1) If $q \not\equiv p$, then $K[p := \sigma \cap \tau]^q \equiv K[p := \sigma]^q$. By the IH,

$$\Pi_1^q : \{ e(H_1^q), \ldots, e(H_n^q) \} \vdash_{LJ} e(K[p := \sigma \cap \tau]^q)$$

is in $D_{LJ}^{\rightarrow}$ and $T_{x_1,\ldots,x_n}(\Pi_1^q) = T_{x_1,\ldots,x_n}(\Pi_1)$, for every $x_1, \ldots, x_n$. It is easy to see that $\Pi^q \equiv \Pi_1^q$, so that Π^q is in $D_{LJ}^{\rightarrow}$ and $T_{x_1,\ldots,x_n}(\Pi^q) = T_{x_1,\ldots,x_n}(\Pi_1^q)$, for every $x_1, \ldots, x_n$. Thus, we have $T_{x_1,\ldots,x_n}(\Pi^q) = T_{x_1,\ldots,x_n}(\Pi_1) = T_{x_1,\ldots,x_n}(\Pi)$, for every $x_1, \ldots, x_n$. (2) If $q \equiv p$, then $K[p := \sigma \cap \tau]^p \equiv \sigma \cap \tau$ and $K[p := \sigma]^p \equiv \sigma$. By the IH, $\Pi_1^p : \{ e(H_1^p), \ldots, e(H_n^p) \} \vdash_{LJ} e(\sigma \cap \tau) \equiv e(\sigma) \wedge e(\tau)$ and, applying $(\wedge E_{LJ}^l)$, we get $\Pi^p : \{ e(H_1^p), \ldots, e(H_n^p) \} \vdash_{LJ} e(\sigma)$. We have that Π^p is in $D_{LJ}^{\rightarrow}$, since Π_1^p is in $D_{LJ}^{\rightarrow}$, by the IH, and Π^p follows from Π_1^p by $(\wedge E)$. In addition, $T_{x_1,\ldots,x_n}(\Pi^p) = T_{x_1,\ldots,x_n}(\Pi_1^p)$, for every $x_1, \ldots, x_n$. Since, by the IH, $T_{x_1,\ldots,x_n}(\Pi_1^p) = T_{x_1,\ldots,x_n}(\Pi_1)$, for every $x_1, \ldots, x_n$, we have $T_{x_1,\ldots,x_n}(\Pi^p) = T_{x_1,\ldots,x_n}(\Pi_1) = T_{x_1,\ldots,x_n}(\Pi)$, for every $x_1, \ldots, x_n$.

For $(\cap E_{pIL}^r)$, we work similarly. ⊣

Remark 3.4.2 The conclusion of theorem 3.4.1

$$T_{x_1,\ldots,x_n}(\Pi^p) = T_{x_1,\ldots,x_n}(\Pi), \text{ for every } x_1, \ldots, x_n \text{ and } p \in P_T(K)$$

implies that $U(\Pi^p) = U(\Pi)$, for every $p \in P_T(K)$. In general, for Π' and Π in $D_{LJ}^{\rightarrow} \cup D_{pIL}$, the following implication holds:

$$\begin{aligned} T_{x_1,\ldots,x_n}(\Pi') &= T_{x_1,\ldots,x_n}(\Pi), \text{ for every } x_1, \ldots, x_n \implies \\ U(\Pi') = \bigcup_{x_1,\ldots,x_n} T_{x_1,\ldots,x_n}(\Pi') &= \bigcup_{x_1,\ldots,x_n} T_{x_1,\ldots,x_n}(\Pi) = U(\Pi) \end{aligned}$$

It is *not* generally true, though, that

$$U(\Pi') = U(\Pi) \implies T_{x_1,\ldots,x_n}(\Pi') = T_{x_1,\ldots,x_n}(\Pi), \text{ for every } x_1, \ldots, x_n$$

For example, consider the axioms $\Pi' : \{\alpha, \beta\} \vdash_{LJ} \alpha$ and $\Pi : \{\alpha, \beta\} \vdash_{LJ} \beta$ in $D_{LJ}^{\rightarrow}$. We have that $U(\Pi') = Var = U(\Pi)$, but, for $x \not\equiv y$, $T_{x,y}(\Pi') = \{x\} \neq \{y\} = T_{x,y}(\Pi)$. (The notation Var stands for the set of all λ-variables.) Hence, "$T_{x_1,\ldots,x_n}(\Pi') = T_{x_1,\ldots,x_n}(\Pi)$, for every $x_1, \ldots, x_n$" is stronger than "$U(\Pi') = U(\Pi)$"; let us call the former claim *"set (or term) equality"* and the latter *"form equality"*.

Definition 3.4.3 *Let* $\Pi : \{H_1, \ldots, H_n\} \vdash_{pIL} K$ *and* $LJ(\Pi) = \{\Pi^p \mid p \in P_T(K)\}$. *Any LJ-deduction in* $LJ(\Pi)$ *shall be refered to as a* LJ-projection *of* Π.

We continue with an example showing the LJ-projections generated by a pIL-deduction according to theorem 3.4.1.

Example 3.4.4 Let $\sigma \equiv \alpha \cap \beta \cap \gamma$ with α, β and γ propositional variables. Then $e(\sigma) = \alpha \wedge \beta \wedge \gamma$, $e(\alpha) = \alpha$, $e(\beta) = \beta$ and $e(\gamma) = \gamma$. Deduction $\Pi : [\sigma, \sigma] \vdash_{pIL} [\gamma, \alpha]$ (on page 36) has two LJ-projections $\Pi^l : e(\sigma) \vdash_{LJ} e(\gamma)$ (on page 37) and $\Pi^r : e(\sigma) \vdash_{LJ} e(\alpha)$ (on page 36).

We can roughly describe a *projection algorithm* for the *construction* of LJ-projections, given a pIL-deduction. We construct Π^l bottom-up. As the theorem states, its conclusion is $e(\sigma) \vdash_{LJ} \gamma$, i.e. we follow path l on the kits in Π's conclusion and apply e. Rule $(\cap E^r)_1$ in Π applied on path l transforms to $(\wedge E^r)_1$ in Π^l, so that the judgement $e(\sigma) \vdash_{LJ} \alpha \wedge \gamma$ is formed right above $e(\sigma) \vdash_{LJ} \gamma$. Rule $(\cap I)_2$ in Π applied on path l transforms to $(\wedge I)_2$ in Π^l with premises $e(\sigma) \vdash_{LJ} \alpha$ and $e(\sigma) \vdash_{LJ} \gamma$. Then, we follow paths ll and lr in Π's judgements to construct subdeductions Π^l_0 and Π^l_1, respectively. Rule $(\rightarrow E)_3$ in Π induces a $(\rightarrow E)$ rule in each of Π^l_0 and Π^l_1; in particular, rule $(\rightarrow E)_{3_0}$ in Π^l_0 and rule $(\rightarrow E)_{3_1}$ in Π^l_1. Focusing on Π^l_1, premises $e(\sigma) \vdash_{LJ} \beta \rightarrow \gamma$ and $e(\sigma) \vdash_{LJ} \beta$ are placed above the conclusion. Rule $(\cap E)_4$ in Π brings about $(\wedge E)_{4_1}$ in Π^l_1, so that $e(\sigma) \vdash_{LJ} e(\sigma)$ sits above $e(\sigma) \vdash_{LJ} \beta$, while rule $(\rightarrow I)_5$ in Π appears as $(\rightarrow I)_{5_1}$ in Π^l_1 and puts $\beta, e(\sigma) \vdash_{LJ} \gamma$ above $e(\sigma) \vdash_{LJ} \beta \rightarrow \gamma$. Continuing this way, rules $(\rightarrow E)_8, (\cap E)_9, (\rightarrow I)_{10}$ and $(\cap E)_{11}$ in Π appear in Π^l_1 as $(\rightarrow E)_{8_1}, (\wedge E)_{9_1}, (\rightarrow I)_{10_1}$ and $(\wedge E^r)_{11_1}$, respectively. Rules $(\cap E^l)_6$ and $(\cap I)_7$ in Π applied on ll do not affect Π^l_1, as the latter follows path lr. They project to Π^l_0 as $(\wedge E^l)_6$ and $(\wedge I)_7$, respectively. Subdeduction Π^l_0 is constructed in a similar manner, starting from premises $e(\sigma) \vdash_{LJ} \beta \rightarrow \alpha$ and $e(\sigma) \vdash_{LJ} \beta$ placed above the conclusion by $(\rightarrow E)_{3_0}$.

Note that intersection elimination rules in Π applied on different terminal paths of the form lp project to conjunction elimination rules in different subdeductions of Π^l and implication introduction (or elimination) rules in Π applied on judgements including kits with n terminal paths of the form lp project to n implication introduction (or elimination) rules in Π^l. On the other hand, if a pIL-deduction Π contains pruning rules, they do not project to LJ.

Both Π^l and Π^r are decoratable by $\mathsf{d}_{\rightarrow}$. In particular, for $s \in \{l, r\}$ and every x, $T_x(\Pi^s) = T_x(\Pi) = \{(\lambda y.(\lambda z.x)x)x \mid y(\not\equiv x), z(\not\equiv x)$ a sequence of distinct variables$\}$.

3.4.1 Characteristics of the set $\mathbf{D}^{\rightarrow}_{\mathbf{LJ}}$

At this point, we will investigate the conditions that a LJ-deduction must satisfy to be decoratable by $\mathsf{d}_{\rightarrow}$ and we will examine *why* the LJ-deductions which are projections of a pIL-deduction indeed satisfy these conditions.

Considering a LJ-deduction with, say, three $(\wedge I)$ rules and tracing it from top to bottom, the following conditions must hold for it to be decoratable by $\mathsf{d}_{\rightarrow}$.

(1) The first $(\wedge I)$ rule must conjunct two LJ-formulas that have been derived from subdeductions which: (i) share both the number of instances and the order of application of the rules $(\rightarrow I)$ and $(\rightarrow E)$, (ii) have the same number of axioms[5] and (iii) have the same context cardinality in corresponding[6] axioms.

[5]An extra axiom in one of the two would imply the existence of an extra binary rule, but they both have the same number of instances of $(\rightarrow E)$ and no instances of $(\wedge I)$.

[6]If we number the axioms of each subdeduction from left to right, the ones identically labelled are *corresponding.*

$$
\begin{prooftree}
\AxiomC{$[[[\beta,\beta],\beta],\gamma],[[[\beta,\beta,],\beta],\alpha],[[[\sigma,\sigma],\sigma],\sigma] \vdash_{pIL} [[[\sigma,\sigma],\sigma],\sigma]$}
\RightLabel{\scriptsize $(\cap E)_{11}$}
\doubleLine
\UnaryInfC{$[[[\beta,\beta],\beta],\gamma],[[[\beta,\beta,],\beta],\alpha],[[[\sigma,\sigma],\sigma],\sigma] \vdash_{pIL} [[[\alpha,\gamma],\gamma],\alpha]$}
\RightLabel{\scriptsize $(\to I)_{10}$}
\UnaryInfC{$[[[\beta,\beta,],\beta],\alpha],[[[\sigma,\sigma],\sigma],\sigma] \vdash_{pIL} [[[\beta\to\alpha,\beta\to\gamma],\beta\to\gamma],\gamma\to\alpha]$}
\AxiomC{$[[[\beta,\beta,],\beta],\alpha],[[[\sigma,\sigma],\sigma],\sigma] \vdash_{pIL} [[[\sigma,\sigma],\sigma],\sigma]$}
\RightLabel{\scriptsize $(\cap E)_{9}$}
\doubleLine
\UnaryInfC{$[[[\beta,\beta,],\beta],\alpha],[[[\sigma,\sigma],\sigma],\sigma] \vdash_{pIL} [[[\beta,\beta],\beta],\gamma]$}
\RightLabel{\scriptsize $(\to E)_{8}$}
\BinaryInfC{$[[[\beta,\beta,],\beta],\alpha],[[[\sigma,\sigma],\sigma],\sigma] \vdash_{pIL} [[[\alpha,\gamma],\gamma],\alpha]$}
\RightLabel{\scriptsize $(\cap I)_{7}$}
\UnaryInfC{$[[\beta,\beta],\alpha],[[\sigma,\sigma],\sigma] \vdash_{pIL} [[\alpha\cap\gamma,\gamma],\alpha]$}
\RightLabel{\scriptsize $(\cap E^{l})_{6}$}
\UnaryInfC{$[[\beta,\beta],\alpha],[[\sigma,\sigma],\sigma] \vdash_{pIL} [[\alpha,\gamma],\alpha]$}
\RightLabel{\scriptsize $(\to I)_{5}$}
\UnaryInfC{$[[\sigma,\sigma],\sigma] \vdash_{pIL} [[\beta\to\alpha,\beta\to\gamma],\alpha\to\alpha]$}
\AxiomC{$[[\sigma,\sigma],\sigma] \vdash_{pIL} [[\sigma,\sigma],\sigma]$}
\RightLabel{\scriptsize $(\cap E)_{4}$}
\doubleLine
\UnaryInfC{$[[\sigma,\sigma],\sigma] \vdash_{pIL} [[\beta,\beta],\alpha]$}
\RightLabel{\scriptsize $(\to E)_{3}$}
\BinaryInfC{$[[\sigma,\sigma],\sigma] \vdash_{pIL} [[\alpha,\gamma],\alpha]$}
\RightLabel{\scriptsize $(\cap I)_{2}$}
\UnaryInfC{$[\sigma,\sigma] \vdash_{pIL} [\alpha\cap\gamma,\alpha]$}
\RightLabel{\scriptsize $(\cap E^{r})_{1}$}
\UnaryInfC{$\Pi : [\sigma,\sigma] \vdash_{pIL} [\gamma,\alpha]$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$\gamma,\alpha,e(\sigma) \vdash_{LJ} e(\sigma)$}
\RightLabel{\scriptsize $(\wedge E)$}
\doubleLine
\UnaryInfC{$\gamma,\alpha,e(\sigma) \vdash_{LJ} \alpha$}
\RightLabel{\scriptsize $(\to I)$}
\UnaryInfC{$\alpha,e(\sigma) \vdash_{LJ} \gamma\to\alpha$}
\AxiomC{$\alpha,e(\sigma) \vdash_{LJ} e(\sigma)$}
\RightLabel{\scriptsize $(\wedge E^{r})$}
\UnaryInfC{$\alpha,e(\sigma) \vdash_{LJ} \gamma$}
\RightLabel{\scriptsize $(\to E)$}
\BinaryInfC{$\alpha,e(\sigma) \vdash_{LJ} \alpha$}
\RightLabel{\scriptsize $(\to I)$}
\UnaryInfC{$e(\sigma) \vdash_{LJ} \alpha\to\alpha$}
\AxiomC{$e(\sigma) \vdash_{LJ} e(\sigma)$}
\RightLabel{\scriptsize $(\wedge E)$}
\doubleLine
\UnaryInfC{$e(\sigma) \vdash_{LJ} \alpha$}
\RightLabel{\scriptsize $(\to E)$}
\BinaryInfC{$\Pi^{r} : e(\sigma) \vdash_{LJ} \alpha$}
\end{prooftree}
$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\beta,\beta,e(\sigma)\vdash_{LJ} e(\sigma)}{\beta,\beta,e(\sigma)\vdash_{LJ}\alpha}(\wedge E)_{11_{00}}}{\beta,e(\sigma)\vdash_{LJ}\beta\to\alpha}(\to I)_{10_{00}} \quad \dfrac{\beta,e(\sigma)\vdash_{LJ} e(\sigma)}{\beta,e(\sigma)\vdash_{LJ}\beta}(\wedge E)_{9_{00}}}{\Pi^l_{00}:\beta,e(\sigma)\vdash_{LJ}\alpha}(\to E)_{8_{00}} \quad \dfrac{\dfrac{\dfrac{\beta,\beta,e(\sigma)\vdash_{LJ} e(\sigma)}{\beta,\beta,e(\sigma)\vdash_{LJ}\gamma}(\wedge E^r)_{11_{01}}}{\beta,e(\sigma)\vdash_{LJ}\beta\to\gamma}(\to I)_{10_{01}} \quad \dfrac{\beta,e(\sigma)\vdash_{LJ} e(\sigma)}{\beta,e(\sigma)\vdash_{LJ}\beta}(\wedge E)_{9_{01}}}{\Pi^l_{01}:\beta,e(\sigma)\vdash_{LJ}\gamma}(\to E)_{8_{01}}}{\beta,e(\sigma)\vdash_{LJ}\alpha\wedge\gamma}(\wedge I)_7}{\beta,e(\sigma)\vdash_{LJ}\alpha}(\wedge E^l)_6}{e(\sigma)\vdash_{LJ}\beta\to\alpha}(\to I)_{5_0} \quad \dfrac{e(\sigma)\vdash_{LJ} e(\sigma)}{e(\sigma)\vdash_{LJ}\beta}(\wedge E)_{4_0}}{\Pi^l_0 : e(\sigma)\vdash_{LJ}\alpha}(\to E)_{3_0}$$

$$\dfrac{\dfrac{\Pi^l_0 : e(\sigma)\vdash_{LJ}\alpha \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\beta,\beta,e(\sigma)\vdash_{LJ} e(\sigma)}{\beta,\beta,e(\sigma)\vdash_{LJ}\gamma}(\wedge E^r)_{11_1}}{\beta,e(\sigma)\vdash_{LJ}\beta\to\gamma}(\to I)_{10_1} \quad \dfrac{\beta,e(\sigma)\vdash_{LJ} e(\sigma)}{\beta,e(\sigma)\vdash_{LJ}\beta}(\wedge E)_{9_1}}{\beta,e(\sigma)\vdash_{LJ}\gamma}(\to E)_{8_1}}{e(\sigma)\vdash_{LJ}\beta\to\gamma}(\to I)_{5_1} \quad \dfrac{e(\sigma)\vdash_{LJ} e(\sigma)}{e(\sigma)\vdash_{LJ}\beta}(\wedge E)_{4_1}}{\Pi^l_1 : e(\sigma)\vdash_{LJ}\gamma}(\to E)_{3_1}}{e(\sigma)\vdash_{LJ}\alpha\wedge\gamma}(\wedge I)_2}{\Pi^l : e(\sigma)\vdash_{LJ}\gamma}(\wedge E^r)_1$$

(2) Identifying the two subdeductions leading to the first $(\wedge I)$ rule with respect to the sequence of **im**plicative rules from top to bottom and the sequence of **ax**ioms[7] from left to right (abbreviated "imax"), the second $(\wedge I)$ rule must conjunct two LJ-formulas derived from subdeductions which satisfy (i), (ii) and (iii).

(3) Identifying, with respect to imax, the two subdeductions leading to the second $(\wedge I)$ rule, which can be done given the identification, with respect to imax, of the two subdeductions leading to the first $(\wedge I)$ rule, the third $(\wedge I)$ rule must conjunct two LJ-formulas derived from subdeductions which satisfy (i), (ii) and (iii).

For LJ-deductions with more $(\wedge I)$ rules, the conditions adapt accordingly.

We can now ask why LJ-projections of a pIL-deduction indeed satisfy the conditions. We resolve this question by examining projection Π^l of example 3.4.4.

The first $(\wedge I)$ combines subdeductions Π^l_{00} and Π^l_{01}, which are themselves LJ-projections of a pIL-deduction $\hat{\Pi}$. In particular, $\Pi^l_{00} \equiv (\hat{\Pi})^l$ and $\Pi^l_{01} \equiv (\hat{\Pi})^r$.

$$
\dfrac{
\dfrac{\dfrac{[\beta,\beta],[\beta,\beta],[\sigma,\sigma] \vdash_{pIL} [\sigma,\sigma]}{[\beta,\beta],[\beta,\beta],[\sigma,\sigma] \vdash_{pIL} [\alpha,\gamma]}\,(\cap E)}{[\beta,\beta],[\sigma,\sigma] \vdash_{pIL} [\beta\to\alpha,\beta\to\gamma]}\,(\to I)
\qquad
\dfrac{[\beta,\beta],[\sigma,\sigma] \vdash_{pIL} [\sigma,\sigma]}{[\beta,\beta],[\sigma,\sigma] \vdash_{pIL} [\beta,\beta]}\,(\cap E)
}{\hat{\Pi} : [\beta,\beta],[\sigma,\sigma] \vdash_{pIL} [\alpha,\gamma]}\,(\to E)
$$

The fact that Π^l_{00} and Π^l_{01} satisfy (i) derives from the fact that the rules $(\to I)$ and $(\to E)$ in $\hat{\Pi}$ are global, so they affect the left and right leaves of the kits in exactly the same way.

Informally speaking, the "left part" of each axiom in $\hat{\Pi}$ gives an axiom in Π^l_{00}, while the "right part" gives an axiom in Π^l_{01}. But, since the kits in $\hat{\Pi}$'s axioms—and even more in any pIL-judgement—are *full* binary, left and right parts appear in pairs, so every axiom in Π^l_{00} can be matched to a unique axiom in Π^l_{01} and vice versa. Hence, Π^l_{00} and Π^l_{01} satisfy (ii).

Corresponding axioms in Π^l_{00} and Π^l_{01} are those that have emanated from the same axiom in $\hat{\Pi}$ as left and right part, respectively. Since the kits in $\hat{\Pi}$'s axioms are full binary, every left context-leaf has a pairing right one; hence, Π^l_{00} and Π^l_{01} satisfy (iii).

The second $(\wedge I)$ combines subdeductions Π^l_0 and Π^l_1, which are LJ-projections of a pIL-deduction $\tilde{\Pi}$. It is $\Pi^l_0 \equiv (\tilde{\Pi})^l$ and $\Pi^l_1 \equiv (\tilde{\Pi})^r$.

$$
\dfrac{
\dfrac{\dfrac{\dfrac{
\dfrac{\dfrac{[[\beta,\beta],\beta],[[\beta,\beta],\beta],[[\sigma,\sigma],\sigma] \vdash_{pIL} [[\sigma,\sigma],\sigma]}{[[\beta,\beta],\beta],[[\beta,\beta],\beta],[[\sigma,\sigma],\sigma] \vdash_{pIL} [[\alpha,\gamma],\gamma]}\,(\cap E)}{[[\beta,\beta],\beta],[[\sigma,\sigma],\sigma] \vdash_{pIL} [[\beta\to\alpha,\beta\to\gamma],\beta\to\gamma]}\,(\to I)
\qquad
\dfrac{[[\beta,\beta],\beta],[[\sigma,\sigma],\sigma] \vdash_{pIL} [[\sigma,\sigma],\sigma]}{[[\beta,\beta],\beta],[[\sigma,\sigma],\sigma] \vdash_{pIL} [[\beta,\beta],\beta]}\,(\cap E)
}{[[\beta,\beta],\beta],[[\sigma,\sigma],\sigma] \vdash_{pIL} [[\alpha,\gamma],\gamma]}\,(\to E)}{[\beta,\beta],[\sigma,\sigma] \vdash_{pIL} [\alpha\cap\gamma,\gamma]}\,(\cap I)}{[\beta,\beta],[\sigma,\sigma] \vdash_{pIL} [\alpha,\gamma]}\,(\cap E^l)}{[\sigma,\sigma] \vdash_{pIL} [\beta\to\alpha,\beta\to\gamma]}\,(\to I)
\qquad
\dfrac{[\sigma,\sigma] \vdash_{pIL} [\sigma,\sigma]}{[\sigma,\sigma] \vdash_{pIL} [\beta,\beta]}\,(\cap E)
}{\tilde{\Pi} : [\sigma,\sigma] \vdash_{pIL} [\alpha,\gamma]}\,(\to E)
$$

[7] The "identification" of corresponding axioms concerns their context cardinality; they are *not* identical. Even more, the two subdeductions are *not* identical; they just exhibit the same sequence of implicative rules.

The identification with respect to imax of Π^l_{00} and Π^l_{01} translates in kit terminology to the identification of terminal paths ll and lr, in all judgements of $\tilde{\Pi}$ before the application of $(\cap I)$. This does not necessarily mean that the leaves at the end of ll and lr are identical; it is an improper identification to explain in kit level why Π^l_0 and Π^l_1 satisfy (i), (ii) and (iii). If paths ll and lr coincide, we can think of all kits in $\tilde{\Pi}$ as having a left and a right part.

The sequence of global rules $(\rightarrow I), (\rightarrow E), (\rightarrow I), (\rightarrow E)$ in $\tilde{\Pi}$ equally affects the left and right parts of the kits; thus, Π^l_0 and Π^l_1 satisfy (i).

The left part of each axiom in $\tilde{\Pi}$ gives an axiom in Π^l_0, while the right part gives one in Π^l_1. Arguing as in the case of $\hat{\Pi}$, we conclude that Π^l_0 and Π^l_1 satisfy (ii) and (iii).

3.5 Strong normalization of IL

In this section, we define a $\diamond$-redex of a pIL-deduction, where $\diamond \in \{P, \rightarrow, \cap\}$, and show how to eliminate redexes. We prove that pIL is strongly normalizable by turning to LJ-projections of a pIL-deduction and invoking the strong normalizability of LJ. We then derive strong normalizability of IL.

We start by showing how the rule (P_{pIL}) can be eliminated from a pIL-deduction. The so-called P-commuting conversions or $\hookrightarrow_P$-normalization steps, defined below, shift an occurence of (P_{pIL}) upwards in the deduction until it reaches an axiom. At that point, its conclusion is by itself an axiom, so we can erase the initial axiom and the rule and take its conclusion as a deduction leaf.

Definition 3.5.1 *(i) The P-commuting conversions or $\hookrightarrow_P$-normalization steps on pIL are the following rewriting rules.*

1. *The axiom (A).*

$$\dfrac{\dfrac{}{\Gamma \cup \{H\} \vdash_{pIL} H}\,(A)}{\Gamma\backslash^p \cup \{H\backslash^p\} \vdash_{pIL} H\backslash^p}\,(P) \quad \hookrightarrow_P \quad \dfrac{}{\Gamma\backslash^p \cup \{H\backslash^p\} \vdash_{pIL} H\backslash^p}\,(A)$$

2. *The introduction of implication $(\rightarrow I)$.*

$$\dfrac{\dfrac{\Gamma \cup \{H\} \vdash_{pIL} K}{\Gamma \vdash_{pIL} H \rightarrow K}\,(\rightarrow I)}{\Gamma\backslash^p \vdash_{pIL} (H \rightarrow K)\backslash^p \equiv H\backslash^p \rightarrow K\backslash^p}\,(P) \quad \hookrightarrow_P \quad \dfrac{\dfrac{\Gamma \cup \{H\} \vdash_{pIL} K}{\Gamma\backslash^p \cup \{H\backslash^p\} \vdash_{pIL} K\backslash^p}\,(P)}{\Gamma\backslash^p \vdash_{pIL} H\backslash^p \rightarrow K\backslash^p}\,(\rightarrow I)$$

3. *The elimination of implication $(\rightarrow E)$.*

$$\dfrac{\dfrac{\Gamma \vdash_{pIL} H \rightarrow K \qquad \Gamma \vdash_{pIL} H}{\Gamma \vdash_{pIL} K}\,(\rightarrow E)}{\Gamma\backslash^p \vdash_{pIL} K\backslash^p}\,(P) \quad \hookrightarrow_P$$

$$\dfrac{\dfrac{\Gamma \vdash_{pIL} H \rightarrow K}{\Gamma\backslash^p \vdash_{pIL} H\backslash^p \rightarrow K\backslash^p}\,(P) \qquad \dfrac{\Gamma \vdash_{pIL} H}{\Gamma\backslash^p \vdash_{pIL} H\backslash^p}\,(P)}{\Gamma\backslash^p \vdash_{pIL} K\backslash^p}\,(\rightarrow E)$$

4. *For the introduction of intersection* $(\cap I)$*, we distinguish two cases. The contexts include kits for* $i \in \{1, \ldots, n\}$.

Case 1: $q \subseteqq p$

$$\dfrac{\dfrac{\{H_i[p := [\sigma_i, \sigma_i]]\} \vdash_{pIL} K[p := [\sigma, \tau]]}{\{H_i[p := \sigma_i]\} \vdash_{pIL} K[p := \sigma \cap \tau]}\ (\cap I)}{\{H_i[p := \sigma_i]\backslash^q \equiv H'_i[p' := \sigma_i]\} \vdash_{pIL} K[p := \sigma \cap \tau]\backslash^q \equiv K'[p' := \sigma \cap \tau]}\ (P) \quad \hookrightarrow_P$$

$$\dfrac{\dfrac{\{H_i[p := [\sigma_i, \sigma_i]]\} \vdash_{pIL} K[p := [\sigma, \tau]]}{\{H_i[p := [\sigma_i, \sigma_i]]\backslash^q \equiv H'_i[p' := [\sigma_i, \sigma_i]]\} \vdash_{pIL} K[p := [\sigma, \tau]]\backslash^q \equiv K'[p' := [\sigma, \tau]]}\ (P)}{\{H'_i[p' := \sigma_i]\} \vdash_{pIL} K'[p' := \sigma \cap \tau]}\ (\cap I)$$

Case 2: q, p *different paths*

$$\dfrac{\dfrac{\{H_i[p := [\sigma_i, \sigma_i]]\} \vdash_{pIL} K[p := [\sigma, \tau]]}{\{H_i[p := \sigma_i]\} \vdash_{pIL} K[p := \sigma \cap \tau]}\ (\cap I)}{\{H_i[p := \sigma_i]\backslash^q \equiv H'_i\} \vdash_{pIL} K[p := \sigma \cap \tau]\backslash^q \equiv K'}\ (P) \quad \hookrightarrow_P$$

$$\dfrac{\{H_i[p := [\sigma_i, \sigma_i]]\} \vdash_{pIL} K[p := [\sigma, \tau]]}{\{H_i[p := [\sigma_i, \sigma_i]]\backslash^q \equiv H'_i\} \vdash_{pIL} K[p := [\sigma, \tau]]\backslash^q \equiv K'}\ (P)$$

5. *For the elimination of intersection* $(\cap E)$*, we consider two cases, as well.*

Case 1: $q \subseteqq p$

$$\dfrac{\dfrac{\Gamma \vdash_{pIL} K[p := \sigma_l \cap \sigma_r]}{\Gamma \vdash_{pIL} K[p := \sigma_s]}\ (\cap E^s)}{\Gamma\backslash^q \vdash_{pIL} K[p := \sigma_s]\backslash^q \equiv K'[p' := \sigma_s]}\ (P) \quad \hookrightarrow_P$$

$$\dfrac{\dfrac{\Gamma \vdash_{pIL} K[p := \sigma_l \cap \sigma_r]}{\Gamma\backslash^q \vdash_{pIL} K[p := \sigma_l \cap \sigma_r]\backslash^q \equiv K'[p' := \sigma_l \cap \sigma_r]}\ (P)}{\Gamma\backslash^q \vdash_{pIL} K'[p' := \sigma_s]}\ (\cap E^s)$$

Case 2: q, p *different paths*

$$\dfrac{\dfrac{\Gamma \vdash_{pIL} K[p := \sigma_l \cap \sigma_r]}{\Gamma \vdash_{pIL} K[p := \sigma_s]}\ (\cap E^s)}{\Gamma\backslash^q \vdash_{pIL} K[p := \sigma_s]\backslash^q \equiv K'}\ (P) \quad \hookrightarrow_P \quad \dfrac{\Gamma \vdash_{pIL} K[p := \sigma_l \cap \sigma_r]}{\Gamma\backslash^q \vdash_{pIL} K[p := \sigma_l \cap \sigma_r]\backslash^q \equiv K'}\ (P)$$

(ii) Every pair of rules to the left of $\hookrightarrow_P$ *is called a* P-redex *and the conversion provides the corresponding* P-reduct.

Remark 3.5.2 (i) In the cases of $(\cap I)$ and $(\cap E)$, for the pruning at path q to have meaning, q has to be defined in $H_i[p := \sigma_i], K[p := \sigma \cap \tau]$ and $K[p := \sigma_s]$, respectively, i.e. in kits where p is terminal. For this reason, it can't be that $p \subset q$, so we consider the cases $q \subsetneqq p$ and q different from p. (ii) In case 2 of $(\cap I)$ and $(\cap E)$, the $\hookrightarrow_P$-normalization step eliminates the intersection rules. This is not a problem, since we are only interested in shifting pruning one place up in the deduction—which is done—and in concluding by the same judgement after the rules have been exchanged—which is attained without applying intersection. Actually, intersection in the P-redexes is applied to this part of the kits which is then pruned, so, if we first do the pruning, there is no longer space for intersection.

Definition 3.5.3 *(i) A pIL-deduction free of occurences of* (P_{pIL}) *is* P-normal.
(ii) A class π *of IL reduces under* $\hookrightarrow_P$ *to another class* π' *of IL* ($\pi \hookrightarrow_P \pi'$)*, if* $\Pi \in \pi$ *and* $\Pi \hookrightarrow_P \Pi'$ *imply that* $\Pi' \in \pi'$.

Using P-commuting conversions, it is easy to see that the following lemma holds.

Lemma 3.5.4 *Every* $\Pi : \Gamma \vdash_{pIL} H$ *can be reduced to a* P*-normal* $\Pi' : \Gamma \vdash_{pIL} H$ *under any strategy.*

Definition 3.5.5 *Let* Π *be a pIL-deduction and* $s \in \{l, r\}$.
(i) A $\rightarrow$*-redex of* Π *is a sequence* $(\rightarrow I_{pIL}), (\rightarrow E_{pIL})$ *in* Π *of the rules introducing and eliminating the implication.*

$$\dfrac{\dfrac{\Gamma \cup \{H\} \vdash_{pIL} K}{\Gamma \vdash_{pIL} H \rightarrow K}(\rightarrow I_{pIL}) \qquad \Gamma \vdash_{pIL} H}{\Gamma \vdash_{pIL} K}(\rightarrow E_{pIL})$$

(ii) A $\cap$*-redex of* Π *is a sequence* $(\cap I_{pIL}), (\cap E^s_{pIL})$ *in* Π *of the rules introducing and eliminating the intersection.*

$$\dfrac{\dfrac{\{H_1[p := [\sigma_1, \sigma_1]], \ldots, H_n[p := [\sigma_n, \sigma_n]]\} \vdash_{pIL} K[p := [\sigma_l, \sigma_r]]}{\{H_1[p := \sigma_1], \ldots, H_n[p := \sigma_n]\} \vdash_{pIL} K[p := \sigma_l \cap \sigma_r]}(\cap I_{pIL})}{\{H_1[p := \sigma_1], \ldots, H_n[p := \sigma_n]\} \vdash_{pIL} K[p := \sigma_s]}(\cap E^s_{pIL})$$

Definition 3.5.6 *Let* $\{H_1, \ldots, H_n\} \vdash_{pIL} K$ *be a judgement in position* x *of a pIL-deduction* Π *consisting of* k *steps* $(0 \leqslant x \leqslant k)$*. If the context-kit* H_i *(or a pruned descendant of* H_i*) doesn't move to the right of* $\vdash_{pIL}$ *by a* $(\rightarrow I)$ *rule in steps* $x+1, \ldots, k$*, it is said to be* open.

Remark 3.5.7 If $\Pi : \{H_1, \ldots, H_n\} \vdash_{pIL} K$ is a pIL-deduction, all context-kits H_i are stable.

The following lemma is used for the elimination of $\rightarrow$-redexes from a pIL-deduction.

Lemma 3.5.8 (Substitution lemma) *Let* $\Pi_0 : \Gamma \cup \{H\} \vdash_{pIL} K$, $\Pi_1 : \Gamma \vdash_{pIL} H$ *be pIL-deductions and* $S(\Pi_1, \Pi_0)$ *be the deductive structure obtained from* Π_0 *by substituting all axioms* $\Gamma' \cup \{H\} \vdash_{pIL} H$ $(\Gamma \subseteq \Gamma')$ *with* H *open by* $\Pi'_1 : \Gamma' \vdash_{pIL} H$*. Then,* $S(\Pi_1, \Pi_0) : \Gamma \vdash_{pIL} K$.

Proof: Use double induction, see [RR01, Pr65, Gi89]. $\dashv$

Remark 3.5.9 If Π_0 and Π_1 are P-normal, then so is $S(\Pi_1, \Pi_0)$.

The next definition shows the normalization procedures called up for eliminating single implication and intersection redexes, i.e. describes single normalization steps.

Definition 3.5.10 *Let Π be a pIL-deduction and $s \in \{l, r\}$.*

(i_a) A $\rightarrow$-rewriting step on Π is a normalization step that eliminates a $\rightarrow$-redex of the deduction.

$$\dfrac{\dfrac{\Pi_0 : \Gamma \cup \{H\} \vdash_{pIL} K}{\Gamma \vdash_{pIL} H \rightarrow K}\,(\rightarrow I_{pIL}) \qquad \Pi_1 : \Gamma \vdash_{pIL} H}{\Gamma \vdash_{pIL} K}\,(\rightarrow E_{pIL}) \quad \hookrightarrow_{\rightarrow}$$

$$S(\Pi_1, \Pi_0) : \Gamma \vdash_{pIL} K$$

(i_b) A class π of IL reduces to another class π' of IL under a $\rightarrow$-rewriting step ($\pi \hookrightarrow_{\rightarrow} \pi'$), if $\Pi \in \pi$ and $\Pi \hookrightarrow_{\rightarrow} \Pi'$ imply that $\Pi' \in \pi'$.

(ii_a) A $\cap$-rewriting step on Π is a normalization step that eliminates a $\cap$-redex of the deduction.

$$\dfrac{\dfrac{\{H_1[p := [\sigma_1, \sigma_1]], \ldots, H_n[p := [\sigma_n, \sigma_n]]\} \vdash_{pIL} K[p := [\sigma_l, \sigma_r]]}{\{H_1[p := \sigma_1], \ldots, H_n[p := \sigma_n]\} \vdash_{pIL} K[p := \sigma_l \cap \sigma_r]}\,(\cap I_{pIL})}{\{H_1[p := \sigma_1], \ldots, H_n[p := \sigma_n]\} \vdash_{pIL} K[p := \sigma_s]}\,(\cap E^s_{pIL}) \quad \hookrightarrow_{\cap}$$

$$\dfrac{\{H_1[p := [\sigma_1, \sigma_1]], \ldots, H_n[p := [\sigma_n, \sigma_n]]\} \vdash_{pIL} K[p := [\sigma_l, \sigma_r]]}{\{(H_1[p := [\sigma_1, \sigma_1]])\backslash^{ps}, \ldots, (H_n[p := [\sigma_n, \sigma_n]])\backslash^{ps}\} \vdash_{pIL} (K[p := [\sigma_l, \sigma_r]])\backslash^{ps}}\,(P_{pIL})$$

It is $(H_i[p := [\sigma_i, \sigma_i]])\backslash^{ps} \equiv H_i[p := \sigma_i]$ $(1 \leqslant i \leqslant n)$ and $(K[p := [\sigma_l, \sigma_r]])\backslash^{ps} \equiv K[p := \sigma_s]$.

(ii_b) A class π of IL reduces to another class π' of IL under a $\cap$-rewriting step ($\pi \hookrightarrow_{\cap} \pi'$), if $\Pi \in \pi$ and $\Pi \hookrightarrow_{\cap} \Pi'$ imply that $\Pi' \in \pi'$.

Remark 3.5.11 If Π is P-normal and $\Pi \hookrightarrow_{\rightarrow} \Pi'$, then Π' is P-normal, too. On the contrary, if $\Pi \hookrightarrow_{\cap} \Pi'$, then Π' is not P-normal, but it can be reduced to one using P-commuting conversions.

Definition 3.5.12 *(i) A pIL-deduction Π is* normal*, if it is P-normal and free of implication and intersection redexes.*

(ii) An IL-deduction π is normal (strongly normalizable)*, if there exists a normal (strongly normalizable) Π in π.*

Remark 3.5.13 We note that, for any IL-deduction π, if there is a normal (strongly normalizable) Π in π, then every Π in π is normal (strongly normalizable).

Theorem 3.5.14 *pIL is strongly normalizable, i.e. every pIL-deduction Π is strongly normalizable.*

Proof: Suppose there exists a pIL-deduction $\Pi : \{H_1, \ldots, H_m\} \vdash_{pIL} K$ which is not strongly normalizable. Then, there is an infinite sequence s of $\diamond$-steps, where $\diamond$ belongs to $\{\hookrightarrow_P, \hookrightarrow_{\rightarrow}, \hookrightarrow_{\cap}\}$, starting from Π. If $P_T(K) = \{p_1, \ldots, p_n\}$, then, by theorem 3.4.1, Π gives n LJ-projections $\Pi^1, \ldots, \Pi^n$ in $D^{\rightarrow}_{LJ}$, where $\Pi^i : \{e(H_1^{p_i}), \ldots, e(H_m^{p_i})\} \vdash_{LJ} e(K^{p_i})$. If $\Pi \hookrightarrow_P \Pi'$, then, for every $i \in \{1, \ldots, n\}$, $\Pi^i \equiv (\Pi')^i$. If $\Pi \hookrightarrow_{\rightarrow} \Pi'$, then, for every i,

$\Pi^i \hookrightarrow^1_{LJ_\rightarrow} \ldots \hookrightarrow^{r_i}_{LJ_\rightarrow} (\Pi')^i$. Finally, if $\Pi \hookrightarrow_\cap \Pi'$, then there is an $i_0 \in \{1, \ldots, n\}$, such that: (1) $\Pi^{i_0} \hookrightarrow_{LJ_\wedge} (\Pi')^{i_0}$ and (2) for every $i \in \{1, \ldots, n\} \setminus \{i_0\}$, it is $\Pi^i \equiv (\Pi')^i$.

Case 1: There are infinitely many $\hookrightarrow_\rightarrow$-steps in s. Then, since each such step generates finitely many $\hookrightarrow_{LJ_\rightarrow}$-steps in each Π^i, we meet infinitely many $\hookrightarrow_{LJ_\rightarrow}$-steps in each Π^i, which contradicts the strong normalization of LJ.

Case 2: There are infinitely many $\hookrightarrow_\cap$-steps in s. In this case, since each such step generates a $\hookrightarrow_{LJ_\wedge}$-step in one of the Π^i, there are infinitely many $\hookrightarrow_{LJ_\wedge}$-steps to be mounted in n LJ-deductions. Consequently, there is an $i \in \{1, \ldots, n\}$, such that we meet infinitely many $\hookrightarrow_{LJ_\wedge}$-steps in Π^i, which contradicts the strong normalization of LJ.

Case 3: There are infinitely many $\hookrightarrow_P$-steps in s. Then, there should be infinitely many $\hookrightarrow_\cap$-steps in s, since the (P) rules initially in Π are eliminated in a finite number of $\hookrightarrow_P$-steps and so is the (P) rule generated by a single $\hookrightarrow_\cap$-step. So, this case reduces to case 2. $\dashv$

Theorem 3.5.15 *IL is strongly normalizable.*

Proof: If there is an IL-deduction π which is not strongly normalizable, then, by definition 3.5.12(ii), if $\Pi \in \pi$, then Π is not strongly normalizable, which contradicts theorem 3.5.14. $\dashv$

3.6 IL and the part of LJ decoratable by $\mathsf{d}_{\rightarrow}$

In section 3.4, we saw that any pIL-deduction Π gives rise to a finite number of LJ-deductions in $D^{\rightarrow}_{LJ}$, called its LJ-projections, that all share the implicative structure of Π. In this section, we start by examining the special case of theorem 3.4.1 where Π is proper and has a single LJ-projection Π^ϵ, which is uniquely determined by the projection algorithm. Throughout the section, we concentrate on P-normal, proper deductions. We observe that equivalent pIL-deductions project to the same LJ-deduction, which is the unique LJ-projection of their equivalence class. Different equivalence classes have different LJ-projections. We continue by showing the inverse of theorem 3.4.1 (*inverse projection theorem for pIL*), i.e. how a finite number of LJ-deductions in $D^{\rightarrow}_{LJ}$ with the same implicative structure merge to give a single pIL-deduction with this very implicative structure. We describe an algorithmic procedure for constructing this pIL-deduction (*simulation algorithm*) and note that it actually uniquely determines an equivalence class of P-normal pIL-deductions, i.e. a P-normal IL-deduction (*inverse projection theorem for IL*). We then restrict the inverse for IL to a single LJ-deduction in $D^{\rightarrow}_{LJ}$, which gives through the simulation argorithm a unique P-normal, proper IL-deduction called its *IL-duplicate*. The fact that projecting and simulating are inverse procedures between P-normal, proper IL-deductions and LJ-deductions in $D^{\rightarrow}_{LJ}$ leads to the main theorem of this section which claims a one-to-one correspondence between such deductions, corresponding deductions sharing the same implicative structure.

If Π of theorem 3.4.1 concludes by a proper judgement $\{\sigma_1, \ldots, \sigma_n\} \vdash_{pIL} \tau$, there is just one terminal path ϵ to be considered, so we get a single LJ-projection $\Pi^\epsilon : \{e(\sigma_1), \ldots, e(\sigma_n)\} \vdash_{LJ} e(\tau)$, which is decoratable by $\mathsf{d}_\rightarrow$ and such that $T_{x_1,\ldots,x_n}(\Pi^\epsilon) = T_{x_1,\ldots,x_n}(\Pi)$, for every $x_1, \ldots, x_n$. Given Π, Π^ϵ is uniquely determined by the projection algorithm. If Π' is the P-normal deduction to which Π can be reduced, then $(\Pi')^\epsilon \equiv \Pi^\epsilon$, since (P_{pIL}) doesn't have an image in LJ. We obtain $(\Pi')^\epsilon$ from Π' by the projection algorithm, as usual.

Let us now concentrate on P-normal, proper pIL-deductions. So far, we have that, for every P-normal, proper pIL-deduction $\Pi : \{\sigma_1, \ldots, \sigma_n\} \vdash_{pIL} \tau$, the projection $\Pi^\epsilon : \{e(\sigma_1), \ldots, e(\sigma_n)\} \vdash_{LJ} e(\tau)$ is unique, belongs to $D_{LJ}^{\rightarrow}$ and has the same implicative structure as Π, i.e. $T_{x_1,\ldots,x_n}(\Pi^\epsilon) = T_{x_1,\ldots,x_n}(\Pi)$, for every $x_1, \ldots, x_n$.

We can make some observations on projecting. An axiom in Π involving kits with m terminal paths generates m axioms in Π^ϵ. In general, a judgement $J \equiv \{H_1, \ldots, H_n\} \vdash_{pIL} K$ in Π involving kits with m terminal paths $p_1, \ldots, p_m$ generates m judgements $J_1, \ldots, J_m$ in Π^ϵ, where, for each $k \in \{1, \ldots, m\}$, it is $J_k \equiv \{e(H_1^{p_k}), \ldots, e(H_n^{p_k})\} \vdash_{LJ} e(K^{p_k})$[8]. The clockwise order of judgements in Π^ϵ is the anticlockwise order of appropriate leaves in Π (see example 3.6.2). The sequence of rules in Π^ϵ copies the one in Π modulo the conversion of intersection to conjunction, the splitting of judgements and the iteration of global rules in isomorphic with respect to $\mathsf{d}_\rightarrow$ subdeductions. Now suppose we draw a full binary tree $\mathcal{T}^\epsilon$ on Π^ϵ by putting an imaginary node on the conclusion of each binary rule and then drawing two branches from each node towards the two premises, so that each branch either meets another node and the procedure is repeated or runs through a subdeduction with no binary rules and ends up to a leaf. Suppose also that there are r axioms $A_1, \ldots, A_r$ in Π involving kits with $n_1, \ldots, n_r$ terminal paths, respectively. Then, $\mathcal{T}^\epsilon$ has $n_1 + \cdots + n_r$ terminal paths. Each $(\cap I)$ rule in Π generates one node in $\mathcal{T}^\epsilon$, while each $(\rightarrow E)$ rule in Π on judgements involving kits with l terminal paths generates l nodes in $\mathcal{T}^\epsilon$. (Recall that $(\cap I)$ is a local rule, while $(\rightarrow E)$ is global.) If we consider the nodes generated by $(\cap I)$ rules in columns of Π that have no column to their left and ignore the rest, we make a full binary tree $\mathcal{T}_l^\epsilon$ on Π^ϵ—in the same manner that we drew $\mathcal{T}^\epsilon$—with the structure of kits in the leftmost axiom of Π inverted.

The following example on projecting proper pIL-deductions illuminates some of the points discussed above.

Example 3.6.1 Let $\sigma \equiv \alpha\cap\beta\cap\gamma$ with α, β and γ propositional variables. Deduction $\Pi : \sigma \vdash_{pIL} \alpha \cap (\gamma \cap (\beta \cap \alpha))$ projects to $\Pi^\epsilon : e(\sigma) \vdash_{LJ} \alpha \wedge (\gamma \wedge (\beta \wedge \alpha))$. The P-normal deduction $\Pi' : \sigma \vdash_{pIL} \alpha \cap (\gamma \cap (\beta \cap \alpha))$ derived from Π projects to $(\Pi')^\epsilon \equiv \Pi^\epsilon$. Deductions Π and Π' are shown on page 45 and Π^ϵ on page 46.

Note: Consider the judgement $[\gamma, [[\gamma, [\gamma, \gamma]], \gamma]], [\sigma, [[\sigma, [\sigma, \sigma]], \sigma]] \vdash_{pIL} [\alpha, [[\gamma, [\beta, \alpha]], \beta]]$ of Π'. The anticlockwise order of leaves in $[\alpha, [[\gamma, [\beta, \alpha]], \beta]]$ is $\alpha, \gamma, \beta, \alpha, \beta$. As a result, in Π^ϵ we clockwise meet judgements J_1, J_2, J_3, J_4, J_5. Consider also the judgement $[\gamma, [[\gamma, \gamma], \gamma]], [\sigma, [[\sigma, \sigma], \sigma]] \vdash_{pIL} [\alpha, [[\gamma, \beta \cap \alpha], \beta]]$ of Π'. The anticlockwise order of leaves in $[\alpha, [[\gamma, \beta \cap \alpha], \beta]]$ is $\alpha, \gamma, \beta \cap \alpha, \beta$. In Π^ϵ we clockwise meet J_1, J_2, J_4^3, J_5. Similarly, since the anticlockwise order of leaves in $[\alpha \cap \gamma, [\gamma, \gamma]]$ in the judgement $[\sigma, [\sigma, \sigma]] \vdash_{pIL} [\alpha \cap \gamma, [\gamma, \gamma]]$ of Π' is $\alpha \cap \gamma, \gamma, \gamma$, we clockwise meet the judgements J_1', J_2', J_3' in Π^ϵ. Likewise, the anticlockwise order $\alpha, \gamma \cap (\beta \cap \alpha), \beta$ of leaves in the kit $[\alpha, [\gamma \cap (\beta \cap \alpha), \beta]]$ in $[\sigma, [\sigma, \sigma]] \vdash_{pIL} [\alpha, [\gamma \cap (\beta \cap \alpha), \beta]]$ of Π' results in the clockwise order J_6, J_7, J_8 of judgements in Π^ϵ. Every judgement of Π' is subject to such a comment.

Now let $\pi : \{\sigma_1, \ldots, \sigma_n\} \vdash_{IL} \tau$ be a P-normal, proper IL-deduction and Π, Π' be different pIL-deductions in π. Then, $\Pi^\epsilon \equiv (\Pi')^\epsilon \equiv \pi^\epsilon$ (1). This is because consecutive intersection rules applied on different paths project to conjunction rules in different subdeductions, so the concept of order disappears. We also have that, for any $\Pi \in \pi$,

[8]If Π is *not* P-normal, an axiom in Π involving kits with m terminal paths generates $m' \leqslant m$ axioms in Π^ϵ, where the inequality sign is needed to exclude paths that will be pruned. In general, a judgement $J \equiv \{H_1, \ldots, H_n\} \vdash_{pIL} K$ in Π involving kits with m terminal paths $p_1, \ldots, p_m$ generates $m' \leqslant m$ judgements $J_{i_1}, \ldots, J_{i_{m'}}$ in Π^ϵ, where $i_1, \ldots, i_{m'}$ is a subsequence of $1, \ldots, m$ and, for each $k \in \{1, \ldots, m'\}$, it is $J_{i_k} \equiv \{e(H_1^{p_{i_k}}), \ldots, e(H_n^{p_{i_k}})\} \vdash_{LJ} e(K^{p_{i_k}})$.

$$
\begin{prooftree}
\AxiomC{$[\gamma,[[[\gamma,[\alpha\cap\beta,\gamma\cap\alpha]],[\gamma,\gamma]],\gamma]],[\sigma,[[[\sigma,[\alpha,\beta]],[\sigma,\sigma]],\sigma]] \vdash_{pIL} [\sigma,[[[\sigma,[\alpha,\beta]],[\sigma,\sigma]],\sigma]]$}
\doubleLine
\RightLabel{\scriptsize $(\cap E)$}
\UnaryInfC{$[\gamma,[[[\gamma,[\alpha\cap\beta,\gamma\cap\alpha]],[\gamma,\gamma]],\gamma]],[\sigma,[[[\sigma,[\alpha,\beta]],[\sigma,\sigma]],\sigma]] \vdash_{pIL} [\alpha,[[[\gamma,[\alpha,\beta]],[\beta,\alpha]],\beta]]$}
\RightLabel{\scriptsize $(\cap I)$}
\UnaryInfC{$[\gamma,[[[\gamma,[\alpha\cap\beta,\gamma\cap\alpha]],\gamma],\gamma]],[\sigma,[[[\sigma,[\alpha,\beta]],\sigma],\sigma]] \vdash_{pIL} [\alpha,[[[\gamma,[\alpha,\beta]],\beta\cap\alpha],\beta]]$}
\RightLabel{\scriptsize (P)}
\UnaryInfC{$[\gamma,[[\gamma,\gamma],\gamma]],[\sigma,[[\sigma,\sigma],\sigma]] \vdash_{pIL} [\alpha,[[\gamma,\beta\cap\alpha],\beta]]$}
\RightLabel{\scriptsize $(\cap I)$}
\UnaryInfC{$[\gamma,[\gamma,\gamma]],[\sigma,[\sigma,\sigma]] \vdash_{pIL} [\alpha,[\gamma\cap(\beta\cap\alpha),\beta]]$}
\RightLabel{\scriptsize $(\rightarrow I)$}
\UnaryInfC{$[\sigma,[\sigma,\sigma]] \vdash_{pIL} [\gamma\rightarrow\alpha,[\gamma\rightarrow\gamma\cap(\beta\cap\alpha),\gamma\rightarrow\beta]]$}
\AxiomC{$[[\sigma,\sigma],[\sigma,[\beta,\sigma]]] \vdash_{pIL} [[\sigma,\sigma],[\sigma,[\beta,\sigma]]]$}
\doubleLine
\RightLabel{\scriptsize $(\cap E)$}
\UnaryInfC{$[[\sigma,\sigma],[\sigma,[\beta,\sigma]]] \vdash_{pIL} [[\alpha,\gamma],[\gamma,[\beta,\gamma]]]$}
\RightLabel{\scriptsize $(\cap I)$}
\UnaryInfC{$[\sigma,[\sigma,[\beta,\sigma]]] \vdash_{pIL} [\alpha\cap\gamma,[\gamma,[\beta,\gamma]]]$}
\RightLabel{\scriptsize $(\cap E)$}
\UnaryInfC{$[\sigma,[\sigma,[\beta,\sigma]]] \vdash_{pIL} [\gamma,[\gamma,[\beta,\gamma]]]$}
\RightLabel{\scriptsize (P)}
\UnaryInfC{$[\sigma,[\sigma,\sigma]] \vdash_{pIL} [\gamma,[\gamma,\gamma]]$}
\RightLabel{\scriptsize $(\rightarrow E)$}
\BinaryInfC{$[\sigma,[\sigma,\sigma]] \vdash_{pIL} [\alpha,[\gamma\cap(\beta\cap\alpha),\beta]]$}
\RightLabel{\scriptsize $(\cap I)$}
\UnaryInfC{$[\sigma,\sigma] \vdash_{pIL} [\alpha,\gamma\cap(\beta\cap\alpha)\cap\beta]$}
\RightLabel{\scriptsize $(\cap E)$}
\UnaryInfC{$[\sigma,\sigma] \vdash_{pIL} [\alpha,\gamma\cap(\beta\cap\alpha)]$}
\RightLabel{\scriptsize $(\cap I)$}
\UnaryInfC{$\Pi : \sigma \vdash_{pIL} \alpha\cap(\gamma\cap(\beta\cap\alpha))$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$[\gamma,[[\gamma,[\gamma,\gamma]],\gamma]],[\sigma,[[\sigma,[\sigma,\sigma]],\sigma]] \vdash_{pIL} [\sigma,[[\sigma,[\sigma,\sigma]],\sigma]]$}
\doubleLine
\RightLabel{\scriptsize $(\cap E)$}
\UnaryInfC{$[\gamma,[[\gamma,[\gamma,\gamma]],\gamma]],[\sigma,[[\sigma,[\sigma,\sigma]],\sigma]] \vdash_{pIL} [\alpha,[[\gamma,[\beta,\alpha]],\beta]]$}
\RightLabel{\scriptsize $(\cap I)$}
\UnaryInfC{$[\gamma,[[\gamma,\gamma],\gamma]],[\sigma,[[\sigma,\sigma],\sigma]] \vdash_{pIL} [\alpha,[[\gamma,\beta\cap\alpha],\beta]]$}
\RightLabel{\scriptsize $(\cap I)$}
\UnaryInfC{$[\gamma,[\gamma,\gamma]],[\sigma,[\sigma,\sigma]] \vdash_{pIL} [\alpha,[\gamma\cap(\beta\cap\alpha),\beta]]$}
\RightLabel{\scriptsize $(\rightarrow I)$}
\UnaryInfC{$[\sigma,[\sigma,\sigma]] \vdash_{pIL} [\gamma\rightarrow\alpha,[\gamma\rightarrow\gamma\cap(\beta\cap\alpha),\gamma\rightarrow\beta]]$}
\AxiomC{$[[\sigma,\sigma],[\sigma,\sigma]] \vdash_{pIL} [[\sigma,\sigma],[\sigma,\sigma]]$}
\doubleLine
\RightLabel{\scriptsize $(\cap E)$}
\UnaryInfC{$[[\sigma,\sigma],[\sigma,\sigma]] \vdash_{pIL} [[\alpha,\gamma],[\gamma,\gamma]]$}
\RightLabel{\scriptsize $(\cap I)$}
\UnaryInfC{$[\sigma,[\sigma,\sigma]] \vdash_{pIL} [\alpha\cap\gamma,[\gamma,\gamma]]$}
\RightLabel{\scriptsize $(\cap E)$}
\UnaryInfC{$[\sigma,[\sigma,\sigma]] \vdash_{pIL} [\gamma,[\gamma,\gamma]]$}
\RightLabel{\scriptsize $(\rightarrow E)$}
\BinaryInfC{$[\sigma,[\sigma,\sigma]] \vdash_{pIL} [\alpha,[\gamma\cap(\beta\cap\alpha),\beta]]$}
\RightLabel{\scriptsize $(\cap I)$}
\UnaryInfC{$[\sigma,\sigma] \vdash_{pIL} [\alpha,\gamma\cap(\beta\cap\alpha)\cap\beta]$}
\RightLabel{\scriptsize $(\cap E)$}
\UnaryInfC{$[\sigma,\sigma] \vdash_{pIL} [\alpha,\gamma\cap(\beta\cap\alpha)]$}
\RightLabel{\scriptsize $(\cap I)$}
\UnaryInfC{$\Pi' : \sigma \vdash_{pIL} \alpha\cap(\gamma\cap(\beta\cap\alpha))$}
\end{prooftree}
$$

$$
\dfrac{
  \dfrac{
    \dfrac{\gamma, e(\sigma) \vdash_{LJ} e(\sigma)}{\gamma, e(\sigma) \vdash_{LJ} \boxed{\gamma}\ (J_2)}\,{\scriptstyle(\wedge E)}
    \quad
    \dfrac{
      \dfrac{\gamma, e(\sigma) \vdash_{LJ} e(\sigma)}{\gamma, e(\sigma) \vdash_{LJ} \boxed{\beta}\ (J_3)}\,{\scriptstyle(\wedge E)}
      \quad
      \dfrac{\gamma, e(\sigma) \vdash_{LJ} e(\sigma)}{\gamma, e(\sigma) \vdash_{LJ} \boxed{\alpha}\ (J_4)}\,{\scriptstyle(\wedge E)}
    }{\gamma, e(\sigma) \vdash_{LJ} \boxed{\beta \wedge \alpha}\ (J_4^3)}\,{\scriptstyle(\wedge I)}
  }{
    \dfrac{\gamma, e(\sigma) \vdash_{LJ} \gamma \wedge (\beta \wedge \alpha)}{e(\sigma) \vdash_{LJ} \gamma \to \gamma \wedge (\beta \wedge \alpha)}\,{\scriptstyle(\to I)}
  }\,{\scriptstyle(\wedge I)}
  \quad
  \dfrac{e(\sigma) \vdash_{LJ} e(\sigma)}{e(\sigma) \vdash_{LJ} \boxed{\gamma}\ (J_2')}\,{\scriptstyle(\wedge E)}
}{\Pi^{\epsilon}_{10} :\ e(\sigma) \vdash_{LJ} \boxed{\gamma \wedge (\beta \wedge \alpha)}\ (J_7)}\,{\scriptstyle(\to E)}
\qquad
\dfrac{
  \dfrac{
    \dfrac{\gamma, e(\sigma) \vdash_{LJ} e(\sigma)}{\gamma, e(\sigma) \vdash_{LJ} \boxed{\beta}\ (J_5)}\,{\scriptstyle(\wedge E)}
  }{e(\sigma) \vdash_{LJ} \gamma \to \beta}\,{\scriptstyle(\to I)}
  \quad
  \dfrac{e(\sigma) \vdash_{LJ} e(\sigma)}{e(\sigma) \vdash_{LJ} \boxed{\gamma}\ (J_3')}\,{\scriptstyle(\wedge E)}
}{\Pi^{\epsilon}_{11} :\ e(\sigma) \vdash_{LJ} \boxed{\beta}\ (J_8)}\,{\scriptstyle(\to E)}
$$

$$
\dfrac{
  \dfrac{\Pi^{\epsilon}_{10} :\ e(\sigma) \vdash_{LJ} \boxed{\gamma \wedge (\beta \wedge \alpha)}\ (J_7) \qquad \Pi^{\epsilon}_{11} :\ e(\sigma) \vdash_{LJ} \boxed{\beta}\ (J_8)}{e(\sigma) \vdash_{LJ} \gamma \wedge (\beta \wedge \alpha) \wedge \beta}\,{\scriptstyle(\wedge I)}
}{\Pi^{\epsilon}_{1} : e(\sigma) \vdash_{LJ} \gamma \wedge (\beta \wedge \alpha)}\,{\scriptstyle(\wedge E)}
$$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\gamma, e(\sigma) \vdash_{LJ} e(\sigma)}{\gamma, e(\sigma) \vdash_{LJ} \boxed{\alpha}\ (J_1)}\,{\scriptstyle(\wedge E)}
    }{e(\sigma) \vdash_{LJ} \gamma \to \alpha}\,{\scriptstyle(\to I)}
    \quad
    \dfrac{
      \dfrac{
        \dfrac{e(\sigma) \vdash_{LJ} e(\sigma)}{e(\sigma) \vdash_{LJ} \alpha}\,{\scriptstyle(\wedge E)}
        \quad
        \dfrac{e(\sigma) \vdash_{LJ} e(\sigma)}{e(\sigma) \vdash_{LJ} \gamma}\,{\scriptstyle(\wedge E)}
      }{e(\sigma) \vdash_{LJ} \boxed{\alpha \wedge \gamma}\ (J_1')}\,{\scriptstyle(\wedge I)}
    }{e(\sigma) \vdash_{LJ} \gamma}\,{\scriptstyle(\wedge E)}
  }{\Pi^{\epsilon}_{0} :\ e(\sigma) \vdash_{LJ} \boxed{\alpha}\ (J_6)}\,{\scriptstyle(\to E)}
  \qquad
  \Pi^{\epsilon}_{1} : e(\sigma) \vdash_{LJ} \gamma \wedge (\beta \wedge \alpha)
}{\Pi^{\epsilon} : e(\sigma) \vdash_{LJ} \alpha \wedge (\gamma \wedge (\beta \wedge \alpha))}\,{\scriptstyle(\wedge I)}
$$

Π^{ϵ} is unique, belongs to $D_{LJ}^{\rightarrow}$ and $T_{x_1,\ldots,x_n}(\Pi^{\epsilon}) \equiv T_{x_1,\ldots,x_n}(\Pi) \equiv T_{x_1,\ldots,x_n}(\pi)$, for every $x_1,\ldots,x_n$ (2). From (1) and (2), we conclude that, for every P-normal, proper IL-deduction $\pi : \{\sigma_1,\ldots,\sigma_n\} \vdash_{IL} \tau$, the projection π^{ϵ} is unique, belongs to $D_{LJ}^{\rightarrow}$ and has the same implicative structure as π, i.e. $T_{x_1,\ldots,x_n}(\pi^{\epsilon}) \equiv T_{x_1,\ldots,x_n}(\pi)$, for every $x_1,\ldots,x_n$. Further on, suppose that π and π' are P-normal, proper IL-deductions, such that $\pi \not\equiv \pi'$ and $\Pi \in \pi$, $\Pi' \in \pi'$. Then, $\pi^{\epsilon} \equiv \Pi^{\epsilon} \not\equiv (\Pi')^{\epsilon} \equiv (\pi')^{\epsilon}$.

Let $D_{IL}^{Pn,p}$ denote the set of P-normal, proper IL-deductions. Resuming the analysis so far, we can say that $\epsilon : D_{IL}^{Pn,p} \rightarrow D_{LJ}^{\rightarrow}$ with $\epsilon(\pi) = \pi^{\epsilon}$ is a one-to-one function, such that every member of $D_{IL}^{Pn,p}$ has an image in $D_{LJ}^{\rightarrow}$ with the same implicative structure.

We illustrate these points by examples.

Example 3.6.2 Let $\sigma \equiv \alpha \cap \beta \cap \gamma$ with α, β and γ propositional variables. The P-normal, proper pIL-deductions Π and Π' are different, but both in the class $\pi : \sigma \vdash_{IL} \alpha \cap \gamma \cap \beta$. It is $\Pi^{\epsilon} \equiv (\Pi')^{\epsilon} \equiv \pi^{\epsilon} : e(\sigma) \vdash_{LJ} \alpha \wedge \gamma \wedge \beta$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[[\sigma,\sigma],\sigma] \vdash_{pIL} [[\sigma,\sigma],\sigma]}{[[\sigma,\sigma],\sigma] \vdash_{pIL} [[\alpha\cap\beta,\sigma],\sigma]}\,(\cap E^l)\text{ on } ll}{[[\sigma,\sigma],\sigma] \vdash_{pIL} [[\alpha,\sigma],\sigma]}\,(\cap E^l)\text{ on } ll}{[[\sigma,\sigma],\sigma] \vdash_{pIL} [[\alpha,\gamma],\sigma]}\,(\cap E^r)\text{ on } lr}{[\sigma,\sigma] \vdash_{pIL} [\alpha\cap\gamma,\sigma]}\,(\cap I)\text{ on } l}{[\sigma,\sigma] \vdash_{pIL} [\alpha\cap\gamma,\alpha\cap\beta]}\,(\cap E^l)\text{ on } r}{[\sigma,\sigma] \vdash_{pIL} [\alpha\cap\gamma,\beta]}\,(\cap E^r)\text{ on } r}{\Pi : \sigma \vdash_{pIL} \alpha\cap\gamma\cap\beta}\,(\cap I)
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[[\sigma,\sigma],\sigma] \vdash_{pIL} [[\sigma,\sigma],\sigma]}{[[\sigma,\sigma],\sigma] \vdash_{pIL} [[\alpha\cap\beta,\sigma],\sigma]}\,(\cap E^l)\text{ on } ll}{[[\sigma,\sigma],\sigma] \vdash_{pIL} [[\alpha\cap\beta,\gamma],\sigma]}\,(\cap E^r)\text{ on } lr}{[[\sigma,\sigma],\sigma] \vdash_{pIL} [[\alpha,\gamma],\sigma]}\,(\cap E^l)\text{ on } ll}{[[\sigma,\sigma],\sigma] \vdash_{pIL} [[\alpha,\gamma],\alpha\cap\beta]}\,(\cap E^l)\text{ on } r}{[\sigma,\sigma] \vdash_{pIL} [\alpha\cap\gamma,\alpha\cap\beta]}\,(\cap I)\text{ on } l}{[\sigma,\sigma] \vdash_{pIL} [\alpha\cap\gamma,\beta]}\,(\cap E^r)\text{ on } r}{\Pi' : \sigma \vdash_{pIL} \alpha\cap\gamma\cap\beta}\,(\cap I)
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{e(\sigma) \vdash_{LJ} e(\sigma)}{e(\sigma) \vdash_{LJ} \alpha\wedge\beta}\,(\wedge E^l)}{e(\sigma) \vdash_{LJ} \alpha}\,(\wedge E^l) \quad \dfrac{e(\sigma) \vdash_{LJ} e(\sigma)}{e(\sigma) \vdash_{LJ} \gamma}\,(\wedge E^r)}{e(\sigma) \vdash_{LJ} \alpha\wedge\gamma}\,(\wedge I) \quad \dfrac{\dfrac{e(\sigma) \vdash_{LJ} e(\sigma)}{e(\sigma) \vdash_{LJ} \alpha\wedge\beta}\,(\wedge E^l)}{e(\sigma) \vdash_{LJ} \beta}\,(\wedge E^r)}{\pi^{\epsilon} : e(\sigma) \vdash_{LJ} \alpha\wedge\gamma\wedge\beta}\,(\wedge I)
$$

Example 3.6.3 Let $\sigma \equiv \alpha \cap \beta \cap \gamma$ with α, β and γ propositional variables. The P-normal, proper pIL-deduction Π'' differs from Π of example 3.6.2 and is in $\pi'' : \sigma \vdash_{IL} \alpha \cap (\gamma \cap \beta)$, which differs from Π's class π. It is $(\pi'')^{\epsilon} \equiv (\Pi'')^{\epsilon} \not\equiv \Pi^{\epsilon} \equiv \pi^{\epsilon}$.

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[\sigma,[\sigma,\sigma]]\vdash_{pIL}[\sigma,[\sigma,\sigma]]}{[\sigma,[\sigma,\sigma]]\vdash_{pIL}[\sigma,[\gamma,\sigma]]}\;(\cap E^r)\text{ on } rl}{[\sigma,[\sigma,\sigma]]\vdash_{pIL}[\alpha\cap\beta,[\gamma,\sigma]]}\;(\cap E^l)\text{ on } l}{[\sigma,[\sigma,\sigma]]\vdash_{pIL}[\alpha\cap\beta,[\gamma,\alpha\cap\beta]]}\;(\cap E^l)\text{ on } rr}{[\sigma,[\sigma,\sigma]]\vdash_{pIL}[\alpha\cap\beta,[\gamma,\beta]]}\;(\cap E^r)\text{ on } rr}{[\sigma,[\sigma,\sigma]]\vdash_{pIL}[\alpha,[\gamma,\beta]]}\;(\cap E^l)\text{ on } l}{[\sigma,\sigma]\vdash_{pIL}[\alpha,\gamma\cap\beta]}\;(\cap I)\text{ on } r}{\Pi'':\sigma\vdash_{pIL}\alpha\cap(\gamma\cap\beta)}\;(\cap I)
$$

$$
\cfrac{\cfrac{\cfrac{e(\sigma)\vdash_{LJ}e(\sigma)}{e(\sigma)\vdash_{LJ}\alpha\wedge\beta}\;(\wedge E^l)}{e(\sigma)\vdash_{LJ}\alpha}\;(\wedge E^l)\qquad \cfrac{\cfrac{e(\sigma)\vdash_{LJ}e(\sigma)}{e(\sigma)\vdash_{LJ}\gamma}\;(\wedge E^r)\qquad \cfrac{\cfrac{e(\sigma)\vdash_{LJ}e(\sigma)}{e(\sigma)\vdash_{LJ}\alpha\wedge\beta}\;(\wedge E^l)}{e(\sigma)\vdash_{LJ}\beta}\;(\wedge E^r)}{e(\sigma)\vdash_{LJ}\gamma\wedge\beta}\;(\wedge I)}{(\pi'')^e:e(\sigma)\vdash_{LJ}\alpha\wedge(\gamma\wedge\beta)}\;(\wedge I)
$$

The following theorem expresses the inverse of theorem 3.4.1. A finite number of LJ-deductions decoratable non-standardly by the same λ-term merge into a single pIL-deduction decoratable non-standardly by this very λ-term.

Theorem 3.6.4 (Inverse projection theorem for pIL) *Let* $n \geqslant 0,\ m \geqslant 1,$

$\Pi_1:\{\sigma_{11},\dots,\sigma_{n1}\}\vdash_{LJ}\tau_1, \Pi_2:\{\sigma_{12},\dots,\sigma_{n2}\}\vdash_{LJ}\tau_2,\dots,\Pi_m:\{\sigma_{1m},\dots,\sigma_{nm}\}\vdash_{LJ}\tau_m$

and $H_1,\dots,H_n,K$ *be* $n+1$ *overlapping kits with* m *terminal paths* $p_1,\dots,p_m$*, such that* $H_i^{p_j}\equiv f(\sigma_{ij}),\ K^{p_j}\equiv f(\tau_j)\ (1\leqslant i\leqslant n,\ 1\leqslant j\leqslant m)$. *If* $\Pi_1,\dots,\Pi_m$ *are all in* $D_{LJ}^{\to}$ *and* $T_{x_1,\dots,x_n}(\Pi_1)=T_{x_1,\dots,x_n}(\Pi_2)=\dots=T_{x_1,\dots,x_n}(\Pi_m)$*, for every* $x_1,\dots,x_n$*, there exists* $\Pi:\{H_1,\dots,H_n\}\vdash_{pIL}K$*, such that* $T_{x_1,\dots,x_n}(\Pi)=T_{x_1,\dots,x_n}(\Pi_1)$*, for every* $x_1,\dots,x_n$.

Proof: By induction on Π_1.

Base: Suppose, without loss of generality, that $\Pi_1:\{\sigma_{11},\dots,\sigma_{n1}\}\vdash_{LJ}\tau_1\equiv\sigma_{11}$. Then, since $T_{x_1,\dots,x_n}(\Pi_j)=T_{x_1,\dots,x_n}(\Pi_1)$, for every $x_1,\dots,x_n$ and $2\leqslant j\leqslant m$, the judgement $\{\sigma_{1j},\dots,\sigma_{nj}\}\vdash_{LJ}\tau_j$ proved by Π_j derives from axioms of the form $\{\sigma_{1j},\dots,\sigma_{nj}\}\vdash_{LJ}\sigma_{1j}$ by a finite number of applications of the rules $(\wedge I),(\wedge E)$. If the number of $(\wedge I)$ instances in Π_j is k_j-1, where $k_j\geqslant 1$, then Π_j involves k_j axioms

$$\Pi_{j1}:\{\sigma_{1j},\dots,\sigma_{nj}\}\vdash_{LJ}\sigma_{1j}\ ,\dots,\ \Pi_{jk_j}:\{\sigma_{1j},\dots,\sigma_{nj}\}\vdash_{LJ}\sigma_{1j}$$

Let $H_1,\dots,H_n,K$ be $n+1$ overlapping kits with m terminal paths $p_1,\dots,p_m$, such that $H_i^{p_j}\equiv f(\sigma_{ij}),\ K^{p_j}\equiv f(\tau_j)\ (1\leqslant i\leqslant n,\ 1\leqslant j\leqslant m)$. Also, for each $j\in\{2,\dots,m\}$, let $L_{1j},L_{2j},\dots,L_{nj},L_j$ be overlapping kits with k_j terminal paths $q_{j1},\dots,q_{jk_j}$, such that $(L_{ij})^{q_{jl_j}}\equiv f(\sigma_{ij}),\ (L_j)^{q_{jl_j}}\equiv f(\sigma_{1j})\ (1\leqslant i\leqslant n,\ 1\leqslant l_j\leqslant k_j)$. Then

$$H_1[p_j:=L_{1j}\mid 2\leqslant j\leqslant m]\,,\dots,\ H_n[p_j:=L_{nj}\mid 2\leqslant j\leqslant m]\,,\ K[p_j:=L_j\mid 2\leqslant j\leqslant m]$$

are $n+1$ overlapping kits with $1+k_2+\dots+k_m$ terminal paths p_1 and

$$\begin{array}{ccc} p_2q_{21} & ,\dots, & p_2q_{2k_2}\\ \vdots & & \vdots\\ p_mq_{m1} & ,\dots, & p_mq_{mk_m}\end{array}$$

such that, for $1\leqslant i\leqslant n,\ 2\leqslant j\leqslant m,\ 1\leqslant l_j\leqslant k_j$:

1. $(H_i[p_j := L_{ij}])^{p_1} \equiv f(\sigma_{i1}),\ (K[p_j := L_j])^{p_1} \equiv f(\sigma_{11})$
2. $(H_i[p_j := L_{ij}])^{p_j q_j l_j} \equiv f(\sigma_{ij}),\ (K[p_j := L_j])^{p_j q_j l_j} \equiv f(\sigma_{1j})$

So, we have $(H_1[p_j := L_{1j}])^{p_1} \equiv f(\sigma_{11}) \equiv (K[p_j := L_j])^{p_1}$ and $(H_1[p_j := L_{1j}])^{p_j q_j l_j} \equiv f(\sigma_{1j}) \equiv (K[p_j := L_j])^{p_j q_j l_j}$, which imply that $H_1[p_j := L_{1j}] \equiv K[p_j := L_j]$. Thus

$$\Pi' : \{H_1[p_j := L_{1j}], \dots, H_n[p_j := L_{nj}]\} \vdash_{pIL} K[p_j := L_j]$$

is an axiom with $T_{x_1,\dots,x_n}(\Pi') = T_{x_1,\dots,x_n}(\Pi_1)$, for every $x_1, \dots, x_n$. Applying $(\cap I_{pIL})$ on a non-terminal path $p_j\, q$ for each $(\wedge I_{LJ})$ in Π_j and $(\cap E^s_{pIL})$ on a terminal path $p_j\, q$ for each $(\wedge E^s_{LJ})$ in Π_j, where $s \in \{l, r\}$, we get $\Pi : \{H_1, \dots, H_n\} \vdash_{pIL} K$. It is $T_{x_1,\dots,x_n}(\Pi) = T_{x_1,\dots,x_n}(\Pi') = T_{x_1,\dots,x_n}(\Pi_1)$, for every $x_1, \dots, x_n$.

Inductive step: We show the most characteristic cases.

- $$\frac{\Pi_{01} : \{\sigma_{11}, \dots, \sigma_{n1}\} \vdash_{LJ} \tau_{01} \qquad \Pi_{11} : \{\sigma_{11}, \dots, \sigma_{n1}\} \vdash_{LJ} \tau_{11}}{\Pi_1 : \{\sigma_{11}, \dots, \sigma_{n1}\} \vdash_{LJ} \tau_{01} \wedge \tau_{11} \equiv \tau_1}\ (\wedge I_{LJ})$$

Let $H_1, \dots, H_n, K$ be $n{+}1$ overlapping kits with m terminal paths $p_1, \dots, p_m$, such that $H_i^{p_j} \equiv f(\sigma_{ij}),\ K^{p_j} \equiv f(\tau_j)$ $(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m)$. Then

$$H_1[p_1 := [f(\sigma_{11}), f(\sigma_{11})]]\, , \dots,\, H_n[p_1 := [f(\sigma_{n1}), f(\sigma_{n1})]]\, , K[p_1 := [f(\tau_{01}), f(\tau_{11})]]$$

are $n{+}1$ overlapping kits with $m{+}1$ terminal paths $p_1 l, p_1 r, p_2, \dots, p_m$, such that:

1. $(H_i[p_1 := [f(\sigma_{i1}), f(\sigma_{i1})]])^{p_1 s} \equiv f(\sigma_{i1})$ $(1 \leqslant i \leqslant n,\ s \in \{l, r\})$
2. $(H_i[p_1 := [f(\sigma_{i1}), f(\sigma_{i1})]])^{p_j} \equiv H_i^{p_j} \equiv f(\sigma_{ij})$ $(1 \leqslant i \leqslant n,\ 2 \leqslant j \leqslant m)$
3. $(K[p_1 := [f(\tau_{01}), f(\tau_{11})]])^{p_1 l} \equiv f(\tau_{01}),\ (K[p_1 := [f(\tau_{01}), f(\tau_{11})]])^{p_1 r} \equiv f(\tau_{11})$
4. $(K[p_1 := [f(\tau_{01}), f(\tau_{11})]])^{p_j} \equiv K^{p_j} \equiv f(\tau_j)$ $(2 \leqslant j \leqslant m)$

Since Π_1 is in $D^{\rightarrow}_{LJ}$, Π_{01} and Π_{11} are in $D^{\rightarrow}_{LJ}$ and $T_{x_1,\dots,x_n}(\Pi_{01}) = T_{x_1,\dots,x_n}(\Pi_{11}) = T_{x_1,\dots,x_n}(\Pi_1)$, for every $x_1, \dots, x_n$. So, $\Pi_{01}, \Pi_{11}, \Pi_2, \dots, \Pi_m$ are all in $D^{\rightarrow}_{LJ}$ and $T_{x_1,\dots,x_n}(\Pi_{01}) = T_{x_1,\dots,x_n}(\Pi_{11}) = T_{x_1,\dots,x_n}(\Pi_2) = \dots = T_{x_1,\dots,x_n}(\Pi_m)$, for every $x_1, \dots, x_n$; hence, by the IH, there exists

$$\Pi' : \{H_1[p_1 := [f(\sigma_{11}), f(\sigma_{11})]], \dots, H_n[p_1 := [f(\sigma_{n1}), f(\sigma_{n1})]]\} \vdash_{pIL} K[p_1 := [f(\tau_{01}), f(\tau_{11})]]$$

such that $T_{x_1,\dots,x_n}(\Pi') = T_{x_1,\dots,x_n}(\Pi_{01})$, for every $x_1, \dots, x_n$. Applying $(\cap I_{pIL})$, we get $\Pi : \{H_1[p_1 := f(\sigma_{11})] \equiv H_1, \dots, H_n[p_1 := f(\sigma_{n1})] \equiv H_n\} \vdash_{pIL} K[p_1 := f(\tau_{01}) \cap f(\tau_{11})]$. But $K[p_1 := f(\tau_{01}) \cap f(\tau_{11})] \equiv K[p_1 := f(\tau_{01} \wedge \tau_{11})] \equiv K[p_1 := f(\tau_1)] \equiv K$, so we get $\Pi : \{H_1, \dots, H_n\} \vdash_{pIL} K$. It is $T_{x_1,\dots,x_n}(\Pi) = T_{x_1,\dots,x_n}(\Pi') = T_{x_1,\dots,x_n}(\Pi_{01}) = T_{x_1,\dots,x_n}(\Pi_1)$, for every $x_1, \dots, x_n$.

- $$\frac{\Pi'_1 : \{\sigma_{11}, \dots, \sigma_{n1}\} \vdash_{LJ} \tau_1 \wedge \tau_2}{\Pi_1 : \{\sigma_{11}, \dots, \sigma_{n1}\} \vdash_{LJ} \tau_1}\ (\wedge E^l_{LJ})$$

Let $H_1, \dots, H_n, K$ be $n{+}1$ overlapping kits with m terminal paths $p_1, \dots, p_m$, such that $H_i^{p_j} \equiv f(\sigma_{ij}),\ K^{p_j} \equiv f(\tau_j)$ $(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m)$. Then

$$H_1\, , \dots,\, H_n\, , K[p_1 := f(\tau_1 \wedge \tau_2)]$$

are $n{+}1$ overlapping kits with m terminal paths $p_1, \dots, p_m$, such that $H_i^{p_j} \equiv f(\sigma_{ij})$ $(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m)$, $(K[p_1 := f(\tau_1 \wedge \tau_2)])^{p_1} \equiv f(\tau_1 \wedge \tau_2)$ and $(K[p_1 := f(\tau_1 \wedge \tau_2)])^{p_j} \equiv$

$K^{p_j} \equiv f(\tau_j)$ $(2 \leqslant j \leqslant m)$. Since Π_1 is in $D_{LJ}^{\rightarrow}$, Π_1' is in $D_{LJ}^{\rightarrow}$ and $T_{x_1,\ldots,x_n}(\Pi_1') = T_{x_1,\ldots,x_n}(\Pi_1)$, for every $x_1,\ldots,x_n$. So, $\Pi_1', \Pi_2, \ldots, \Pi_m$ are all in $D_{LJ}^{\rightarrow}$ and $T_{x_1,\ldots,x_n}(\Pi_1') = T_{x_1,\ldots,x_n}(\Pi_2) = \ldots = T_{x_1,\ldots,x_n}(\Pi_m)$, for every $x_1,\ldots,x_n$; hence, by the IH, there exists

$$\Pi' : \{H_1,\ldots,H_n\} \vdash_{pIL} K[p_1 := f(\tau_1 \wedge \tau_2)] \equiv K[p_1 := f(\tau_1) \cap f(\tau_2)]$$

such that $T_{x_1,\ldots,x_n}(\Pi') = T_{x_1,\ldots,x_n}(\Pi_1')$, for every $x_1,\ldots,x_n$. Applying $(\cap E^l_{pIL})$, we get $\Pi : \{H_1,\ldots,H_n\} \vdash_{pIL} K[p_1 := f(\tau_1)] \equiv K$ with $T_{x_1,\ldots,x_n}(\Pi) = T_{x_1,\ldots,x_n}(\Pi') = T_{x_1,\ldots,x_n}(\Pi_1') = T_{x_1,\ldots,x_n}(\Pi_1)$, for every $x_1,\ldots,x_n$.

The case of $(\wedge E^r_{LJ})$ is analogous to $(\wedge E^l_{LJ})$.

- $$\frac{\Pi_1' : \{\sigma_{11},\ldots,\sigma_{n1},\tau_1'\} \vdash_{LJ} \tau_1''}{\Pi_1 : \{\sigma_{11},\ldots,\sigma_{n1}\} \vdash_{LJ} \tau_1' \rightarrow \tau_1'' \equiv \tau_1}\ (\rightarrow I_{LJ})$$

We have that

$$T_{x_1,\ldots,x_n}(\Pi_1) = \bigcup_x (\lambda x.T_{x_1,\ldots,x_n,x}(\Pi_1')) = T_{x_1,\ldots,x_n}(\Pi_2) = \ldots = T_{x_1,\ldots,x_n}(\Pi_m)$$

for every $x_1,\ldots,x_n$. Consequently, the judgement $\{\sigma_{1j},\ldots,\sigma_{nj}\} \vdash_{LJ} \tau_j$ proved by Π_j $(2 \leqslant j \leqslant m)$ derives, by $k_j - 1$ $(k_j \geqslant 1)$ applications of $(\wedge I)$ and a finite number of applications of $(\wedge E)$, from k_j deductions

$$\Pi_{j1} : \{\sigma_{1j},\ldots,\sigma_{nj}\} \vdash_{LJ} \tau_{j1}\,, \ldots, \,\Pi_{jk_j} : \{\sigma_{1j},\ldots,\sigma_{nj}\} \vdash_{LJ} \tau_{jk_j}$$

which, in turn, derive, by $(\rightarrow I)$, from

$$\Pi_{j1}' : \{\sigma_{1j},\ldots,\sigma_{nj},\tau_{j1}'\} \vdash_{LJ} \tau_{j1}''\,, \ldots, \,\Pi_{jk_j}' : \{\sigma_{1j},\ldots,\sigma_{nj},\tau_{jk_j}'\} \vdash_{LJ} \tau_{jk_j}''$$

respectively. For each j, it is $T_{x_1,\ldots,x_n}(\Pi_{j1}) = \ldots = T_{x_1,\ldots,x_n}(\Pi_{jk_j}) = T_{x_1,\ldots,x_n}(\Pi_j) = T_{x_1,\ldots,x_n}(\Pi_1)$, for every $x_1,\ldots,x_n$. So, the sets $T_{x_1,\ldots,x_n}(\Pi_1)$ and

$$\begin{array}{ccc} T_{x_1,\ldots,x_n}(\Pi_{21}) & ,\ldots, & T_{x_1,\ldots,x_n}(\Pi_{2k_2}) \\ \vdots & & \vdots \\ T_{x_1,\ldots,x_n}(\Pi_{m1}) & ,\ldots, & T_{x_1,\ldots,x_n}(\Pi_{mk_m}) \end{array}$$

are equal, for every $x_1,\ldots,x_n$. For $2 \leqslant j \leqslant m$ and $1 \leqslant l_j \leqslant k_j$, the following implication holds:

$$\begin{array}{rcl} T_{x_1,\ldots,x_n}(\Pi_1) & = & T_{x_1,\ldots,x_n}(\Pi_{jl_j}),\ \text{for every } x_1,\ldots,x_n \implies^{9} \\ T_{x_1,\ldots,x_n,x}(\Pi_1') & = & T_{x_1,\ldots,x_n,x}(\Pi_{jl_j}'),\ \text{for every } x_1,\ldots,x_n,x \end{array}$$

So, the sets $T_{x_1,\ldots,x_n,x}(\Pi_1')$ and

$$\begin{array}{ccc} T_{x_1,\ldots,x_n,x}(\Pi_{21}') & ,\ldots, & T_{x_1,\ldots,x_n,x}(\Pi_{2k_2}') \\ \vdots & & \vdots \\ T_{x_1,\ldots,x_n,x}(\Pi_{m1}') & ,\ldots, & T_{x_1,\ldots,x_n,x}(\Pi_{mk_m}') \end{array}$$

[9] Let $j = 2$ and $l_2 = 1$. Suppose $N \in T_{x_1,\ldots,x_n,x}(\Pi_1')$. Then $\lambda x.N \in \lambda x.T_{x_1,\ldots,x_n,x}(\Pi_1') \subseteq \bigcup_x(\lambda x.T_{x_1,\ldots,x_n,x}(\Pi_1')) = T_{x_1,\ldots,x_n}(\Pi_1) = T_{x_1,\ldots,x_n}(\Pi_{21}) = \bigcup_x(\lambda x.T_{x_1,\ldots,x_n,x}(\Pi_{21}'))$. So, $\lambda x.N \in \bigcup_x(\lambda x.T_{x_1,\ldots,x_n,x}(\Pi_{21}'))$, which implies that $\lambda x.N \in \lambda y.T_{x_1,\ldots,x_n,y}(\Pi_{21}')$, for some y distinct from x_i $(1 \leqslant i \leqslant n)$. Hence, $\lambda x.N \equiv \lambda y.N'$, for some $N' \in T_{x_1,\ldots,x_n,y}(\Pi_{21}')$ $(*)$, which implies that $x \equiv y$ and $N \equiv N'$, so that $(*)$ becomes $N \in T_{x_1,\ldots,x_n,x}(\Pi_{21}')$. This shows that $T_{x_1,\ldots,x_n,x}(\Pi_1') \subseteq T_{x_1,\ldots,x_n,x}(\Pi_{21}')$. The other inclusion is shown similarly.

are equal, for every $x_1, \ldots, x_n, x$.

Let $H_1, \ldots, H_n, K$ be $n+1$ overlapping kits with m terminal paths $p_1, \ldots, p_m$, such that $H_i^{p_j} \equiv f(\sigma_{ij})$, $K^{p_j} \equiv f(\tau_j)$ $(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m)$. Also, for each $j \in \{2, \ldots, m\}$, let $L_{1j}, L_{2j}, \ldots, L_{nj}, L_j, L'_j, L''_j$ be overlapping kits with k_j terminal paths $q_{j1}, \ldots, q_{jk_j}$, such that $(L_{ij})^{q_{jl_j}} \equiv f(\sigma_{ij})$, $(L_j)^{q_{jl_j}} \equiv f(\tau_{jl_j})$, $(L'_j)^{q_{jl_j}} \equiv f(\tau'_{jl_j})$ and $(L''_j)^{q_{jl_j}} \equiv f(\tau''_{jl_j})$ $(1 \leqslant i \leqslant n,\ 1 \leqslant l_j \leqslant k_j)$. Then

$$H_1[p_j := L_{1j}]\,, \ldots,\ H_n[p_j := L_{nj}]\,,\ K[p_1 := f(\tau'_1), p_j := L'_j]\,,\ K[p_1 := f(\tau''_1), p_j := L''_j]$$

are $n+2$ overlapping kits with $1 + k_2 + \cdots + k_m$ terminal paths p_1 and

$$\begin{array}{ccc} p_2 q_{21} & ,\ldots, & p_2 q_{2k_2} \\ \vdots & & \vdots \\ p_m q_{m1} & ,\ldots, & p_m q_{mk_m} \end{array}$$

such that, for $1 \leqslant i \leqslant n,\ 2 \leqslant j \leqslant m,\ 1 \leqslant l_j \leqslant k_j$:

1. $(H_i[p_j := L_{ij}])^{p_1} \equiv f(\sigma_{i1})$
2. $(K[p_1 := f(\tau'_1), p_j := L'_j])^{p_1} \equiv f(\tau'_1)$, $(K[p_1 := f(\tau''_1), p_j := L''_j])^{p_1} \equiv f(\tau''_1)$
3. $(H_i[p_j := L_{ij}])^{p_j q_{jl_j}} \equiv f(\sigma_{ij})$
4. $(K[p_1{:=}f(\tau'_1), p_j{:=}L'_j])^{p_j q_{jl_j}} \equiv f(\tau'_{jl_j})$, $(K[p_1{:=}f(\tau''_1), p_j{:=}L''_j])^{p_j q_{jl_j}} \equiv f(\tau''_{jl_j})$

The analysis so far implies that, by the IH, there exists

$$\Pi' : \{H_1[p_j := L_{1j}], \ldots, H_n[p_j := L_{nj}], K[p_1 := f(\tau'_1), p_j := L'_j]\} \vdash_{pIL} K[p_1 := f(\tau''_1), p_j := L''_j]$$

such that $T_{x_1,\ldots,x_n,x}(\Pi') = T_{x_1,\ldots,x_n,x}(\Pi'_1)$, for every $x_1, \ldots, x_n, x$. Applying $(\rightarrow I_{pIL})$, we get $\Pi_0 : \{H_1[p_j := L_{1j}], \ldots, H_n[p_j := L_{nj}]\} \vdash_{pIL} K[p_j := L_j]$ with $T_{x_1,\ldots,x_n}(\Pi_0) = T_{x_1,\ldots,x_n}(\Pi_1)$, for every $x_1, \ldots, x_n$ (the details are as in the $(\rightarrow I)$ case in the proof of theorem 3.4.1). Then, applying $(\cap I_{pIL})$ on a non-terminal path $p_j\, q$ for each $(\wedge I_{LJ})$ in Π_j and $(\cap E^s_{pIL})$ on a terminal path $p_j\, q$ for each $(\wedge E^s_{LJ})$ in Π_j , where $s \in \{l, r\}$, we get $\Pi : \{H_1, \ldots, H_n\} \vdash_{pIL} K$. It is $T_{x_1,\ldots,x_n}(\Pi) = T_{x_1,\ldots,x_n}(\Pi_0) = T_{x_1,\ldots,x_n}(\Pi_1)$, for every $x_1, \ldots, x_n$.

In a similar manner, we tackle the case of $(\rightarrow E_{LJ})$. The details are left to the reader. $\dashv$

Remark 3.6.5 The $n+1$ overlapping kits $H_1, \ldots, H_n, K$ of theorem 3.6.4 can be of *any kit-structure* corresponding to m terminal paths (with leaves as required). For example, there are 14 different kit-structures corresponding to 5 terminal paths.

$$\begin{array}{ccc} [[[[\sigma,\sigma],\sigma],\sigma],\sigma] & [[[\sigma,[\sigma,\sigma]],\sigma],\sigma] & [[[\sigma,\sigma],[\sigma,\sigma]],\sigma] \\ [[[\sigma,\sigma],\sigma],[\sigma,\sigma]] & [[\sigma,[[\sigma,\sigma],\sigma]],\sigma] & [[\sigma,[\sigma,[\sigma,\sigma]]],\sigma] \\ [[\sigma,[\sigma,\sigma]],[\sigma,\sigma]] & [[\sigma,\sigma],[[\sigma,\sigma],\sigma]] & [[\sigma,\sigma],[\sigma,[\sigma,\sigma]]] \\ [\sigma,[[\sigma,\sigma],[\sigma,\sigma]]] & [\sigma,[\sigma,[[\sigma,\sigma],\sigma]]] & [\sigma,[\sigma,[\sigma,[\sigma,\sigma]]]] \end{array}$$

$$[\sigma,[[[\sigma,\sigma],\sigma],\sigma]] \quad [\sigma,[[\sigma,[\sigma,\sigma]],\sigma]]$$

Example 3.6.6 Let $\sigma \equiv \rho_1 \wedge \upsilon_1 \wedge \omega_1$ and $\tau \equiv \rho_2 \wedge \upsilon_2 \wedge \omega_2$. Deductions

$$\Pi_1 : \{\sigma\} \vdash_{LJ} \upsilon_1 \to \rho_1 \qquad \Pi_2 : \{\tau\} \vdash_{LJ} (\omega_2 \to \upsilon_2) \wedge (\rho_2 \to \omega_2)$$

(shown decorated[10] by $\mathsf{d}_{\to}$ on page 53) are such that

$$T_x(\Pi_1) = T_x(\Pi_2) = \{\, \lambda y.(\lambda z.x)y \mid y, z \text{ distinct variables } \not\equiv x\,\}$$

for every x. Let $\sigma' \equiv f(\sigma) \equiv f(\rho_1) \cap f(\upsilon_1) \cap f(\omega_1) \equiv \rho_1' \cap \upsilon_1' \cap \omega_1'$ and $\tau' \equiv f(\tau) \equiv f(\rho_2) \cap f(\upsilon_2) \cap f(\omega_2) \equiv \rho_2' \cap \upsilon_2' \cap \omega_2'$. We will roughly describe an algorithmic procedure, called the *simulation algorithm*, for the *construction* of

$$\Pi : [\sigma', \tau'] \vdash_{pIL} [\upsilon_1' \to \rho_1', (\omega_2' \to \upsilon_2') \cap (\rho_2' \to \omega_2')]$$

(on page 54), whose *existence* is proved in theorem 3.6.4.

We construct Π bottom-up. As the theorem states, its conclusion is (J_1) with kits of the form $[-,-]$. The leaves at path l (r) are generated by formulas in Π_1's (Π_2's) conclusion via the function f. Rule $(\wedge I)^1$ imposes a wedge ($\bigwedge$) at path r of the kits in (J_1), so that premise (J_2), which gives (J_1) by application of a $(\cap I)$ rule, includes kits of the form $[-,[-,-]]$. The leaves at path rl (rr) are generated by formulas in Π_{20}'s (Π_{21}'s) conclusion via the function f. Conjunction—in formulas and rules—is converted to intersection. Rules $(\to I)^1, (\to I)^2$ and $(\to I)^3$ translate to a single $(\to I)$ rule with premise the judgement (J_3). Rule $(\wedge E)^1$ induces a $(\cap E)$ rule on path l, while $(\wedge E)^2$ induces a $(\cap E)$ rule on path rl. These $(\cap E)$ rules can appear in any order; different orders correspond to equivalent pIL-deductions, but, since we are actually trying to create a class of equivalent pIL-deductions, i.e. an IL-deduction, which can be identified with any of its members, any order of the two $(\cap E)$ rules is acceptable. The doubleline $(\cap E)$ with premise (J_4) summarizes the two $(\cap E)$ rules brought about by $(\wedge E)^1$ and $(\wedge E)^2$. Rules $(\to E)^1, (\to E)^2$ and $(\to E)^3$ condense to a single $(\to E)$ rule with premises (J_5) and (J_6). The fact that $T_x(\Pi_1) = T_x(\Pi_2)$, for every x, secures that (J_6) is a pIL-axiom. It joins together axioms $\{\upsilon_1, \sigma\} \vdash_{LJ} \upsilon_1$ of Π_1 and $\{\tau, \omega_2\} \vdash_{LJ} \omega_2$, $\{\tau, \rho_2\} \vdash_{LJ} \rho_2$ of Π_2. Rules $(\to I)^4, (\to I)^5$ and $(\to I)^6$ disguise in a single $(\to I)$ rule with premise (J_7). Rules $(\wedge I)^2$ and $(\wedge I)^3$ impose wedges at paths l and rl of the kits in (J_7), respectively, so that premise (J_9), which gives (J_7) by application of two $(\cap I)$ rules (one on path l and one on rl), involves kits of the form $[[-,-],[[-,-],-]]$. The leaves at path ll (lr, rll, rlr) are generated by formulas in Π_{10}'s (Π_{11}'s, Π_{200}'s, Π_{201}'s) conclusion using f. The two $(\cap I)$ rules in question can be interchanged for reasons already cited. Finally, the two $(\wedge E)$ rules denoted $(\wedge E)^3$ induce two $(\cap E)$ rules on path ll, rule $(\wedge E)^4$ induces a $(\cap E)$ rule on path lr, the two $(\wedge E)$ rules denoted $(\wedge E)^5$ induce two $(\cap E)$ rules on path rll and rules $(\wedge E)^6, (\wedge E)^7$ induce a $(\cap E)$ rule each on paths rlr and rr, respectively. The doubleline $(\cap E)$ with premise (J_{10}) sums up the seven $(\cap E)$ rules brought about by $(\wedge E)^3 - (\wedge E)^7$. The order in which these seven $(\cap E)$ rules can occur is not unique. Judgement (J_{10}) is a pIL-axiom for the same reason that (J_6) is; it groups together axioms $\{\upsilon_1, \upsilon_1, \sigma\} \vdash_{LJ} \sigma$ (1), $\{\upsilon_1, \upsilon_1, \sigma\} \vdash_{LJ} \sigma$ (2) of Π_1, $\{\omega_2, \tau, \omega_2\} \vdash_{LJ} \tau$ (1), $\{\omega_2, \tau, \omega_2\} \vdash_{LJ} \tau$ (2) of Π_{20} and $\{\tau, \rho_2, \rho_2\} \vdash_{LJ} \tau$ of Π_{21}. The construction of a member Π of the equivalence class $\pi : [\sigma', \tau'] \vdash_{IL} [\upsilon_1' \to \rho_1', (\omega_2' \to \upsilon_2') \cap (\rho_2' \to \omega_2')]$ is now completed. It is $T_x(\pi) = T_x(\Pi) = T_x(\Pi_1)$, for every x.

Deductions Π_1 and Π_2 *satisfy the set equality*, i.e. $T_x(\Pi_1) = T_x(\Pi_2)$, for every x. This implies that:

[10] We start by decorating the conclusion context and move contextwise to the top. When we reach axioms, the decoration passes to the right of $\vdash$ and then descends to the conclusion.

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\{y:\upsilon_1,z:\upsilon_1,x:\sigma\}\vdash_{LJ} x:\sigma\quad(1)}{\Pi_{10}:\{y:\upsilon_1,z:\upsilon_1,x:\sigma\}\vdash_{LJ} x:\rho_1}{\scriptstyle(\wedge E)^3}
      \qquad
      \dfrac{\{y:\upsilon_1,z:\upsilon_1,x:\sigma\}\vdash_{LJ} x:\sigma\quad(2)}{\Pi_{11}:\{y:\upsilon_1,z:\upsilon_1,x:\sigma\}\vdash_{LJ} x:\omega_1}{\scriptstyle(\wedge E)^4}
    }{\{y:\upsilon_1,z:\upsilon_1,x:\sigma\}=\{y:\upsilon_1,x:\sigma\}\cup\{z:\upsilon_1\}\vdash_{LJ} x:\rho_1\wedge\omega_1}{\scriptstyle(\wedge I)^2}
  }{\{y:\upsilon_1,x:\sigma\}\vdash_{LJ} \lambda z.x:\upsilon_1\rightarrow\rho_1\wedge\omega_1}{\scriptstyle(\rightarrow I)^4}
  \qquad
  \{y:\upsilon_1,x:\sigma\}\vdash_{LJ} y:\upsilon_1
}{
  \dfrac{
    \dfrac{\{y:\upsilon_1,x:\sigma\}\vdash_{LJ}(\lambda z.x)y:\rho_1\wedge\omega_1}{\{y:\upsilon_1,x:\sigma\}=\{x:\sigma\}\cup\{y:\upsilon_1\}\vdash_{LJ}(\lambda z.x)y:\rho_1}{\scriptstyle(\wedge E)^1}
  }{\Pi_1:\{x:\sigma\}\vdash_{LJ}\lambda y.(\lambda z.x)y:\upsilon_1\rightarrow\rho_1}{\scriptstyle(\rightarrow I)^1}
}{\scriptstyle(\rightarrow E)^1}
$$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\{z':\omega_2,x:\tau,y':\omega_2\}\vdash_{LJ} x:\tau\quad(1)}{\Pi_{200}:\{z':\omega_2,x:\tau,y':\omega_2\}\vdash_{LJ} x:\upsilon_2}{\scriptstyle(\wedge E)^5}
      \qquad
      \dfrac{\{z':\omega_2,x:\tau,y':\omega_2\}\vdash_{LJ} x:\tau\quad(2)}{\Pi_{201}:\{z':\omega_2,x:\tau,y':\omega_2\}\vdash_{LJ} x:\omega_2}{\scriptstyle(\wedge E)^6}
    }{\{z':\omega_2,x:\tau,y':\omega_2\}=\{x:\tau,y':\omega_2\}\cup\{z':\omega_2\}\vdash_{LJ} x:\upsilon_2\wedge\omega_2}{\scriptstyle(\wedge I)^3}
  }{\{x:\tau,y':\omega_2\}\vdash_{LJ} \lambda z'.x:\omega_2\rightarrow\upsilon_2\wedge\omega_2}{\scriptstyle(\rightarrow I)^5}
  \qquad
  \{x:\tau,y':\omega_2\}\vdash_{LJ} y':\omega_2
}{
  \dfrac{
    \dfrac{\{x:\tau,y':\omega_2\}\vdash_{LJ}(\lambda z'.x)y':\upsilon_2\wedge\omega_2}{\{x:\tau,y':\omega_2\}=\{x:\tau\}\cup\{y':\omega_2\}\vdash_{LJ}(\lambda z'.x)y':\upsilon_2}{\scriptstyle(\wedge E)^2}
  }{\Pi_{20}:\{x:\tau\}\vdash_{LJ}\lambda y'.(\lambda z'.x)y':\omega_2\rightarrow\upsilon_2}{\scriptstyle(\rightarrow I)^2}
}{\scriptstyle(\rightarrow E)^2}
$$

$$
\dfrac{
  \dfrac{
    \dfrac{\{x:\tau,y'':\rho_2,z'':\rho_2\}\vdash_{LJ} x:\tau}{\{x:\tau,y'':\rho_2,z'':\rho_2\}=\{x:\tau,y'':\rho_2\}\cup\{z'':\rho_2\}\vdash_{LJ} x:\omega_2}{\scriptstyle(\wedge E)^7}
  }{\{x:\tau,y'':\rho_2\}\vdash_{LJ}\lambda z''.x:\rho_2\rightarrow\omega_2}{\scriptstyle(\rightarrow I)^6}
  \qquad
  \{x:\tau,y'':\rho_2\}\vdash_{LJ} y'':\rho_2
}{
  \dfrac{\{x:\tau,y'':\rho_2\}=\{x:\tau\}\cup\{y'':\rho_2\}\vdash_{LJ}(\lambda z''.x)y'':\omega_2}{\Pi_{21}:\{x:\tau\}\vdash_{LJ}\lambda y''.(\lambda z''.x)y'':\rho_2\rightarrow\omega_2}{\scriptstyle(\rightarrow I)^3}
}{\scriptstyle(\rightarrow E)^3}
$$

$$
\dfrac{\Pi_{20}\qquad(\,\lambda y'.(\lambda z'.x)y' =_{\alpha} \lambda y''.(\lambda z''.x)y''\,)\qquad\Pi_{21}}{\Pi_2:\{x:\tau\}\vdash_{LJ}\lambda y'.(\lambda z'.x)y':(\omega_2\rightarrow\upsilon_2)\wedge(\rho_2\rightarrow\omega_2)}\;{\scriptstyle(\wedge I)^1}
$$

$$
\dfrac{
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\dfrac{
\dfrac{
\dfrac{
[[\sigma',\sigma'],[[\tau',\tau'],\tau']],[[\upsilon_1',\upsilon_1'],[[\omega_2',\omega_2'],\rho_2']],[[\upsilon_1',\upsilon_1'],[[\omega_2',\omega_2'],\rho_2']] \vdash_{pIL} [[\sigma',\sigma'],[[\tau',\tau'],\tau']] \quad (J_{10})
}{
[[\sigma',\sigma'],[[\tau',\tau'],\tau']][[\upsilon_1',\upsilon_1'],[[\omega_2',\omega_2'],\rho_2']],[[\upsilon_1',\upsilon_1'],[[\omega_2',\omega_2'],\rho_2']] \vdash_{pIL} [[\rho_1',\omega_1'],[[\upsilon_2',\omega_2'],\omega_2']] \quad (J_9)
}\,(\cap E)
}{
[\sigma',[[\tau',\tau'],\tau']],[\upsilon_1',[[\omega_2',\omega_2'],\rho_2']],[\upsilon_1',[[\omega_2',\omega_2'],\rho_2']] \vdash_{pIL} [\rho_1'\cap\omega_1',[[\upsilon_2',\omega_2'],\omega_2']] \quad (J_8)
}\,(\cap I)
}{
[\sigma',[\tau',\tau']],[\upsilon_1',[\omega_2',\rho_2']],[\upsilon_1',[\omega_2',\rho_2']] \vdash_{pIL} [\rho_1'\cap\omega_1',[\upsilon_2'\cap\omega_2',\omega_2']] \quad (J_7)
}\,(\cap I)
}{
[\sigma',[\tau',\tau']],[\upsilon_1',[\omega_2',\rho_2']] \vdash_{pIL} [\upsilon_1'\to\rho_1'\cap\omega_1',[\omega_2'\to\upsilon_2'\cap\omega_2',\rho_2'\to\omega_2']] \quad (J_5)
}\,(\to I)
\qquad
[\sigma',[\tau',\tau']],[\upsilon_1',[\omega_2',\rho_2']] \vdash_{pIL} [\upsilon_1',[\omega_2',\rho_2']] \quad (J_6)
}{
\dfrac{
\dfrac{
\dfrac{
[\sigma',[\tau',\tau']],[\upsilon_1',[\omega_2',\rho_2']] \vdash_{pIL} [\rho_1'\cap\omega_1',[\upsilon_2'\cap\omega_2',\omega_2']] \quad (J_4)
}{
[\sigma',[\tau',\tau']],[\upsilon_1',[\omega_2',\rho_2']] \vdash_{pIL} [\rho_1',[\upsilon_2',\omega_2']] \quad (J_3)
}\,(\cap E)
}{
[\sigma',[\tau',\tau']] \vdash_{pIL} [\upsilon_1'\to\rho_1',[\omega_2'\to\upsilon_2',\rho_2'\to\omega_2']] \quad (J_2)
}\,(\to I)
}{
\Pi : [\sigma',\tau'] \vdash_{pIL} [\upsilon_1'\to\rho_1',(\omega_2'\to\upsilon_2')\cap(\rho_2'\to\omega_2')] \quad (J_1)
}\,(\cap I)
}\,(\to E)
$$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\{y:\upsilon_1,z:\upsilon_1,x:\sigma\} \vdash_{LJ} x:\sigma \quad (1)
}{
\{y:\upsilon_1,z:\upsilon_1,x:\sigma\} \vdash_{LJ} x:\rho_1
}\,(\wedge E)
\qquad
\dfrac{
\{y:\upsilon_1,z:\upsilon_1,x:\sigma\} \vdash_{LJ} x:\sigma \quad (2)
}{
\{y:\upsilon_1,z:\upsilon_1,x:\sigma\} \vdash_{LJ} x:\omega_1
}\,(\wedge E)
}{
\{y:\upsilon_1,z:\upsilon_1,x:\sigma\} = \{y:\upsilon_1,x:\sigma\}\cup\{z:\upsilon_1\} \vdash_{LJ} x:\rho_1\wedge\omega_1
}\,(\wedge I)
}{
\{y:\upsilon_1,x:\sigma\} \vdash_{LJ} \lambda z.x:\upsilon_1\to\rho_1\wedge\omega_1
}\,(\to I)
\qquad
\dfrac{
\{y:\upsilon_1,x:\sigma\} \vdash_{LJ} x:\sigma
}{
\{y:\upsilon_1,x:\sigma\} \vdash_{LJ} x:\upsilon_1
}\,(\wedge E)
}{
\dfrac{
\dfrac{
\{y:\upsilon_1,x:\sigma\} \vdash_{LJ} (\lambda z.x)x:\rho_1\wedge\omega_1
}{
\{y:\upsilon_1,x:\sigma\} = \{x:\sigma\}\cup\{y:\upsilon_1\} \vdash_{LJ} (\lambda z.x)x:\rho_1
}\,(\wedge E)
}{
\Pi_1' : \{x:\sigma\} \vdash_{LJ} \lambda y.(\lambda z.x)x:\upsilon_1\to\rho_1
}\,(\to I)
}\,(\to E)
$$

$$
\dfrac{
\begin{array}{c}\uparrow\\ [\sigma',[\tau',\tau']],[\upsilon_1',[\omega_2',\rho_2']] \vdash_{pIL} [\upsilon_1'\to\rho_1'\cap\omega_1',[\omega_2'\to\upsilon_2'\cap\omega_2',\rho_2'\to\omega_2']]\end{array}
\qquad
\dfrac{
[\sigma',[\tau',\tau']],[\upsilon_1',[\omega_2',\rho_2']] \vdash_{pIL} [\sigma',[\omega_2',\rho_2']] \quad (J)
}{
[\sigma',[\tau',\tau']],[\upsilon_1',[\omega_2',\rho_2']] \vdash_{pIL} [\upsilon_1',[\omega_2',\rho_2']] \quad (J')
}\,(\cap E)^*
}{
\dfrac{
\dfrac{
\dfrac{
[\sigma',[\tau',\tau']],[\upsilon_1',[\omega_2',\rho_2']] \vdash_{pIL} [\rho_1'\cap\omega_1',[\upsilon_2'\cap\omega_2',\omega_2']]
}{
[\sigma',[\tau',\tau']],[\upsilon_1',[\omega_2',\rho_2']] \vdash_{pIL} [\rho_1',[\upsilon_2',\omega_2']]
}\,(\cap E)
}{
[\sigma',[\tau',\tau']] \vdash_{pIL} [\upsilon_1'\to\rho_1',[\omega_2'\to\upsilon_2',\rho_2'\to\omega_2']]
}\,(\to I)
}{
\bar{\Pi} : [\sigma',\tau'] \vdash_{pIL} [\upsilon_1'\to\rho_1',(\omega_2'\to\upsilon_2')\cap(\rho_2'\to\omega_2')]
}\,(\cap I)
}\,(\to E)
$$

(i) They *bear the same sequence of implicative rules* $(\rightarrow I, \rightarrow E, \rightarrow I)$ modulo repetitions due to $(\wedge I)$ rules (see Π_2)[11], which is a necessary condition for their union in a single pIL-deduction[12]. Since implicative pIL-rules are global, in order for two LJ-deductions Π_1 and Π_2 in $D_{LJ}^{\rightarrow}$ to mate in kits of a pIL-deduction Π, it is necessary that they grow identically with respect to implication, i.e. that they bear the same sequence $(R_1, \ldots, R_n)$ of implicative rules, repetitions put aside. If they do, then their implicative rules can be gathered in n groups $G_1, \ldots, G_n$, so that, for each i, group G_i contains rules of the same kind (introduction or elimination) which all correspond to R_i. The rules of each such group merge to a single implicative rule of the group's kind in Π, which then bears $(R_1, \ldots, R_n)$ without repetitions[13]. On the contrary, as far as conjunctive rules are concerned, the two deductions are allowed to evolve separately; this is because the image of conjunction in pIL, i.e. intersection, functions locally. If Π_1 is settled at path l in Π's conclusion and Π_2 at path r and if conjunction C_1 in Π_1 transforms to intersection I_1 in Π, then I_1 is applied on a path lq. Consequently, it doesn't at all affect leaves originating from Π_2, which occupy paths of the form rq. A similar argument holds for a C_2 in Π_2 transforming to I_2 in Π.

(ii) *Judgements* of the two deductions *in the same group*, i.e. that are to be joined together to form a single judgement of Π, *have the same context cardinality*, which is also the context cardinality of the pIL-judgement formed.

(iii) *Axioms* of the two deductions *in the same group are decorated by corresponding variables*, which secures that the algorithmic procedure adopted for the creation of Π winds up to pIL-axioms. For example, axioms $\{\upsilon_1, \sigma\} \vdash_{LJ} \upsilon_1$ of Π_1 and $\{\tau, \omega_2\} \vdash_{LJ} \omega_2$, $\{\tau, \rho_2\} \vdash_{LJ} \rho_2$ of Π_2 are decorated by y, y', y'', respectively, i.e. by corresponding variables, which is responsible for the fact that $[\upsilon_1', [\omega_2', \rho_2']]$ is one of the kits in the set $\{[\sigma', [\tau', \tau']], [\upsilon_1', [\omega_2', \rho_2']]\}$, so that (J_6) is a pIL-axiom. To make this point more intelligible, consider $\Pi_1' : \{\sigma\} \vdash_{LJ} \upsilon_1 \rightarrow \rho_1$ (shown decorated by $\mathsf{d}_{\rightarrow}$ on page 54) in place of Π_1, where σ is as in Π_1, and suppose we try to construct a pIL-deduction $\bar{\Pi}$ from Π_1' and Π_2 following the simulation algorithm. It is $T_x(\Pi_1') = \{\lambda y.(\lambda z.x)x \mid y, z \text{ distinct variables } \not\equiv x\} \neq T_x(\Pi_2)$, for every x, so Π_1' and Π_2 do not satisfy the set equality, although they (i) bear the same (modulo repetitions) sequence of implicative rules $(\rightarrow I, \rightarrow E, \rightarrow I)$ and (ii) have the same context cardinality in judgements to be grouped together. Hence, conditions (i) and (ii) are not sufficient for the set equality to hold; we also need to have condition (iii). Indeed, the point where Π_1' and Π_2 differ, a difference which implies the inequality of $T_x(\Pi_1')$ and $T_x(\Pi_2)$, is the decoration of axiom $\{\upsilon_1, \sigma\} \vdash_{LJ} \sigma$ on one hand and axioms $\{\tau, \omega_2\} \vdash_{LJ} \omega_2$ and $\{\tau, \rho_2\} \vdash_{LJ} \rho_2$ on the other by non-corresponding variables (x doesn't correspond to y' or y''). Consequently, we see (on page 54) that the construction of $\bar{\Pi}$ reaches rule $(\cap E)^*$ and stops, since (J)—which is expected to gather the three LJ-axioms in one—is *not* a pIL-axiom. Nevertheless, since axioms $\{\upsilon_1, \upsilon_1, \sigma\} \vdash_{LJ} \sigma$ (1), $\{\upsilon_1, \upsilon_1, \sigma\} \vdash_{LJ} \sigma$ (2) of Π_1' and $\{\omega_2, \tau, \omega_2\} \vdash_{LJ} \tau$ (1), $\{\omega_2, \tau, \omega_2\} \vdash_{LJ} \tau$ (2), $\{\tau, \rho_2, \rho_2\} \vdash_{LJ} \tau$ of Π_2 *are* decorated by corresponding—in fact by identical—variables (all by x), if we continued the algorithm to the direction of the arrow, we would reach a pIL-axiom, namely (J_{10}). So, one might argue that, since

[11] Observing Π_2, we see that the whole implicative sequence $(\rightarrow I, \rightarrow E, \rightarrow I)$ is repeated in Π_{20} and Π_{21} due to rule $(\wedge I)^1$, which has no implicative rules beneath it. It can also happen, though, that parts of the implicative sequence are repeated due to $(\wedge I)$ rules in between implicative ones.

[12] This condition is not sufficient, as we shall soon ascertain (see (iii) of this example).

[13] For the particular Π_1 and Π_2 of our example, it is $n = 3$, $(R_1, R_2, R_3) = (\rightarrow I, \rightarrow E, \rightarrow I)$ (bottom to top) and group G_1 contains $(\rightarrow I)^1, (\rightarrow I)^2, (\rightarrow I)^3$, group G_2 contains $(\rightarrow E)^1, (\rightarrow E)^2, (\rightarrow E)^3$, group G_3 contains $(\rightarrow I)^4, (\rightarrow I)^5, (\rightarrow I)^6$.

judgement (J') is a pIL-axiom, the construction of $\bar{\Pi}$ could stop at (J') as far as the right column before $(\to E)$ is concerned. This is not correct, though, as theorem 3.6.4 requires that $T_x(\bar{\Pi}) = T_x(\Pi'_1) = T_x(\Pi_2)$, for every x. The fact that $T_x(\Pi'_1) \neq T_x(\Pi_2)$, for every x, zeroes the chances of constructing a $\bar{\Pi}$ joining them according to 3.6.4.

Given Π_1 and Π_2, the simulation algorithm constructs a unique equivalence class π, which is P-normal. Obviously, we can construct many different individual members of π, but the construction of any such member is actually seen as the construction of π.

Theorem 3.6.4 can now be restated as follows:

Theorem 3.6.7 (Inverse projection theorem for IL) *Let $n \geqslant 0,\ m \geqslant 1$,*

$$\Pi_1 : \{\sigma_{11}, \dots, \sigma_{n1}\} \vdash_{LJ} \tau_1, \Pi_2 : \{\sigma_{12}, \dots, \sigma_{n2}\} \vdash_{LJ} \tau_2, \dots, \Pi_m : \{\sigma_{1m}, \dots, \sigma_{nm}\} \vdash_{LJ} \tau_m$$

and $H_1, \dots, H_n, K$ be $n+1$ overlapping kits with m terminal paths $p_1, \dots, p_m$, such that $H_i^{p_j} \equiv f(\sigma_{ij})$, $K^{p_j} \equiv f(\tau_j)$ $(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m)$. If $\Pi_1, \dots, \Pi_m$ are all in $D_{LJ}^{\rightarrow}$ and $T_{x_1,\dots,x_n}(\Pi_1) = T_{x_1,\dots,x_n}(\Pi_2) = \dots = T_{x_1,\dots,x_n}(\Pi_m)$, for every $x_1, \dots, x_n$, there exists a unique equivalence class $\pi : \{H_1, \dots, H_n\} \vdash_{IL} K$, which is P-normal and such that $T_{x_1,\dots,x_n}(\pi) = T_{x_1,\dots,x_n}(\Pi_1)$, for every $x_1, \dots, x_n$.

Corollary 3.6.8 *For every $\Pi : \{\sigma_1, \dots, \sigma_n\} \vdash_{LJ} \tau$ in $D_{LJ}^{\rightarrow}$, there exists a unique equivalence class $\Pi^\delta : \{f(\sigma_1), \dots, f(\sigma_n)\} \vdash_{IL} f(\tau)$, which is P-normal and such that $T_{x_1,\dots,x_n}(\Pi^\delta) = T_{x_1,\dots,x_n}(\Pi)$, for every $x_1, \dots, x_n$.*

Proof: Apply theorem 3.6.7 for $m = 1$. ⊣

Definition 3.6.9 *Let $\Pi : \{\sigma_1, \dots, \sigma_n\} \vdash_{LJ} \tau$ belong to $D_{LJ}^{\rightarrow}$. The P-normal, proper IL-deduction $\Pi^\delta : \{f(\sigma_1), \dots, f(\sigma_n)\} \vdash_{IL} f(\tau)$ which copies Π in IL will be called the* IL-duplicate *of* Π.

Given Π, the class Π^δ is obtained using the simulation algorithm. We give an example of simulation in IL of a single LJ-deduction decoratable non-standardly. Ccompare this to example 3.6.6 where we simulated in IL a pair of LJ-deductions decoratable non-standardly by the same λ-term.

Example 3.6.10 Let $\sigma \equiv \alpha \wedge \beta \wedge \gamma$ with α, β and γ propositional variables. Deductions $\Pi_1 : \sigma \vdash_{LJ} \alpha \to \beta$, $\Pi_2 : \sigma \vdash_{LJ} \beta \to \alpha$ and $\Pi_3 : \sigma \vdash_{LJ} \gamma \to \alpha$ (on page 57) are isomorphic with respect to $\mathsf{d}_{\to}$ and combine under $(\wedge I)$ to give $\Pi : \sigma \vdash_{LJ} (\alpha \to \beta) \wedge (\beta \to \alpha) \wedge (\gamma \to \alpha)$ (on page 58), which is decoratable by $\mathsf{d}_{\to}$ and simulated in IL by $\Pi^\delta : f(\sigma) \equiv \tau \vdash_{IL} (\alpha \to \beta) \cap (\beta \to \alpha) \cap (\gamma \to \alpha)$ (on page 58). It is $T_x(\Pi) = T_x(\Pi^\delta) = \{\, \lambda y.(\lambda z.(\lambda u.x)y)((\lambda w.x)y) \mid y, z, u, w \text{ distinct variables } \not\equiv x \,\}$, for every x.

So far, we have that for every P-normal, proper IL-deduction π, there is a unique projection π^ϵ in $D_{LJ}^{\rightarrow}$, such that the set (term) equality holds. Given π, we choose a pIL-deduction Π in π and apply the projection algorithm to obtain $\Pi^\epsilon \equiv \pi^\epsilon$. Also, for every LJ-deduction Π in $D_{LJ}^{\rightarrow}$, there is a unique duplicate Π^δ in $D_{IL}^{Pn,p}$, such that the set (term) equality holds. Given Π, the class Π^δ is attained using the simulation algorithm.

From the examples described, it is easy to see that projecting and simulating are inverse procedures between the sets $D_{IL}^{Pn,p}$ and $D_{LJ}^{\rightarrow}$. This means that for every π in $D_{IL}^{Pn,p}$, it is $(\pi^\epsilon)^\delta \equiv \pi$ and for every Π in $D_{LJ}^{\rightarrow}$, it is $(\Pi^\delta)^\epsilon \equiv \Pi$.

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  \dfrac{
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    \dfrac{
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      \dfrac{\sigma,\alpha,\gamma,\alpha\vdash_{LJ}\sigma}{\sigma,\alpha,\gamma,\alpha\vdash_{LJ}\beta}\,(\wedge E)
      \quad
      \dfrac{\sigma,\alpha,\gamma,\alpha\vdash_{LJ}\sigma}{\sigma,\alpha,\gamma,\alpha\vdash_{LJ}\gamma}\,(\wedge E)
     }{\sigma,\alpha,\gamma,\alpha\vdash_{LJ}\beta\wedge\gamma}\,(\wedge I)
    }{\sigma,\alpha,\gamma\vdash_{LJ}\alpha\to\beta\wedge\gamma}\,(\to I)
    \quad \sigma,\alpha,\gamma\vdash_{LJ}\alpha
   }{\sigma,\alpha,\gamma\vdash_{LJ}\beta\wedge\gamma}\,(\to E)
  }{\sigma,\alpha,\gamma\vdash_{LJ}\beta}\,(\wedge E)
 }{\sigma,\alpha\vdash_{LJ}\gamma\to\beta}\,(\to I)
 \quad
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    \dfrac{
     \dfrac{\sigma,\alpha,\alpha\vdash_{LJ}\sigma}{\sigma,\alpha,\alpha\vdash_{LJ}\beta}\,(\wedge E)
     \quad
     \dfrac{\sigma,\alpha,\alpha\vdash_{LJ}\sigma}{\sigma,\alpha,\alpha\vdash_{LJ}\gamma}\,(\wedge E)
    }{\sigma,\alpha,\alpha\vdash_{LJ}\beta\wedge\gamma}\,(\wedge I)
   }{\sigma,\alpha\vdash_{LJ}\alpha\to\beta\wedge\gamma}\,(\to I)
   \quad \sigma,\alpha\vdash_{LJ}\alpha
  }{\sigma,\alpha\vdash_{LJ}\beta\wedge\gamma}\,(\to E)
 }{\sigma,\alpha\vdash_{LJ}\gamma}\,(\wedge E)
}{\sigma,\alpha\vdash_{LJ}\beta}\,(\to E)}{\Pi_1:\sigma\vdash_{LJ}\alpha\to\beta}\,(\to I)
$$

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    \dfrac{
     \dfrac{\sigma,\beta,\gamma,\beta\vdash_{LJ}\sigma}{\sigma,\beta,\gamma,\beta\vdash_{LJ}\alpha\wedge\beta}\,(\wedge E)
    }{\sigma,\beta,\gamma\vdash_{LJ}\beta\to\alpha\wedge\beta}\,(\to I)
    \quad \sigma,\beta,\gamma\vdash_{LJ}\beta
   }{\sigma,\beta,\gamma\vdash_{LJ}\alpha\wedge\beta}\,(\to E)
  }{\sigma,\beta,\gamma\vdash_{LJ}\alpha}\,(\wedge E)
 }{\sigma,\beta\vdash_{LJ}\gamma\to\alpha}\,(\to I)
 \quad
 \dfrac{
  \dfrac{
   \dfrac{\sigma,\beta,\beta\vdash_{LJ}\sigma}{\sigma,\beta,\beta\vdash_{LJ}\gamma}\,(\wedge E)
  }{\sigma,\beta\vdash_{LJ}\beta\to\gamma}\,(\to I)
  \quad \sigma,\beta\vdash_{LJ}\beta
 }{\sigma,\beta\vdash_{LJ}\gamma}\,(\to E)
}{\sigma,\beta\vdash_{LJ}\alpha}\,(\to E)}{\Pi_2:\sigma\vdash_{LJ}\beta\to\alpha}\,(\to I)
$$

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     \dfrac{
      \dfrac{\sigma,\gamma,\beta,\gamma\vdash_{LJ}\sigma}{\sigma,\gamma,\beta,\gamma\vdash_{LJ}\alpha}\,(\wedge E)
      \quad
      \dfrac{\sigma,\gamma,\beta,\gamma\vdash_{LJ}\sigma}{\sigma,\gamma,\beta,\gamma\vdash_{LJ}\gamma}\,(\wedge E)
     }{\sigma,\gamma,\beta,\gamma\vdash_{LJ}\alpha\wedge\gamma}\,(\wedge I)
    }{\sigma,\gamma,\beta\vdash_{LJ}\gamma\to\alpha\wedge\gamma}\,(\to I)
    \quad \sigma,\gamma,\beta\vdash_{LJ}\gamma
   }{\sigma,\gamma,\beta\vdash_{LJ}\alpha\wedge\gamma}\,(\to E)
  }{\sigma,\gamma,\beta\vdash_{LJ}\alpha}\,(\wedge E)
 }{\sigma,\gamma\vdash_{LJ}\beta\to\alpha}\,(\to I)
 \quad
 \dfrac{
  \dfrac{
   \dfrac{\sigma,\gamma,\gamma\vdash_{LJ}\sigma}{\sigma,\gamma,\gamma\vdash_{LJ}\beta}\,(\wedge E)
  }{\sigma,\gamma\vdash_{LJ}\gamma\to\beta}\,(\to I)
  \quad \sigma,\gamma\vdash_{LJ}\gamma
 }{\sigma,\gamma\vdash_{LJ}\beta}\,(\to E)
}{\sigma,\gamma\vdash_{LJ}\alpha}\,(\to E)}{\Pi_3:\sigma\vdash_{LJ}\gamma\to\alpha}\,(\to I)
$$

$$\dfrac{\dfrac{\Pi_1 \qquad \Pi_2}{\sigma \vdash_{LJ} (\alpha \to \beta) \wedge (\beta \to \alpha)}\,(\wedge I) \qquad \Pi_3}{\Pi : \sigma \vdash_{LJ} (\alpha \to \beta) \wedge (\beta \to \alpha) \wedge (\gamma \to \alpha)}\,(\wedge I)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[[[\tau,\tau],\tau],[\tau,\tau]],[[[\alpha,\alpha],\beta],[\gamma,\gamma]],[[[\gamma,\gamma],\gamma],[\beta,\beta]],[[[\alpha,\alpha],\beta],[\gamma,\gamma]] \vdash_{IL} [[[\tau,\tau],\tau],[\tau,\tau]]}{[[[\tau,\tau],\tau],[\tau,\tau]],[[[\alpha,\alpha],\beta],[\gamma,\gamma]],[[[\gamma,\gamma],\gamma],[\beta,\beta]],[[[\alpha,\alpha],\beta],[\gamma,\gamma]] \vdash_{IL} [[[\beta,\gamma],\alpha \cap \beta],[\alpha,\gamma]]}\,(\cap E)}{[[\tau,\tau],\tau],[[\alpha,\beta],\gamma],[[\gamma,\gamma],\beta],[[\alpha,\beta],\gamma] \vdash_{IL} [[\beta \cap \gamma, \alpha \cap \beta], \alpha \cap \gamma]}\,(\cap I)}{[[\tau,\tau],\tau],[[\alpha,\beta],\gamma],[[\gamma,\gamma],\beta] \vdash_{IL} [[\alpha \to \beta \cap \gamma, \beta \to \alpha \cap \beta], \gamma \to \alpha \cap \gamma]}\,(\to I) \qquad [[\tau,\tau],\tau],[[\alpha,\beta],\gamma],[[\gamma,\gamma],\beta] \vdash_{IL} [[\alpha,\beta],\gamma]}{\dfrac{\dfrac{[[\tau,\tau],\tau],[[\alpha,\beta],\gamma],[[\gamma,\gamma],\beta] \vdash_{IL} [[\beta \cap \gamma, \alpha \cap \beta], \alpha \cap \gamma]}{[[\tau,\tau],\tau],[[\alpha,\beta],\gamma],[[\gamma,\gamma],\beta] \vdash_{IL} [[\beta,\alpha],\alpha]}\,(\cap E)}{\Pi_0^\delta : [[\tau,\tau],\tau],[[\alpha,\beta],\gamma] \vdash_{IL} [[\gamma \to \beta, \gamma \to \alpha], \beta \to \alpha]}\,(\to I)}\,(\to E)$$

$$\dfrac{\Pi_0^\delta \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[[[\tau,\tau],\tau],\tau],[[[\alpha,\alpha],\beta],\gamma],[[[\alpha,\alpha],\beta],\gamma] \vdash_{IL} [[[\tau,\tau],\tau],\tau]}{[[[\tau,\tau],\tau],\tau],[[[\alpha,\alpha],\beta],\gamma],[[[\alpha,\alpha],\beta],\gamma] \vdash_{IL} [[[\beta,\gamma],\gamma],\beta]}\,(\cap E)}{[[\tau,\tau],\tau],[[\alpha,\beta],\gamma],[[\alpha,\beta],\gamma] \vdash_{IL} [[\beta \cap \gamma,\gamma],\beta]}\,(\cap I)}{[[\tau,\tau],\tau],[[\alpha,\beta],\gamma] \vdash_{IL} [[\alpha \to \beta \cap \gamma, \beta \to \gamma], \gamma \to \beta]}\,(\to I) \qquad [[\tau,\tau],\tau],[[\alpha,\beta],\gamma] \vdash_{IL} [[\alpha,\beta],\gamma]}{[[\tau,\tau],\tau],[[\alpha,\beta],\gamma] \vdash_{IL} [[\beta \cap \gamma,\gamma],\beta]}\,(\to E)}{\Pi_1^\delta : [[\tau,\tau],\tau],[[\alpha,\beta],\gamma] \vdash_{IL} [[\gamma,\gamma],\beta]}\,(\cap E)}{\dfrac{\dfrac{\dfrac{[[\tau,\tau],\tau],[[\alpha,\beta],\gamma] \vdash_{IL} [[\beta,\alpha],\alpha]}{[[\tau,\tau],\tau] \vdash_{IL} [[\alpha \to \beta, \beta \to \alpha], \gamma \to \alpha]}\,(\to I)}{[\tau,\tau] \vdash_{IL} [(\alpha \to \beta) \cap (\beta \to \alpha), \gamma \to \alpha]}\,(\cap I)}{\Pi^\delta : \tau \vdash_{IL} (\alpha \to \beta) \cap (\beta \to \alpha) \cap (\gamma \to \alpha)}\,(\cap I)}\,(\to E)$$

We can now demonstrate the existence of a bijection between $D_{IL}^{Pn,p}$ and $D_{LJ}^{\rightarrow}$.

Theorem 3.6.11 *There exists a bijection between the sets $D_{IL}^{Pn,p}$ and $D_{LJ}^{\rightarrow}$. The corresponding deductions share the same implicative structure.*

Proof sketch: As already discussed, $\epsilon : D_{IL}^{Pn,p} \to D_{LJ}^{\rightarrow}$ defined as $\epsilon(\pi) = \pi^{\epsilon}$ is a function, such that argument and image share the same implicative structure. Since, for any Π in $D_{LJ}^{\rightarrow}$, there is a unique Π^{δ} in $D_{IL}^{Pn,p}$, such that $\epsilon(\Pi^{\delta}) = (\Pi^{\delta})^{\epsilon} \equiv \Pi$, ϵ is one-to-one and onto, i.e. a bijection. $\dashv$

By inference, we can say that IL expresses the part of LJ decoratable by $\mathsf{d}_{\rightarrow}$. That's why it is indeed appropriate for the logical foundation of IT. Since this part of LJ comprises a proper subset of LJ, we are addressing a proper embedding of IL in LJ.

3.7 IL and IT

In this section, we show the relation betweem pIL and the intersection-types system IT, namely the way to attain IT-deductions from pIL-deductions and vice versa. Then, we derive the connection between IL and IT deductions, which establishes IL's appropriateness for the logical foundation of IT.

Theorem 3.7.1 (pIL and IT) *(i) If $\Pi : \{H_1, \ldots, H_m\} \vdash_{pIL} K$, the terminal paths of K are $p_1, \ldots, p_n$ and $H_j^{p_i} \equiv \sigma_j^i$, $K^{p_i} \equiv \tau_i$ $(1 \leqslant j \leqslant m,\ 1 \leqslant i \leqslant n)$, then*

$$\{x_1 : \sigma_1^i, \ldots, x_m : \sigma_m^i\} \vdash_{IT} T_{x_1,\ldots,x_m}(\Pi) : \tau_i$$

for every $i \in \{1, \ldots, n\}$ and every sequence $x_1, \ldots, x_m$ of distinct variables. So, for a proper deduction $\Pi : \{\sigma_1, \ldots, \sigma_m\} \vdash_{pIL} \tau$, we have that

$$\{x_1 : \sigma_1, \ldots, x_m : \sigma_m\} \vdash_{IT} T_{x_1,\ldots,x_m}(\Pi) : \tau$$

for every $x_1, \ldots, x_m$.

(ii) Suppose that $x_1, \ldots, x_m$ is a fixed, but arbitrary sequence of distinct variables. If, for every $i \in \{1, \ldots, n\}$, $\Pi_i : \{x_1 : \sigma_1^i, \ldots, x_m : \sigma_m^i\} \vdash_{IT} M : \tau_i$ and $H_1, \ldots, H_m, K$ are $m+1$ overlapping kits with n terminal paths $p_1, \ldots, p_n$, such that $H_j^{p_i} \equiv \sigma_j^i$ and $K^{p_i} \equiv \tau_i$ $(1 \leqslant j \leqslant m,\ 1 \leqslant i \leqslant n)$, then there exists

$$\Pi : \{H_1, \ldots, H_m\} \vdash_{pIL} K$$

such that $T_{x_1,\ldots,x_m}(\Pi) \equiv M$. So, if $\Pi_1 : \{x_1 : \sigma_1^1, \ldots, x_m : \sigma_m^1\} \vdash_{IT} M : \tau_1$, then there exists $\Pi : \{\sigma_1^1, \ldots, \sigma_m^1\} \vdash_{pIL} \tau_1$, such that $T_{x_1,\ldots,x_m}(\Pi) \equiv M$.

Proof: (i) By theorem 3.4.1, for every $i \in \{1, \ldots, n\}$, we have that

$$\Pi^{p_i} : \{e(\sigma_1^i), \ldots, e(\sigma_m^i)\} \vdash_{LJ} e(\tau_i)$$

is in $D_{LJ}^{\rightarrow}$ and such that $T_{x_1,\ldots,x_m}(\Pi^{p_i}) \equiv T_{x_1,\ldots,x_m}(\Pi)$, for every $x_1, \ldots, x_m$[14]. So, for every $i \in \{1, \ldots, n\}$, we have, by theorem 2.6.2, that

$$\{x_1 : f(e(\sigma_1^i)), \ldots, x_m : f(e(\sigma_m^i))\} \vdash_{IT} T_{x_1,\ldots,x_m}(\Pi^{p_i}) : f(e(\tau_i))$$

[14]For convenience, we identify α-equivalent λ-terms, so we consider $T_{x_1,\ldots,x_m}(\Pi^{p_i})$ and $T_{x_1,\ldots,x_m}(\Pi)$ to be λ-terms, for every $x_1, \ldots, x_m$.

for every $x_1, \ldots, x_m$. Hence, for every $i \in \{1, \ldots, n\}$ and every $x_1, \ldots, x_m$, it is $\{x_1 : \sigma_1^i, \ldots, x_m : \sigma_m^i\} \vdash_{IT} T_{x_1,\ldots,x_m}(\Pi) : \tau_i$, as required.

(ii) By theorem 2.6.3, if we apply the erasing function E on Π_i, we get

$$E(\Pi_i) : \{e(\sigma_1^i), \ldots, e(\sigma_m^i)\} \vdash_{LJ} e(\tau_i)$$

which is in $D_{LJ}^{\rightarrow}$ and such that $T_{x_1,\ldots,x_m}(E(\Pi_i)) \equiv M$. So, we have n LJ-deductions $E(\Pi_1), \ldots, E(\Pi_n)$ in $D_{LJ}^{\rightarrow}$, such that $T_{x_1,\ldots,x_m}(E(\Pi_1)) \equiv \ldots \equiv T_{x_1,\ldots,x_m}(E(\Pi_n))$, for an arbitrary $x_1, \ldots, x_m$. We also have $m+1$ overlapping kits $H_1, \ldots, H_m, K$ with n terminal paths $p_1, \ldots, p_n$, such that $H_j^{p_i} \equiv \sigma_j^i \equiv f(e(\sigma_j^i))$, $K^{p_i} \equiv \tau_i \equiv f(e(\tau_i))$ $(1 \leqslant j \leqslant m,\ 1 \leqslant i \leqslant n)$. Then, by theorem 3.6.4, there exists Π: $\{H_1, \ldots, H_m\} \vdash_{pIL} K$, such that $T_{x_1,\ldots,x_m}(\Pi) \equiv T_{x_1,\ldots,x_m}(E(\Pi_1)) \equiv M$. $\dashv$

Corollary 3.7.2 (IL and IT) *(i) For every proper* $\pi : \{\sigma_1, \ldots, \sigma_m\} \vdash_{IL} \tau$*, there exists* $\Pi : \{x_1 : \sigma_1, \ldots, x_m : \sigma_m\} \vdash_{IT} T_{x_1,\ldots,x_m}(\pi) : \tau$*, for every* $x_1, \ldots, x_m$*.*

(ii) If $x_1, \ldots, x_m$ *is fixed, but arbitrary, for every*

$$\Pi : \{x_1 : \sigma_1, \ldots, x_m : \sigma_m\} \vdash_{IT} M : \tau$$

there exists $\pi : \{\sigma_1, \ldots, \sigma_m\} \vdash_{IL} \tau$*, such that* $T_{x_1,\ldots,x_m}(\pi) \equiv M$*.*

Proof: (i) By considering a Π in the class π and applying theorem 3.7.1(i). Recall that, for every Π in π, it is $T_{x_1,\ldots,x_m}(\Pi) \equiv T_{x_1,\ldots,x_m}(\pi)$.

(ii) By theorem 3.7.1(ii), there exists $\Pi' : \{\sigma_1, \ldots, \sigma_m\} \vdash_{pIL} \tau$, such that $T_{x_1,\ldots,x_m}(\Pi') \equiv M$. But Π' belongs to a class $\pi : \{\sigma_1, \ldots, \sigma_m\} \vdash_{pIL} \tau$ with $T_{x_1,\ldots,x_m}(\pi) \equiv T_{x_1,\ldots,x_m}(\Pi')$. $\dashv$

The transition from IT to pIL-deductions, stated by theorem 3.7.1, allows to derive for free the property of strong normalization of λ-terms typable in IT (one direction of theorem 2.5.7).

Theorem 3.7.3 *Let* $M \in \Lambda$*. If* M *is typable in IT, then* M *is strongly normalizable with respect to* β*-reduction.*

Proof: Suppose M is typable in IT, i.e. there exists

$$\Pi_0 : \{x_1 : \sigma_1, \ldots, x_m : \sigma_m\} \vdash_{IT} M : \sigma$$

By theorem 3.7.1(ii), Π_0 can be embedded in pIL to get $\Pi_0' : \{\sigma_1, \ldots, \sigma_m\} \vdash_{pIL} \sigma$, such that $T_{x_1,\ldots,x_m}(\Pi_0') \equiv M$. Any redex of M corresponds to a $\rightarrow$-redex in Π_0'. If $M \rightarrow_\beta M_1$, there exists $\Pi_1 : \{x_1 : \sigma_1, \ldots, x_m : \sigma_m\} \vdash_{IT} M_1 : \sigma$ (see [Kr93], p. 50). Deduction Π_1 is embedded in $\Pi_1' : \{\sigma_1, \ldots, \sigma_m\} \vdash_{pIL} \sigma$, which is such that $\Pi_0' \hookrightarrow_* \Pi_1'$, where $\hookrightarrow_*$ denotes a finite number of pIL normalization steps that include exactly one $\hookrightarrow_\rightarrow$-step. Suppose now that M is not strongly normalizable, i.e. that there exists an infinite sequence of β-reductions starting from M.

$$M \equiv M_0 \rightarrow_\beta M_1 \rightarrow_\beta M_2 \rightarrow_\beta \ldots$$

The IT-deductions $\Pi_0, \Pi_1, \Pi_2, \ldots$, which assign type σ to $M_0, M_1, M_2, \ldots$, respectively, are embedded in the pIL-deductions $\Pi_0', \Pi_1', \Pi_2', \ldots$, respectively, which are such that

$$\Pi_0' \hookrightarrow_* \Pi_1' \hookrightarrow_* \Pi_2' \hookrightarrow_* \ldots$$

But then Π_0' is not strongly normalizable, which contradicts theorem 3.5.14. $\dashv$

Example 3.7.4 Deduction Π_2 of example 2.4.5 is an IT-deduction typing the term $M \equiv (\lambda x.xx)\lambda x.x$. Let us call it Π_0 for the purpose of this example. It is $M \to_\beta M_1 \equiv (\lambda x.x)\lambda x.x \to_\beta M_2 \equiv \lambda x.x$. We show $\Pi'_0, \Pi_1, \Pi'_1, \Pi_2$ and Π'_2 below.

$$\dfrac{\dfrac{\dfrac{\dfrac{\tau \vdash_{pIL} \tau}{\tau \vdash_{pIL} (\alpha \to \alpha) \to \alpha \to \alpha}\,(\cap E^l) \quad \dfrac{\tau \vdash_{pIL} \tau}{\tau \vdash_{pIL} \alpha \to \alpha}\,(\cap E^r)}{\tau \vdash_{pIL} \alpha \to \alpha}\,(\to E)}{\vdash_{pIL} \tau \to \alpha \to \alpha}\,(\to I) \quad \dfrac{\dfrac{[\alpha \to \alpha\,,\,\alpha] \vdash_{pIL} [\alpha \to \alpha\,,\,\alpha]}{\vdash_{pIL} [(\alpha \to \alpha) \to \alpha \to \alpha\,,\,\alpha \to \alpha]}\,(\to I)}{\vdash_{pIL} \tau}\,(\cap I)}{\Pi'_0 : \vdash_{pIL} \alpha \to \alpha}\,(\to E)$$

$$\dfrac{\dfrac{\{x : \alpha \to \alpha\} \vdash_{IT} x : \alpha \to \alpha}{\vdash_{IT} \lambda x.x : (\alpha \to \alpha) \to \alpha \to \alpha}\,(\to I) \quad \dfrac{\{x : \alpha\} \vdash_{IT} x : \alpha}{\vdash_{IT} \lambda x.x : \alpha \to \alpha}\,(\to I)}{\Pi_1 : \vdash_{IT} (\lambda x.x)\lambda x.x : \alpha \to \alpha}\,(\to E)$$

$$\dfrac{\dfrac{\{\alpha \to \alpha\} \vdash_{pIL} \alpha \to \alpha}{\vdash_{pIL} (\alpha \to \alpha) \to \alpha \to \alpha}\,(\to I) \quad \dfrac{\{\alpha\} \vdash_{pIL} \alpha}{\vdash_{pIL} \alpha \to \alpha}\,(\to I)}{\Pi'_1 : \vdash_{pIL} \alpha \to \alpha}\,(\to E)$$

$$\dfrac{\{x : \alpha\} \vdash_{IT} x : \alpha}{\Pi_2 : \vdash_{IT} \lambda x.x : \alpha \to \alpha}\,(\to I) \qquad \dfrac{\{\alpha\} \vdash_{pIL} \alpha}{\Pi'_2 : \vdash_{pIL} \alpha \to \alpha}\,(\to I)$$

Note that Π'_1 derives from Π'_0 by one $\hookrightarrow_\to$-step, two $\hookrightarrow_\cap$-steps and four $\hookrightarrow_P$-steps, while Π'_2 derives from Π'_1 by one $\hookrightarrow_\to$-step. The term $\lambda x.x$ is in normal form and Π'_2 is normal. The procedure stops there.

Chapter 4

Intersection Synchronous Logic (ISL)

4.1 Definition of ISL

In this section, we present the logical system ISL as defined by E. Pimentel in [PR05]. It is given including three connectives: implication, intersection and conjunction. We define its basic building blocks (atoms, molecules) and exhibit its deductive rules. We then restrict it to implication and intersection and display a decoration of its deductions with untyped λ-terms that encode the implication only.

Definition 4.1.1 (ISL) *(i) The set of formulas $\mathcal{F}_{ISL}$ of ISL is generated by the grammar: $\sigma ::= \alpha \mid \sigma \to \sigma \mid \sigma \cap \sigma \mid \sigma \wedge \sigma$, where α belongs to a denumerable set of propositional variables.*

(ii) An atom *is a pair $(\Gamma; \sigma)$, where the* ISL-context *Γ is a finite sequence of formulas and σ is a formula.*

(iii) A molecule *is a finite multiset of atoms, such that the contexts in all atoms have the same cardinality; $[\mathcal{A}_1, \ldots, \mathcal{A}_n]$ denotes a molecule consisting of the atoms $\mathcal{A}_1, \ldots, \mathcal{A}_n$. Letters $\mathcal{M}, \mathcal{N}$ will range over molecules.*

(iv) The logical system ISL derives molecules. Its rules are shown in Figure 4.1. Writing $\Pi : \mathcal{M}$ means that the ISL-deduction Π concludes by proving $\mathcal{M}$. Writing $\vdash_{ISL} \mathcal{M}$ denotes the existence of an ISL-deduction $\Pi : \mathcal{M}$.

Remark 4.1.2 (i) Formulas of ISL are formulas (types) of LJr, i.e. $\mathcal{F}_{ISL} = \mathcal{F}_{LJr}$. (ii) In the rule (P_{ISL}), $\cup$ is the multiset union. (iii) The rules $(\to I), (\to E)$ and $(\wedge I), (\wedge E)$ are *global* rules, in the sense that they affect all the atoms of the involved molecules, while $(\cap I), (\cap E)$ are *local*, since they modify only particular atoms of the premise.

Example 4.1.3 Let $\rho \equiv \alpha \to \alpha$, $\sigma \equiv (\rho \to \rho) \wedge \rho$ and $\tau \equiv (\rho \to \rho) \cap \rho$. The LJr-deductions Π_1 and Π_2 of example 2.4.4 can be developed inside ISL as $\hat{\Pi}_1$ and $\hat{\Pi}_2$, respectively, without the need of λ-terms.

$$\frac{}{[(\sigma_i;\sigma_i) \mid 1 \le i \le n]}\,(A_{ISL}) \qquad \frac{\mathcal{M} \cup \mathcal{N}}{\mathcal{M}}\,(P_{ISL})$$

$$\frac{[(\Gamma_i;\sigma_i) \mid 1 \le i \le n]}{[(\Gamma_i,\tau_i;\sigma_i) \mid 1 \le i \le n]}\,(W_{ISL}) \qquad \frac{[(\Gamma_1^i,\tau_i,\rho_i,\Gamma_2^i;\sigma_i) \mid 1 \le i \le n]}{[(\Gamma_1^i,\rho_i,\tau_i,\Gamma_2^i;\sigma_i) \mid 1 \le i \le n]}\,(X_{ISL})$$

$$\frac{[(\Gamma_i,\sigma_i;\tau_i) \mid 1 \le i \le n]}{[(\Gamma_i;\sigma_i \to \tau_i) \mid 1 \le i \le n]}\,(\to I_{ISL})$$

$$\frac{[(\Gamma_i;\sigma_i \to \tau_i) \mid 1 \le i \le n] \qquad [(\Gamma_i;\sigma_i) \mid 1 \le i \le n]}{[(\Gamma_i;\tau_i) \mid 1 \le i \le n]}\,(\to E_{ISL})$$

$$\frac{\mathcal{M} \cup [(\Gamma;\sigma),(\Gamma;\tau)]}{\mathcal{M} \cup [(\Gamma;\sigma \cap \tau)]}\,(\cap I_{ISL}) \qquad \frac{\mathcal{M} \cup [(\Gamma;\sigma_l \cap \sigma_r)]}{\mathcal{M} \cup [(\Gamma;\sigma_s)]}\,(\cap E^s_{ISL},\ s \in \{l,r\})$$

$$\frac{[(\Gamma_i;\sigma_i) \mid 1 \le i \le n] \qquad [(\Gamma_i;\tau_i) \mid 1 \le i \le n]}{[(\Gamma_i;\sigma_i \wedge \tau_i) \mid 1 \le i \le n]}\,(\wedge I_{ISL})$$

$$\frac{[(\Gamma_i;\sigma_i \wedge \tau_i) \mid 1 \le i \le n]}{[(\Gamma_i;\sigma_i) \mid 1 \le i \le n]}\,(\wedge E^l_{ISL}) \qquad \frac{[(\Gamma_i;\sigma_i \wedge \tau_i) \mid 1 \le i \le n]}{[(\Gamma_i;\tau_i) \mid 1 \le i \le n]}\,(\wedge E^r_{ISL})$$

Figure 4.1: The rules of ISL.

$$\dfrac{\dfrac{\dfrac{\dfrac{[(\sigma;\sigma)]}{[(\sigma;\rho\to\rho)]}(\wedge E^l)\quad\dfrac{[(\sigma;\sigma)]}{[(\sigma;\rho)]}(\wedge E^r)}{[(\sigma;\rho)]}(\to E)}{[(\emptyset;\sigma\to\rho)]}(\to I)\qquad\dfrac{\dfrac{[(\rho;\rho)]}{[(\emptyset;\rho\to\rho)]}(\to I)\quad\dfrac{[(\alpha;\alpha)]}{[(\emptyset;\rho)]}(\to I)}{[(\emptyset;\sigma)]}(\wedge I)}{\hat{\Pi}_1:[(\emptyset;\rho)]}(\to E)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[(\tau;\tau)]}{[(\tau;\rho\to\rho)]}(\cap E^l)\quad\dfrac{[(\tau;\tau)]}{[(\tau;\rho)]}(\cap E^r)}{[(\tau;\rho)]}(\to E)}{[(\emptyset;\tau\to\rho)]}(\to I)\qquad\dfrac{\dfrac{[(\rho;\rho),(\alpha;\alpha)]}{[(\emptyset;\rho\to\rho),(\emptyset;\rho)]}(\to I)}{[(\emptyset;\tau)]}(\cap I)}{\hat{\Pi}_2:[(\emptyset;\rho)]}(\to E)$$

Let us now restrict ISL to $\to$ and $\cap$. We follow [PR05] in presenting a non-standard decoration of ISL-deductions, denoted $\mathsf{sd}_\to$ in this thesis, which encodes the implicative rules only. This decoration will be used in the following sections to relate ISL with LJ and IT.

Definition 4.1.4 ($\mathsf{sd}_\to$: non-standard decoration of ISL) *(i) Consider an ISL-context $\Gamma \equiv \sigma_1,\dots,\sigma_m$. A decoration $(\Gamma)^s$ of Γ with respect to a sequence $s \equiv x_1,\dots,x_m$ of distinct λ-variables is a sequence of assignments $x_1:\sigma_1,\dots,x_m:\sigma_m$.*

(ii) Every $\Pi : \mathcal{M} \equiv [(\sigma_1^i,\dots,\sigma_m^i;\tau_i) \mid 1\leqslant i\leqslant n]$ can be associated through an inductive algorithm to a decorated deduction

$$\vdash^\star_{ISL} M_{x_1,\dots,x_m}(\Pi) : (\mathcal{M})_{x_1,\dots,x_m} \equiv [(x_1:\sigma_1^i,\dots,x_m:\sigma_m^i;\tau_i)\mid 1\leqslant i\leqslant n]$$

assigning the decorated molecule $(\mathcal{M})_s \equiv [((\Gamma_i)^s;\tau_i)\mid 1\leqslant i\leqslant n]$ to the λ-term $M_s(\Pi)$ in Λ, where $\Gamma_i \equiv \sigma_1^i,\dots,\sigma_m^i$ and $s\equiv x_1,\dots,x_m$ is a sequence of distinct variables.

- $\dfrac{}{\Pi:[(\sigma_i;\sigma_i)\mid 1\leqslant i\leqslant n]}(A_{ISL}) \;\Rightarrow\; \dfrac{}{\vdash^\star_{ISL} x:[(x:\sigma_i;\sigma_i)\mid 1\leqslant i\leqslant n]}(A^\star_{ISL})$

  and $M_x(\Pi)\equiv x$.

- $\dfrac{\Pi_1:[(\Gamma_i;\sigma_i)\mid 1\leqslant i\leqslant n]}{\Pi:[(\Gamma_i,\tau_i;\sigma_i)\mid 1\leqslant i\leqslant n]}(W_{ISL}) \;\Rightarrow$

  $$\dfrac{\vdash^\star_{ISL} M_s(\Pi_1):[((\Gamma_i)^s;\sigma_i)\mid 1\leqslant i\leqslant n] \quad x\notin s}{\vdash^\star_{ISL} M_{s,x}(\Pi)\equiv M_s(\Pi_1):[((\Gamma_i)^s,x:\tau_i;\sigma_i)\mid 1\leqslant i\leqslant n]}(W^\star_{ISL})$$

- $\dfrac{\Pi_1:[(\Gamma_1^i,\tau_i,\rho_i,\Gamma_2^i;\sigma_i)\mid 1\leqslant i\leqslant n]}{\Pi:[(\Gamma_1^i,\rho_i,\tau_i,\Gamma_2^i;\sigma_i)\mid 1\leqslant i\leqslant n]}(X_{ISL}) \;\Rightarrow$

  $$\dfrac{\vdash^\star_{ISL} M_{s_1,y,x,s_2}(\Pi_1):[((\Gamma_1^i)^{s_1},y:\tau_i,x:\rho_i,(\Gamma_2^i)^{s_2};\sigma_i)\mid 1\leqslant i\leqslant n]}{\vdash^\star_{ISL} M_{s_1,x,y,s_2}(\Pi):[((\Gamma_1^i)^{s_1},x:\rho_i,y:\tau_i,(\Gamma_2^i)^{s_2};\sigma_i)\mid 1\leqslant i\leqslant n]}(X^\star_{ISL})$$

  where $M_{s_1,x,y,s_2}(\Pi)\equiv M_{s_1,y,x,s_2}(\Pi_1)$.

- $$\dfrac{\Pi_1 : [(\Gamma_i, \sigma_i; \tau_i) \mid 1 \leqslant i \leqslant n]}{\Pi : [(\Gamma_i; \sigma_i \to \tau_i) \mid 1 \leqslant i \leqslant n]} \, (\to I_{ISL}) \quad \Rightarrow$$

$$\dfrac{\vdash^{\star}_{ISL} M_{s,x}(\Pi_1) : [((\Gamma_i)^s, x : \sigma_i \,; \tau_i) \mid 1 \leqslant i \leqslant n]}{\vdash^{\star}_{ISL} M_s(\Pi) \equiv \lambda x.M_{s,x}(\Pi_1) : [((\Gamma_i)^s; \sigma_i \to \tau_i) \mid 1 \leqslant i \leqslant n]} \, (\to I^{\star}_{ISL})$$

- $$\dfrac{\Pi_1 : [(\Gamma_i; \sigma_i \to \tau_i) \mid 1 \leqslant i \leqslant n] \qquad \Pi_2 : [(\Gamma_i; \sigma_i) \mid 1 \leqslant i \leqslant n]}{\Pi : [(\Gamma_i; \tau_i) \mid 1 \leqslant i \leqslant n]} \, (\to E_{ISL}) \quad \Rightarrow$$

$$\dfrac{\vdash^{\star}_{ISL} M_1 : [((\Gamma_i)^s; \sigma_i \to \tau_i) \mid 1 \leqslant i \leqslant n] \quad \vdash^{\star}_{ISL} M_2 : [((\Gamma_i)^s; \sigma_i) \mid 1 \leqslant i \leqslant n]}{\vdash^{\star}_{ISL} M_s(\Pi) \equiv M_1 M_2 : [((\Gamma_i)^s; \tau_i) \mid 1 \leqslant i \leqslant n]} \, (\to E^{\star}_{ISL})$$

where $M_1 \equiv M_s(\Pi_1)$ *and* $M_2 \equiv M_s(\Pi_2)$.

- $$\dfrac{\Pi_1 : \mathcal{M}_1}{\Pi : \mathcal{M}_2} \, (R_{ISL}) \quad \Rightarrow \quad \dfrac{\vdash^{\star}_{ISL} M_s(\Pi_1) : (\mathcal{M}_1)_s}{\vdash^{\star}_{ISL} M_s(\Pi) \equiv M_s(\Pi_1) : (\mathcal{M}_2)_s} \, (R^{\star}_{ISL})$$

where $R \in \{(P), (\cap I), (\cap E^l), (\cap E^r)\}$.

4.2 ISL and LJ

In this section, we work with ISL$\lceil\{\to, \cap\}$. We survey the relation between ISL-deductions and LJ-deductions in $D^{\to}_{LJ}$. ISL, as IL, is defined to realize the part of LJ decoratable by $\mathsf{d}_{\to}$.

Remark 4.2.1 Let $\Pi : \{\sigma_1, \ldots, \sigma_m\} \vdash_{LJ} \tau$ be a LJ-deduction in $D^{\to}_{LJ}$. Then Π can be decorated by $\mathsf{d}_{\to}$ to give $\Pi^* : \{x_1 : \sigma_1, \ldots, x_m : \sigma_m\} \vdash^*_{LJ} T_{x_1,\ldots,x_m}(\Pi) : \tau$, where, for every sequence $x_1, \ldots, x_m$ of distinct variables, $T_{x_1,\ldots,x_m}(\Pi)$ is a set of α-equivalent λ-terms. In what follows, though, we identify α-equivalent λ-terms, so that $T_{x_1,\ldots,x_m}(\Pi)$ is a λ-term, for every $x_1, \ldots, x_m$.

Theorem 4.2.2 (From ISL to LJ) *Let* $\Pi : \mathcal{M} \equiv [(\sigma^i_1, \ldots, \sigma^i_m; \tau_i) \mid 1 \leqslant i \leqslant n]$. *For every* $i \in \{1, \ldots, n\}$, *we have that* $\Pi^i : \{e(\sigma^i_1), \ldots, e(\sigma^i_m)\} \vdash_{LJ} e(\tau_i)$ *is a LJ-deduction in* $D^{\to}_{LJ}$, *such that* $T_{x_1,\ldots,x_m}(\Pi^i) \equiv M_{x_1,\ldots,x_m}(\Pi)$, *for every sequence of distinct variables* $x_1, \ldots, x_m$.

Proof: By induction on Π.

Base: For $\Pi : \mathcal{M} \equiv [(\tau_i; \tau_i) \mid 1 \leqslant i \leqslant n]$ an ISL-axiom and $i \in \{1, \ldots, n\}$, we have that $\Pi^i : \{e(\tau_i)\} \vdash_{LJ} e(\tau_i)$ is an axiom of LJ—hence in $D^{\to}_{LJ}$—with $T_x(\Pi^i) \equiv x \equiv M_x(\Pi)$, for every x.

Inductive step: We examine all ISL-rules.

- $$\dfrac{\Pi_1 : [(\sigma^i_1, \ldots, \sigma^i_m; \tau_i) \mid 1 \leqslant i \leqslant n] \cup [(\sigma^i_1, \ldots, \sigma^i_m; \tau_i) \mid n+1 \leqslant i \leqslant n+k]}{\Pi : [(\sigma^i_1, \ldots, \sigma^i_m; \tau_i) \mid 1 \leqslant i \leqslant n]} \, (P_{ISL})$$

By the IH, for every $i \in \{1, \ldots, n+k\}$, we have that

$$\Pi^i_1 : \{e(\sigma^i_1), \ldots, e(\sigma^i_m)\} \vdash_{LJ} e(\tau_i)$$

is a LJ-deduction in $D_{LJ}^{\rightarrow}$, such that $T_{x_1,\ldots,x_m}(\Pi_1^i) \equiv M_{x_1,\ldots,x_m}(\Pi_1)$, for every $x_1,\ldots,x_m$. Hence, the same holds for every $i \in \{1,\ldots,n\}$. But then $\Pi_1^i \equiv \Pi^i$ and $T_{x_1,\ldots,x_m}(\Pi_1^i) \equiv T_{x_1,\ldots,x_m}(\Pi^i)$. We also have that $M_{x_1,\ldots,x_m}(\Pi_1) \equiv M_{x_1,\ldots,x_m}(\Pi)$ and the required result follows.

- $\dfrac{\Pi_1 : [(\sigma_1^i,\ldots,\sigma_m^i;\tau_i) \mid 1 \leqslant i \leqslant n]}{\Pi : [(\sigma_1^i,\ldots,\sigma_m^i,\rho_i;\tau_i) \mid 1 \leqslant i \leqslant n]}\ (W_{ISL})$

  By the IH, for every $i \in \{1,\ldots,n\}$, we have that

$$\Pi_1^i : \{e(\sigma_1^i),\ldots,e(\sigma_m^i)\} \vdash_{LJ} e(\tau_i)$$

is a LJ-deduction in $D_{LJ}^{\rightarrow}$, such that $T_{x_1,\ldots,x_m}(\Pi_1^i) \equiv M_{x_1,\ldots,x_m}(\Pi_1)$, for every $x_1,\ldots,x_m$. By the weakening property for LJ (see proposition 2.1.3), for every formula $e(\rho_i)$, there exists $\Pi^i : \{e(\sigma_1^i),\ldots,e(\sigma_m^i),e(\rho_i)\} \vdash_{LJ} e(\tau_i)$. Since Π_1^i is in $D_{LJ}^{\rightarrow}$, Π^i is in $D_{LJ}^{\rightarrow}$ and furthermore $T_{x_1,\ldots,x_m,x}(\Pi^i) \equiv T_{x_1,\ldots,x_m}(\Pi_1^i)$, for every $x_1,\ldots,x_m$ and every $x \notin x_1,\ldots,x_m$. Hence, it is $T_{x_1,\ldots,x_m,x}(\Pi^i) \equiv M_{x_1,\ldots,x_m}(\Pi_1) \equiv M_{x_1,\ldots,x_m,x}(\Pi)$, for every $x_1,\ldots,x_m,x$.

- $\dfrac{\Pi_1 : [(\sigma_1^i,\ldots,\sigma_m^i,\tau_i,\rho_i,\upsilon_1^i,\ldots,\upsilon_l^i;\omega_i) \mid 1 \leqslant i \leqslant n]}{\Pi : [(\sigma_1^i,\ldots,\sigma_m^i,\rho_i,\tau_i,\upsilon_1^i,\ldots,\upsilon_l^i;\omega_i) \mid 1 \leqslant i \leqslant n]}\ (X_{ISL})$

  By the IH, for every $i \in \{1,\ldots,n\}$, we have that

$$\Pi_1^i : \{e(\sigma_1^i),\ldots,e(\sigma_m^i),e(\tau_i),e(\rho_i),e(\upsilon_1^i),\ldots,e(\upsilon_l^i)\} \vdash_{LJ} e(\omega_i)$$

is a LJ-deduction in $D_{LJ}^{\rightarrow}$, s.t. $T_{x_1,\ldots,x_m,y,x,y_1,\ldots,y_l}(\Pi_1^i) \equiv M_{x_1,\ldots,x_m,y,x,y_1,\ldots,y_l}(\Pi_1)$, for every $(x_1,\ldots,x_m,y,x,y_1,\ldots,y_l)$. But $\{e(\sigma_1^i),\ldots,e(\sigma_m^i),e(\tau_i),e(\rho_i),e(\upsilon_1^i),\ldots,e(\upsilon_l^i)\} = \{e(\sigma_1^i),\ldots,e(\sigma_m^i),e(\rho_i),e(\tau_i),e(\upsilon_1^i),\ldots,e(\upsilon_l^i)\}$, so $\Pi_1^i \equiv \Pi^i$ and $T_{x_1,\ldots,x_m,x,y,y_1,\ldots,y_l}(\Pi^i) \equiv T_{x_1,\ldots,x_m,y,x,y_1,\ldots,y_l}(\Pi_1^i) \equiv M_{x_1,\ldots,x_m,y,x,y_1,\ldots,y_l}(\Pi_1) \equiv M_{x_1,\ldots,x_m,x,y,y_1,\ldots,y_l}(\Pi)$, for every $(x_1,\ldots,x_m,x,y,y_1,\ldots,y_l)$.

- $\dfrac{\Pi_1 : [(\sigma_1^i,\ldots,\sigma_m^i,\tau_i;\rho_i) \mid 1 \leqslant i \leqslant n]}{\Pi : [(\sigma_1^i,\ldots,\sigma_m^i;\tau_i \rightarrow \rho_i) \mid 1 \leqslant i \leqslant n]}\ (\rightarrow I_{ISL})$

  By the IH, for every $i \in \{1,\ldots,n\}$, we have that

$$\Pi_1^i : \{e(\sigma_1^i),\ldots,e(\sigma_m^i),e(\tau_i)\} \vdash_{LJ} e(\rho_i)$$

is a LJ-deduction in $D_{LJ}^{\rightarrow}$, such that $T_{x_1,\ldots,x_m,x}(\Pi_1^i) \equiv M_{x_1,\ldots,x_m,x}(\Pi_1)$, for every $x_1,\ldots,x_m,x$. Applying $(\rightarrow I_{LJ})$ on Π_1^i, we get

$$\Pi^i : \{e(\sigma_1^i),\ldots,e(\sigma_m^i)\} \vdash_{LJ} e(\tau_i) \rightarrow e(\rho_i) \equiv e(\tau_i \rightarrow \rho_i)$$

which is in $D_{LJ}^{\rightarrow}$, since Π_1^i is in $D_{LJ}^{\rightarrow}$ and Π^i results from Π_1^i by $(\rightarrow I)$. We also have that $T_{x_1,\ldots,x_m}(\Pi^i) \equiv \lambda x.T_{x_1,\ldots,x_m,x}(\Pi_1^i) \equiv \lambda x.M_{x_1,\ldots,x_m,x}(\Pi_1) \equiv M_{x_1,\ldots,x_m}(\Pi)$, for every $x_1,\ldots,x_m$.

- $\dfrac{\Pi_1 : [(\sigma_1^i,\ldots,\sigma_m^i;\tau_i \rightarrow \rho_i) \mid 1 \leqslant i \leqslant n] \qquad \Pi_2 : [(\sigma_1^i,\ldots,\sigma_m^i;\tau_i) \mid 1 \leqslant i \leqslant n]}{\Pi : [(\sigma_1^i,\ldots,\sigma_m^i;\rho_i) \mid 1 \leqslant i \leqslant n]}\ (\rightarrow E_{ISL})$

  By the IH, for every $i \in \{1,\ldots,n\}$, we have that

$$\Pi_1^i : \{e(\sigma_1^i),\ldots,e(\sigma_m^i)\} \vdash_{LJ} e(\tau_i \rightarrow \rho_i) \equiv e(\tau_i) \rightarrow e(\rho_i)$$

$$\Pi_2^i : \{e(\sigma_1^i), \ldots, e(\sigma_m^i)\} \vdash_{LJ} e(\tau_i)$$

are LJ-deductions in $D_{LJ}^{\rightarrow}$, such that $T_{x_1,\ldots,x_m}(\Pi_1^i) \equiv M_{x_1,\ldots,x_m}(\Pi_1)$ and $T_{x_1,\ldots,x_m}(\Pi_2^i) \equiv M_{x_1,\ldots,x_m}(\Pi_2)$, for every $x_1, \ldots, x_m$. Applying $(\rightarrow E_{LJ})$ on Π_1^i, Π_2^i, we get

$$\Pi^i : \{e(\sigma_1^i), \ldots, e(\sigma_m^i)\} \vdash_{LJ} e(\rho_i)$$

which is in $D_{LJ}^{\rightarrow}$, since Π_1^i, Π_2^i are in $D_{LJ}^{\rightarrow}$ and Π^i results from them by $(\rightarrow E)$. We also have $T_{x_1,\ldots,x_m}(\Pi^i) \equiv T_{x_1,\ldots,x_m}(\Pi_1^i)T_{x_1,\ldots,x_m}(\Pi_2^i) \equiv M_{x_1,\ldots,x_m}(\Pi_1)M_{x_1,\ldots,x_m}(\Pi_2) \equiv M_{x_1,\ldots,x_m}(\Pi)$, for every $x_1, \ldots, x_m$.

- $$\frac{\Pi_1 : [(\sigma_1^i, \ldots, \sigma_m^i; \tau_i) \mid 1 \leqslant i \leqslant n] \cup [(\sigma_1^{n+1}, \ldots, \sigma_m^{n+1}; \tau_{n+1}), (\sigma_1^{n+1}, \ldots, \sigma_m^{n+1}; \tau_{n+2})]}{\Pi : [(\sigma_1^i, \ldots, \sigma_m^i; \tau_i) \mid 1 \leqslant i \leqslant n] \cup [(\sigma_1^{n+1}, \ldots, \sigma_m^{n+1}; \tau_{n+1} \cap \tau_{n+2})]} (\cap I)$$

  By the IH, for every $i \in \{1, \ldots, n\}$, we have that

$$\Pi_1^i : \{e(\sigma_1^i), \ldots, e(\sigma_m^i)\} \vdash_{LJ} e(\tau_i)$$

is a LJ-deduction in $D_{LJ}^{\rightarrow}$, such that $T_{x_1,\ldots,x_m}(\Pi_1^i) \equiv M_{x_1,\ldots,x_m}(\Pi_1)$, for every $x_1, \ldots, x_m$. We also have that

$$\Pi_1^{n+1} : \{e(\sigma_1^{n+1}), \ldots, e(\sigma_m^{n+1})\} \vdash_{LJ} e(\tau_{n+1})$$

$$\Pi_1^{n+2} : \{e(\sigma_1^{n+1}), \ldots, e(\sigma_m^{n+1})\} \vdash_{LJ} e(\tau_{n+2})$$

are LJ-deductions in $D_{LJ}^{\rightarrow}$, such that $T_{x_1,\ldots,x_m}(\Pi_1^{n+1}) \equiv M_{x_1,\ldots,x_m}(\Pi_1) \equiv T_{x_1,\ldots,x_m}(\Pi_1^{n+2})$, for every $x_1, \ldots, x_m$.

So, for every $i \in \{1, \ldots, n\}$, $\Pi^i \equiv \Pi_1^i$ and $T_{x_1,\ldots,x_m}(\Pi^i) \equiv T_{x_1,\ldots,x_m}(\Pi_1^i) \equiv M_{x_1,\ldots,x_m}(\Pi_1) \equiv M_{x_1,\ldots,x_m}(\Pi)$, for every $x_1, \ldots, x_m$.

Applying $(\wedge I_{LJ})$ on Π_1^{n+1}, Π_1^{n+2}, we get

$$\Pi^{n+1} : \{e(\sigma_1^{n+1}), \ldots, e(\sigma_m^{n+1})\} \vdash_{LJ} e(\tau_{n+1}) \wedge e(\tau_{n+2}) \equiv e(\tau_{n+1} \cap \tau_{n+2})$$

which is in $D_{LJ}^{\rightarrow}$, since Π_1^{n+1}, Π_1^{n+2} are in $D_{LJ}^{\rightarrow}$ and $T_{x_1,\ldots,x_m}(\Pi_1^{n+1}) \equiv T_{x_1,\ldots,x_m}(\Pi_1^{n+2})$, for every $x_1, \ldots, x_m$. Furthermore, $T_{x_1,\ldots,x_m}(\Pi^{n+1}) \equiv T_{x_1,\ldots,x_m}(\Pi_1^{n+1}) \equiv M_{x_1,\ldots,x_m}(\Pi_1) \equiv M_{x_1,\ldots,x_m}(\Pi)$, for every $x_1, \ldots, x_m$.

- $$\frac{\Pi_1 : [(\sigma_1^i, \ldots, \sigma_m^i; \tau_i) \mid 1 \leqslant i \leqslant n] \cup [(\sigma_1^{n+1}, \ldots, \sigma_m^{n+1}; \tau_{n+1}^l \cap \tau_{n+1}^r)]}{\Pi : [(\sigma_1^i, \ldots, \sigma_m^i; \tau_i) \mid 1 \leqslant i \leqslant n] \cup [(\sigma_1^{n+1}, \ldots, \sigma_m^{n+1}; \tau_{n+1}^s)]} (\cap E_{ISL}^s, s \in \{l, r\})$$

  For $i \in \{1, \ldots, n\}$, we work as in the case of $(\cap I_{ISL})$.

  By the IH, we have that

$$\Pi_1^{n+1} : \{e(\sigma_1^{n+1}), \ldots, e(\sigma_m^{n+1})\} \vdash_{LJ} e(\tau_{n+1}^l \cap \tau_{n+1}^r) \equiv e(\tau_{n+1}^l) \wedge e(\tau_{n+1}^r)$$

is a LJ-deduction in $D_{LJ}^{\rightarrow}$, such that $T_{x_1,\ldots,x_m}(\Pi_1^{n+1}) \equiv M_{x_1,\ldots,x_m}(\Pi_1)$, for every $x_1, \ldots, x_m$. Applying $(\wedge E_{LJ}^s)$ on Π_1^{n+1}, we get

$$\Pi^{n+1} : \{e(\sigma_1^{n+1}), \ldots, e(\sigma_m^{n+1})\} \vdash_{LJ} e(\tau_{n+1}^s)$$

which is in $D_{LJ}^{\rightarrow}$, since Π_1^{n+1} is in $D_{LJ}^{\rightarrow}$ and Π^{n+1} follows from it by $(\wedge E)$. It is $T_{x_1,\ldots,x_m}(\Pi^{n+1}) \equiv T_{x_1,\ldots,x_m}(\Pi_1^{n+1}) \equiv M_{x_1,\ldots,x_m}(\Pi_1) \equiv M_{x_1,\ldots,x_m}(\Pi)$, for every $x_1, \ldots, x_m$. $\dashv$

Theorem 4.2.3 (From LJ to ISL) *Let* $m \geqslant 0,\ n \geqslant 1$ *and*

$$\Pi_1 : \{\sigma_{11}, \ldots, \sigma_{m1}\} \vdash_{LJ} \tau_1, \Pi_2 : \{\sigma_{12}, \ldots, \sigma_{m2}\} \vdash_{LJ} \tau_2, \ldots, \Pi_n : \{\sigma_{1n}, \ldots, \sigma_{mn}\} \vdash_{LJ} \tau_n$$

be LJ-deductions in $D_{LJ}^{\rightarrow}$*, such that* $T_{x_1,\ldots,x_m}(\Pi_1) \equiv T_{x_1,\ldots,x_m}(\Pi_2) \equiv \ldots \equiv T_{x_1,\ldots,x_m}(\Pi_n)$*, for every* $x_1, \ldots, x_m$*. Then, there exists*

$$\Pi : [(f(\sigma_{11}), \ldots, f(\sigma_{m1}); f(\tau_1)), \ldots, (f(\sigma_{1n}), \ldots, f(\sigma_{mn}); f(\tau_n))]$$

such that $M_{x_1,\ldots,x_m}(\Pi) \equiv T_{x_1,\ldots,x_m}(\Pi_1)$*, for every* $x_1, \ldots, x_m$*.*

Proof: By induction on Π_1. For convenience, we prove the case with two LJ-deductions Π_1 and Π_2. We note, though, that the proof should be formally given for a n-tuple of LJ-deductions $\Pi_1, \ldots, \Pi_n$, since the inductive hypothesis is applied to more than two deductions.

Base: Suppose that $\Pi_1 : \{\sigma_{11}, \ldots, \sigma_{m1}\} \vdash_{LJ} \tau_1 \equiv \sigma_{11}$ is a LJ-axiom. Then, since $T_{x_1,\ldots,x_m}(\Pi_2) \equiv T_{x_1,\ldots,x_m}(\Pi_1)$, for every $x_1, \ldots, x_m$, the judgement proved by Π_2 derives from axioms of the form $\{\sigma_{12}, \ldots, \sigma_{m2}\} \vdash_{LJ} \sigma_{12}$ by a finite number of applications of the rules $(\wedge I), (\wedge E)$. If the number of $(\wedge I)$ instances in Π_2 is $k-1$, where $k \geqslant 1$, then Π_2 involves k axioms

$$\Pi_{21} : \{\sigma_{12}, \ldots, \sigma_{m2}\} \vdash_{LJ} \sigma_{12}, \ldots, \Pi_{2k} : \{\sigma_{12}, \ldots, \sigma_{m2}\} \vdash_{LJ} \sigma_{12}$$

Let $\sigma'_{ij} \equiv f(\sigma_{ij})$ $(1 \leqslant i \leqslant m,\ j = 1, 2)$. Then

$$\Pi'' : [(\sigma'_{11}; \sigma'_{11}), (\sigma'_{12}; \sigma'_{12}), \ldots, (\sigma'_{12}; \sigma'_{12})]$$

with k atoms $(\sigma'_{12}; \sigma'_{12})$ is an ISL-axiom from which we get

$$\Pi' : [(\sigma'_{11}, \ldots, \sigma'_{m1}; \sigma'_{11}), (\sigma'_{12}, \ldots, \sigma'_{m2}; \sigma'_{12}), \ldots, (\sigma'_{12}, \ldots, \sigma'_{m2}; \sigma'_{12})]$$

by $m-1$ applications of (W_{ISL}). It is $M_{x_1,\ldots,x_m}(\Pi') \equiv x_1 \equiv T_{x_1,\ldots,x_m}(\Pi_1)$, for every $x_1, \ldots, x_m$. Applying $(\cap I_{ISL})$ on appropriate atoms for each $(\wedge I)$ in Π_2 and $(\cap E^s_{ISL})$ for each $(\wedge E^s)$, where $s \in \{l, r\}$, we get

$$\Pi : [(\sigma'_{11}, \ldots, \sigma'_{m1}; \sigma'_{11} \equiv \tau'_1), (\sigma'_{12}, \ldots, \sigma'_{m2}; \tau'_2)]$$

with $M_{x_1,\ldots,x_m}(\Pi) \equiv M_{x_1,\ldots,x_m}(\Pi') \equiv T_{x_1,\ldots,x_m}(\Pi_1)$, for every $x_1, \ldots, x_m$.

Inductive step: We show the most characteristic cases.

- $$\dfrac{\Pi_{01} : \{\sigma_{11}, \ldots, \sigma_{m1}\} \vdash_{LJ} \tau_{01} \qquad \Pi_{11} : \{\sigma_{11}, \ldots, \sigma_{m1}\} \vdash_{LJ} \tau_{11}}{\Pi_1 : \{\sigma_{11}, \ldots, \sigma_{m1}\} \vdash_{LJ} \tau_{01} \wedge \tau_{11} \equiv \tau_1}\ (\wedge I_{LJ})$$

Since Π_1 is in $D_{LJ}^{\rightarrow}$, Π_{01} and Π_{11} are in $D_{LJ}^{\rightarrow}$ and $T_{x_1,\ldots,x_m}(\Pi_{01}) \equiv T_{x_1,\ldots,x_m}(\Pi_{11}) \equiv T_{x_1,\ldots,x_m}(\Pi_1)$, for every $x_1, \ldots, x_m$. Consequently, Π_{01}, Π_{11} and Π_2 are all in $D_{LJ}^{\rightarrow}$ and $T_{x_1,\ldots,x_m}(\Pi_{01}) \equiv T_{x_1,\ldots,x_m}(\Pi_{11}) \equiv T_{x_1,\ldots,x_m}(\Pi_2)$, for every $x_1, \ldots, x_m$. Hence, by the IH, there exists

$$\Pi' : [(\sigma'_{11}, \ldots, \sigma'_{m1}; \tau'_{01}), (\sigma'_{11}, \ldots, \sigma'_{m1}; \tau'_{11}), (\sigma'_{12}, \ldots, \sigma'_{m2}; \tau'_2)]$$

such that $M_{x_1,\ldots,x_m}(\Pi') \equiv T_{x_1,\ldots,x_m}(\Pi_{01})$, for every $x_1, \ldots, x_m$. Applying $(\cap I_{ISL})$, we get

$$\Pi : [(\sigma'_{11}, \ldots, \sigma'_{m1}; \tau'_{01} \cap \tau'_{11} \equiv \tau'_1), (\sigma'_{12}, \ldots, \sigma'_{m2}; \tau'_2)]$$

with $M_{x_1,\ldots,x_m}(\Pi) \equiv M_{x_1,\ldots,x_m}(\Pi') \equiv T_{x_1,\ldots,x_m}(\Pi_1)$, for every $x_1, \ldots, x_m$.

- $$\dfrac{\Pi_1' : \{\sigma_{11}, \ldots, \sigma_{m1}\} \vdash_{LJ} \tau_1 \wedge \rho_1}{\Pi_1 : \{\sigma_{11}, \ldots, \sigma_{m1}\} \vdash_{LJ} \tau_1} \; (\wedge E^l_{LJ})$$

Since Π_1 is in $D_{LJ}^{\vec{}}$, Π_1' is in $D_{LJ}^{\vec{}}$ and $T_{x_1,\ldots,x_m}(\Pi_1') \equiv T_{x_1,\ldots,x_m}(\Pi_1)$, for every $x_1, \ldots, x_m$. So, Π_1' and Π_2 are in $D_{LJ}^{\vec{}}$ and $T_{x_1,\ldots,x_m}(\Pi_1') \equiv T_{x_1,\ldots,x_m}(\Pi_2)$, for every $x_1, \ldots, x_m$. Hence, by the IH, there exists

$$\Pi' : [(\sigma_{11}', \ldots, \sigma_{m1}'; (\tau_1 \wedge \rho_1)' \equiv \tau_1' \cap \rho_1'), (\sigma_{12}', \ldots, \sigma_{m2}'; \tau_2')]$$

such that $M_{x_1,\ldots,x_m}(\Pi') \equiv T_{x_1,\ldots,x_m}(\Pi_1')$, for every $x_1, \ldots, x_m$. Applying $(\cap E^l_{ISL})$, we get

$$\Pi : [(\sigma_{11}', \ldots, \sigma_{m1}'; \tau_1'), (\sigma_{12}', \ldots, \sigma_{m2}'; \tau_2')]$$

with $M_{x_1,\ldots,x_m}(\Pi) \equiv M_{x_1,\ldots,x_m}(\Pi') \equiv T_{x_1,\ldots,x_m}(\Pi_1)$, for every $x_1, \ldots, x_m$.

The case of $(\wedge E^r_{LJ})$ is analogous to $(\wedge E^l_{LJ})$.

- $$\dfrac{\Pi_1' : \{\sigma_{11}, \ldots, \sigma_{m1}, \rho_1\} \vdash_{LJ} \upsilon_1}{\Pi_1 : \{\sigma_{11}, \ldots, \sigma_{m1}\} \vdash_{LJ} \rho_1 \rightarrow \upsilon_1 \equiv \tau_1} \; (\rightarrow I_{LJ})$$

It is $T_{x_1,\ldots,x_m}(\Pi_1) \equiv \lambda x.T_{x_1,\ldots,x_m.x}(\Pi_1') \equiv T_{x_1,\ldots,x_m}(\Pi_2)$, for every $x_1, \ldots, x_m$. This means that the judgement proved by Π_2 derives, by $k-1$ ($k \geqslant 1$) applications of $(\wedge I)$ and a finite number of applications of $(\wedge E)$, from k deductions

$$\Pi_{21} : \{\sigma_{12}, \ldots, \sigma_{m2}\} \vdash_{LJ} \tau_{21}, \ldots, \Pi_{2k} : \{\sigma_{12}, \ldots, \sigma_{m2}\} \vdash_{LJ} \tau_{2k}$$

which, in turn, derive, by $(\rightarrow I)$, from

$$\Pi_{21}' : \{\sigma_{12}, \ldots, \sigma_{m2}, \rho_{21}\} \vdash_{LJ} \upsilon_{21}, \ldots, \Pi_{2k}' : \{\sigma_{12}, \ldots, \sigma_{m2}, \rho_{2k}\} \vdash_{LJ} \upsilon_{2k}$$

respectively. For each $j \in \{1, \ldots, k\}$, it is

$$\begin{array}{rcl} T_{x_1,\ldots,x_m}(\Pi_2) & \equiv & T_{x_1,\ldots,x_m}(\Pi_{2j}), \text{ for every } x_1, \ldots, x_m \Longrightarrow \\ \lambda x.T_{x_1,\ldots,x_m,x}(\Pi_1') & \equiv & \lambda x.T_{x_1,\ldots,x_m,x}(\Pi_{2j}'), \text{ for every } x_1, \ldots, x_m \\ & & \qquad\qquad \text{and every } x \notin x_1, \ldots, x_m \Longrightarrow \\ T_{x_1,\ldots,x_m,x}(\Pi_1') & \equiv & T_{x_1,\ldots,x_m,x}(\Pi_{2j}'), \text{ for every } x_1, \ldots, x_m, x \end{array}$$

Hence, by the IH, there exists

$$\Pi'' : [(\sigma_{11}', \ldots, \sigma_{m1}', \rho_1'; \upsilon_1'), (\sigma_{12}', \ldots, \sigma_{m2}', \rho_{21}'; \upsilon_{21}'), \ldots, (\sigma_{12}', \ldots, \sigma_{m2}', \rho_{2k}'; \upsilon_{2k}')]$$

such that $M_{x_1,\ldots,x_m,x}(\Pi'') \equiv T_{x_1,\ldots,x_m,x}(\Pi_1')$, for every $x_1, \ldots, x_m, x$. Applying $(\rightarrow I_{ISL})$, we get

$$\Pi' : [(\sigma_{11}', \ldots, \sigma_{m1}'; \tau_1'), (\sigma_{12}', \ldots, \sigma_{m2}'; \tau_{21}'), \ldots, (\sigma_{12}', \ldots, \sigma_{m2}'; \tau_{2k}')]$$

with $M_{x_1,\ldots,x_m}(\Pi') \equiv \lambda x.M_{x_1,\ldots,x_m,x}(\Pi'') \equiv \lambda x.T_{x_1,\ldots,x_m,x}(\Pi_1') \equiv T_{x_1,\ldots,x_m}(\Pi_1)$, for every $x_1, \ldots, x_m$. Finally, applying $(\cap I_{ISL})$ on appropriate atoms for each $(\wedge I)$ in Π_2 and $(\cap E^s_{ISL})$ for each $(\wedge E^s)$, where $s \in \{l, r\}$, we get

$$\Pi : [(\sigma_{11}', \ldots, \sigma_{m1}'; \tau_1'), (\sigma_{12}', \ldots, \sigma_{m2}'; \tau_2')]$$

with $M_{x_1,\ldots,x_m}(\Pi) \equiv M_{x_1,\ldots,x_m}(\Pi') \equiv T_{x_1,\ldots,x_m}(\Pi_1)$, for every $x_1, \ldots, x_m$.

The case of $(\rightarrow E_{LJ})$ is tackled in a similar manner. $\dashv$

Corollary 4.2.4 *(i) If* $\Pi : \mathcal{M} \equiv [(\sigma_1, \ldots, \sigma_m; \tau)]$ *is an ISL-deduction, there exists*

$$\Pi^1 : \{e(\sigma_1), \ldots, e(\sigma_m)\} \vdash_{LJ} e(\tau)$$

in $D^{\rightarrow}_{LJ}$*, such that* $T_{x_1,\ldots,x_m}(\Pi^1) \equiv M_{x_1,\ldots,x_m}(\Pi)$*, for every* $x_1, \ldots, x_m$*.*

(ii) If $\Pi_1 : \{\sigma_1, \ldots, \sigma_m\} \vdash_{LJ} \tau$ *is a LJ-deduction in* $D^{\rightarrow}_{LJ}$*, there exists*

$$\Pi : [(f(\sigma_1), \ldots, f(\sigma_m); f(\tau))]$$

such that $M_{x_1,\ldots,x_m}(\Pi) \equiv T_{x_1,\ldots,x_m}(\Pi_1)$*, for every* $x_1, \ldots, x_m$*.*

Proof: (i) Special case of theorem 4.2.2. For $n = 1$, $\mathcal{M}$ consists of a single atom.
(ii) Special case of theorem 4.2.3. For $n = 1$, we have a single LJ-deduction in $D^{\rightarrow}_{LJ}$.$\dashv$

4.3 ISL and IT

In this section, we continue to work with ISL$\restriction\{\rightarrow, \cap\}$. We prove a theorem relating ISL to IT, thanks to which ISL can be proposed as the logic for IT. We also discuss characteristic features of ISL, namely the importance of having explicit srtuctural rules and contexts defined as sequences of formulas.

Theorem 4.3.1 *(i) If* $\Pi : \mathcal{M} \equiv [(\sigma^i_1, \ldots, \sigma^i_m; \tau_i) \mid 1 \leqslant i \leqslant n]$*, then*

$$\{x_1 : \sigma^i_1, \ldots, x_m : \sigma^i_m\} \vdash_{IT} M_{x_1,\ldots,x_m}(\Pi) : \tau_i$$

for every $i \in \{1, \ldots, n\}$ *and every sequence* $x_1, \ldots, x_m$ *of distinct variables. So, for a single-atom molecule* $\mathcal{M} \equiv [(\sigma_1, \ldots, \sigma_m; \tau)]$*, we have that, if* $\Pi : \mathcal{M}$*, then*

$$\{x_1 : \sigma_1, \ldots, x_m : \sigma_m\} \vdash_{IT} M_{x_1,\ldots,x_m}(\Pi) : \tau$$

for every $x_1, \ldots, x_m$*.*

(ii) Suppose that $x_1, \ldots, x_m$ *is a fixed, but arbitrary sequence of distinct variables. If, for every* $i \in \{1, \ldots, n\}$*,* $\Pi_i : \{x_1 : \sigma^i_1, \ldots, x_m : \sigma^i_m\} \vdash_{IT} M : \tau_i$*, there exists* $\Pi' : [(\sigma^i_1, \ldots, \sigma^i_m; \tau_i) \mid 1 \leqslant i \leqslant n]$*, such that* $M_{x_1,\ldots,x_m}(\Pi') \equiv M$*. So, if* $\Pi : \{x_1 : \sigma_1, \ldots, x_m : \sigma_m\} \vdash_{IT} M : \tau$*, there exists* $\Pi' : \mathcal{M} \equiv [(\sigma_1, \ldots, \sigma_m; \tau)]$*, such that* $M_{x_1,\ldots,x_m}(\Pi') \equiv M$*.*

Proof: (i) By theorem 4.2.2, for every $i \in \{1, \ldots, n\}$, we have that

$$\Pi^i : \{e(\sigma^i_1), \ldots, e(\sigma^i_m)\} \vdash_{LJ} e(\tau_i)$$

is in $D^{\rightarrow}_{LJ}$ and such that $T_{x_1,\ldots,x_m}(\Pi^i) \equiv M_{x_1,\ldots,x_m}(\Pi)$, for every $x_1, \ldots, x_m$. So, for every $i \in \{1, \ldots, n\}$ and every $x_1, \ldots, x_m$, we have, by therem 2.6.2 , that

$$\{x_1 : f(e(\sigma^i_1)), \ldots, x_m : f(e(\sigma^i_m))\} \vdash_{IT} T_{x_1,\ldots,x_m}(\Pi^i) : f(e(\tau_i))$$

i.e. that $\{x_1 : \sigma^i_1, \ldots, x_m : \sigma^i_m\} \vdash_{IT} M_{x_1,\ldots,x_m}(\Pi) : \tau_i$.

(ii) By theorem 2.6.3, if we apply the erasing function E on Π_i, we get

$$E(\Pi_i) : \{e(\sigma^i_1), \ldots, e(\sigma^i_m)\} \vdash_{LJ} e(\tau_i)$$

which is in $D^{\rightarrow}_{LJ}$ and such that $T_{x_1,\ldots,x_m}(E(\Pi_i)) \equiv M$. So, we have n LJ-deductions $E(\Pi_1), \ldots, E(\Pi_n)$ in $D^{\rightarrow}_{LJ}$, such that $T_{x_1,\ldots,x_m}(E(\Pi_1)) \equiv \ldots \equiv T_{x_1,\ldots,x_m}(E(\Pi_n))$, for an arbitrary $x_1, \ldots, x_m$. Then, by theorem 4.2.3, there exists

$$\Pi' : [(f(e(\sigma^i_1)), \ldots, f(e(\sigma^i_m)); f(e(\tau_i))) \mid 1 \leqslant i \leqslant n]$$

i.e. $\Pi' : [(\sigma^i_1, \ldots, \sigma^i_m; \tau_i)]$, such that $M_{x_1,\ldots,x_m}(\Pi') \equiv T_{x_1,\ldots,x_m}(E(\Pi_1)) \equiv M$. $\dashv$

4.3.1 The role of sequences and structural rules

All systems introduced in chapter 2 are given in two equivalent versions: the set and sequence versions. The former defines contexts as sets and contains no structural rules, while the latter considers contexts as sequences and includes rules for context weakening and exchange. The logical system ISL$\restriction\{\to,\cap\}$ involves atoms whose contexts are sequences of formulas and includes the structural rules (W_{ISL}) and (X_{ISL}). Could we define equivalent set formulations of ISL? Could we exclude the structural rules?

Consider ISL$'$ with contexts defined as sets, the axiom (A_{ISL}) replaced by

$$[(\Gamma_i \cup \{\sigma_i\};\sigma_i) \mid 1 \leqslant i \leqslant n] \quad (A'_{ISL})$$

where, for every i, Γ_i is a set of formulas and the structural rules excluded. All other rules are included changing contexts from sequences to sets. Weakening is implicit in (A'_{ISL}), while exchange is no longer needed.

In ISL$'$, the molecules

$$\mathcal{M}_1 \equiv [(\emptyset;(\alpha\cap\beta\to\gamma)\to(\alpha\to\beta\to\gamma))]$$

$$\mathcal{M}_2 \equiv [(\emptyset;(\alpha\to\beta\to\gamma)\to(\alpha\cap\beta\to\gamma))]$$

can be proved, as shown below.

$$\dfrac{\dfrac{\dfrac{}{[(\{\alpha\cap\beta\to\gamma,\alpha,\beta\};\alpha\cap\beta\to\gamma)]}(A') \quad \dfrac{\dfrac{}{[(\{\alpha\cap\beta\to\gamma,\alpha,\beta\};\alpha),(\{\alpha\cap\beta\to\gamma,\alpha,\beta\};\beta)]}(A')}{[(\{\alpha\cap\beta\to\gamma,\alpha,\beta\};\alpha\cap\beta)]}(\cap I')}{[(\{\alpha\cap\beta\to\gamma,\alpha,\beta\};\gamma)]}(\to E')}{\Pi'_1 : \mathcal{M}_1 \equiv [(\emptyset;(\alpha\cap\beta\to\gamma)\to(\alpha\to\beta\to\gamma))]}(\to I')\times 3$$

$$\dfrac{\dfrac{}{[(\{\alpha\to\beta\to\gamma,\alpha\cap\beta\};\alpha\to\beta\to\gamma)]}(A') \quad \dfrac{\dfrac{}{[(\{\alpha\to\beta\to\gamma,\alpha\cap\beta\};\alpha\cap\beta)]}(A')}{[(\{\alpha\to\beta\to\gamma,\alpha\cap\beta\};\alpha)]}(\cap E'_l)}{\Pi'_{20} : [(\{\alpha\to\beta\to\gamma,\alpha\cap\beta\};\beta\to\gamma)]}(\to E')$$

$$\dfrac{\dfrac{\Pi'_{20} : [(\{\alpha\to\beta\to\gamma,\alpha\cap\beta\};\beta\to\gamma)] \quad \dfrac{\dfrac{}{[(\{\alpha\to\beta\to\gamma,\alpha\cap\beta\};\alpha\cap\beta)]}(A')}{\Pi'_{21} : [(\{\alpha\to\beta\to\gamma,\alpha\cap\beta\};\beta)]}(\cap E'_r)}{[(\{\alpha\to\beta\to\gamma,\alpha\cap\beta\};\gamma)]}(\to E')}{\Pi'_2 : \mathcal{M}_2 \equiv [(\emptyset;(\alpha\to\beta\to\gamma)\to(\alpha\cap\beta\to\gamma))]}(\to I')\times 2$$

Hence, in ISL$'$, we have the equivalence of $\alpha\cap\beta\to\gamma$ and $\alpha\to\beta\to\gamma$, which means that $\cap$ collapses to $\wedge$, since $\alpha\to\beta\to\gamma$ is equivalent to $\alpha\wedge\beta\to\gamma$ (in both ISL and ISL$'$). In ISL, on the other hand, the formulas $\alpha\cap\beta\to\gamma$ and $\alpha\to\beta\to\gamma$ are not equivalent, since we can prove $\mathcal{M}_2$ but not $\mathcal{M}_1$. Consequently, ISL$'$ cannot be proposed as an equivalent formulation of ISL.

Now consider ISL$''$ with contexts defined as sets, the axiom (A_{ISL}) replaced by

$$[(\{\sigma_i\};\sigma_i) \mid 1 \leqslant i \leqslant n] \quad (A''_{ISL})$$

weakening explicitly included, exchange excluded and the rest of the rules as in ISL$'$.

In ISL$''$, we can derive $\mathcal{M} \equiv [(\{\alpha, \beta\}; \alpha \cap \beta)]$, as shown below.

$$\dfrac{\dfrac{\dfrac{}{[(\{\alpha\}; \alpha), (\{\beta\}; \beta)]}\,(A'')}{[(\{\alpha, \beta\}; \alpha), (\{\beta, \alpha\}; \beta)]}\,(W'')}{\Pi'' : \mathcal{M} \equiv [(\{\alpha, \beta\}; \alpha \cap \beta)]}\,(\cap I'')\ (\{\alpha,\beta\}=\{\beta,\alpha\})$$

If ISL$''$ is equivalent to ISL, then, by theorem 4.3.1, Π'' should correspond to an IT-deduction $\{x : \alpha, y : \beta\} \vdash_{IT} M_{x,y}(\Pi'') : \alpha \cap \beta$. But, if we try to apply a non-standard decoration to Π'', we get

$$\dfrac{\dfrac{\dfrac{}{x : [(\{x : \alpha\}; \alpha), (\{x : \beta\}; \beta)]}\,(A'')^\star}{x : [(\{x : \alpha, y : \beta\}; \alpha), (\{x : \beta, y : \alpha\}; \beta)]}\,(W'')^\star}{? : [(\{? : \alpha, ? : \beta\}; \alpha \cap \beta)]}\,(\cap I'')^\star$$

The decoration cannot proceed to the introduction of the intersection, since the decorated contexts $\{x : \alpha, y : \beta\}$ and $\{x : \beta, y : \alpha\}$ are not identical. Hence, $M_{x,y}(\Pi'')$ is not defined and thus Π'' does not correspond to any IT-deduction. Consequently, ISL$''$ cannot be introduced as an equivalent formulation of ISL, either.

ISL, as opposed to ISL$'$ and ISL$''$, captures correctly the behaviour of the intersection connective. To this end, it is necessary that we define contexts as sequences and that we include explicit structural rules.

4.4 Properties of ISL

In this section, we consider ISL as defined in 4.1.1, i.e. including three connectives $\{\to, \cap, \wedge\}$. We define a $\diamond$-redex of an ISL-deduction, where $\diamond \in \{\to, \cap, \wedge\}$, and show how to eliminate redexes. We prove strong normalization of ISL by reduction to strong normalization of LJ. We also state the sub-formula property of normal ISL-deductions.

4.4.1 Strong normalization

We start by noting that the rule (P_{ISL}) can be eliminated from an ISL-deduction by a finite number of $\hookrightarrow_P$-normalization steps which exchange it with the rule above it, thus moving it up in the deduction until it reaches an axiom. At that point, its conclusion is by itself an axiom, so the initial axiom and the rule can be abolished. The procedure is analogous to the one described for pIL in definition 3.5.1.

Definition 4.4.1 *An ISL-deduction is said to be in* pre-normal *form, if it doesn't have any occurences of the rule* (P).

Lemma 4.4.2 *For every* $\Pi : \mathcal{M}$*, there is a* $\Pi' : \mathcal{M}$ *in pre-normal form.*

Definition 4.4.3 *Let Π be an ISL-deduction.*

(i) A $\rightarrow$-redex of Π is a sequence $(\rightarrow I_{ISL}), (\rightarrow E_{ISL})$ in Π of the rules introducing and eliminating the implication.

$$\dfrac{\dfrac{[(\Gamma_i, \sigma_i; \tau_i) \mid 1 \leqslant i \leqslant n]}{[(\Gamma_i; \sigma_i \rightarrow \tau_i) \mid 1 \leqslant i \leqslant n]}\,(\rightarrow I) \qquad [(\Gamma_i; \sigma_i) \mid 1 \leqslant i \leqslant n]}{[(\Gamma_i; \tau_i) \mid 1 \leqslant i \leqslant n]}\,(\rightarrow E)$$

(ii) A $\cap$-redex of Π is a sequence $(\cap I_{ISL}), (\cap E^l_{ISL})$ or $(\cap I_{ISL}), (\cap E^r_{ISL})$ in Π of the rules introducing and eliminating the intersection.

$$\dfrac{\dfrac{\mathcal{M} \cup [(\Gamma; \sigma), (\Gamma; \tau)]}{\mathcal{M} \cup [(\Gamma; \sigma \cap \tau)]}\,(\cap I)}{\mathcal{M} \cup [(\Gamma; \sigma)]}\,(\cap E^l) \qquad \dfrac{\dfrac{\mathcal{M} \cup [(\Gamma; \sigma), (\Gamma; \tau)]}{\mathcal{M} \cup [(\Gamma; \sigma \cap \tau)]}\,(\cap I)}{\mathcal{M} \cup [(\Gamma; \tau)]}\,(\cap E^r)$$

(iii) A $\wedge$-redex of Π is a sequence $(\wedge I_{ISL}), (\wedge E^l_{ISL})$ or $(\wedge I_{ISL}), (\wedge E^r_{ISL})$ in Π of the rules introducing and eliminating the conjunction.

$$\dfrac{\dfrac{[(\Gamma_i; \sigma_i) \mid 1 \leqslant i \leqslant n] \qquad [(\Gamma_i; \tau_i) \mid 1 \leqslant i \leqslant n]}{[(\Gamma_i; \sigma_i \wedge \tau_i) \mid 1 \leqslant i \leqslant n]}\,(\wedge I)}{[(\Gamma_i; \sigma_i) \mid 1 \leqslant i \leqslant n]}\,(\wedge E^l)$$

$$\dfrac{\dfrac{[(\Gamma_i; \sigma_i) \mid 1 \leqslant i \leqslant n] \qquad [(\Gamma_i; \tau_i) \mid 1 \leqslant i \leqslant n]}{[(\Gamma_i; \sigma_i \wedge \tau_i) \mid 1 \leqslant i \leqslant n]}\,(\wedge I)}{[(\Gamma_i; \tau_i) \mid 1 \leqslant i \leqslant n]}\,(\wedge E^r)$$

Remark 4.4.4 It is easy to see that the structural rules can be moved up when between the rules of a redex, so that the redex in question is formed as definition 4.4.3 requires. In general, suitable arrangements can be made so that the structural rules do not interfere with the normalization process.

Definition 4.4.5 *Let $\mathcal{M} \equiv [(\sigma^i_1, \ldots, \sigma^i_m; \tau_i) \mid 1 \leqslant i \leqslant n]$ be a molecule in position x of an ISL-deduction Π consisting of k steps $(0 \leqslant x \leqslant k)$. The context-formula σ^i_j is* open*, if, in steps $x+1, \ldots, k$ which follow, its atom is not abolished by a (P) rule and it doesn't move to the right of ; by a $(\rightarrow I)$ rule.*

Remark 4.4.6 If $\Pi : \mathcal{M}$ is an ISL-deduction, all context-formulas of $\mathcal{M}$ are stable.

The following lemma is used for the elimination of $\rightarrow$-redexes from an ISL-deduction.

Lemma 4.4.7 (Substitution lemma) *Suppose that $\Pi_0 : [(\Gamma_i, \sigma_i; \tau_i) \mid 1 \leqslant i \leqslant n]$ and $\Pi_1 : [(\Gamma_i; \sigma_i) \mid 1 \leqslant i \leqslant n]$ are ISL-deductions and let $S(\Pi_1, \Pi_0)$ be the deductive structure obtained from Π_0 by substituting Π_1 for all axioms $[(\sigma_i; \sigma_i) \mid 1 \leqslant i \leqslant n]$ with σ_i open and by eliminating all occurences of weakening over open σ_i and all occurences of weakening over open members of Γ_i. Then, $S(\Pi_1, \Pi_0) : [(\Gamma_i; \tau_i) \mid 1 \leqslant i \leqslant n]$.*

Proof: Use double induction, see [PR05, Pr65, Gi89]. $\dashv$

Remark 4.4.8 If Π_0 and Π_1 are both pre-normal, then $S(\Pi_1, \Pi_0)$ is pre-nornal, as well.

The definition of one-step normalization procedures eliminating $\diamond$-redexes of an ISL-deduction, where $\diamond \in \{\rightarrow, \cap, \wedge\}$, is now in order.

Definition 4.4.9 *Let Π be an ISL-deduction.*

(i) A $\rightarrow$-rewriting step on Π is a normalization step that eliminates a $\rightarrow$-redex of the deduction.

$$\dfrac{\dfrac{\Pi_0 : [(\Gamma_i, \sigma_i; \tau_i) \mid 1 \leqslant i \leqslant n]}{[(\Gamma_i; \sigma_i \rightarrow \tau_i) \mid 1 \leqslant i \leqslant n]}\,(\rightarrow I) \qquad \Pi_1 : [(\Gamma_i; \sigma_i) \mid 1 \leqslant i \leqslant n]}{[(\Gamma_i; \tau_i) \mid 1 \leqslant i \leqslant n]}\,(\rightarrow E)$$

$$\hookrightarrow_{\rightarrow} \ S(\Pi_1, \Pi_0) : [(\Gamma_i; \tau_i) \mid 1 \leqslant i \leqslant n]$$

(ii) A $\cap$-rewriting step on Π is a normalization step that eliminates a $\cap$-redex of the deduction.

$$\dfrac{\dfrac{\mathcal{M} \cup [(\Gamma; \sigma), (\Gamma; \tau)]}{\mathcal{M} \cup [(\Gamma; \sigma \cap \tau)]}\,(\cap I)}{\mathcal{M} \cup [(\Gamma; \sigma)]}\,(\cap E^l) \quad \hookrightarrow_{\cap} \quad \dfrac{\mathcal{M} \cup [(\Gamma; \sigma), (\Gamma; \tau)]}{\mathcal{M} \cup [(\Gamma; \sigma)]}\,(P)$$

The case of $(\cap E^r)$ is analogous.

(iii) A $\wedge$-rewriting step on Π is a normalization step that eliminates a $\wedge$-redex of the deduction.

$$\dfrac{\dfrac{\Pi_1 : [(\Gamma_i; \sigma_i) \mid 1 \leqslant i \leqslant n] \qquad \Pi_2 : [(\Gamma_i; \tau_i) \mid 1 \leqslant i \leqslant n]}{[(\Gamma_i; \sigma_i \wedge \tau_i) \mid 1 \leqslant i \leqslant n]}\,(\wedge I)}{[(\Gamma_i; \sigma_i) \mid 1 \leqslant i \leqslant n]}\,(\wedge E^l)$$

$$\hookrightarrow_{\wedge} \ \Pi_1 : [(\Gamma_i; \sigma_i) \mid 1 \leqslant i \leqslant n]$$

The case of $(\wedge E^r)$ is analogous.

Remark 4.4.10 If Π is in pre-normal form and Π' derives from Π by a $\rightarrow$-rewriting step ($\Pi \hookrightarrow_{\rightarrow} \Pi'$) or a $\wedge$-rewriting step ($\Pi \hookrightarrow_{\wedge} \Pi'$), then Π' is in pre-normal form, too. On the other hand, if Π' derives from Π by a $\cap$-rewriting step ($\Pi \hookrightarrow_{\cap} \Pi'$), it is *not* in pre-normal form, since (P) has appeared. Nevertheless, (P) can be eliminated again by a finite number of $\hookrightarrow_P$-steps.

Definition 4.4.11 *An ISL-deduction is in* normal *form, if it is in pre-normal form and contains no $\diamond$-redexes, where $\diamond \in \{\rightarrow, \cap, \wedge\}$.*

Theorem 4.4.12 *ISL is strongly normalizable, i.e. every ISL-deduction $\Pi : \mathcal{M}$ is strongly normalizable.*

Proof: Suppose there exists an ISL-deduction $\Pi : \mathcal{M} \equiv [(\sigma_1^i, \ldots, \sigma_m^i; \tau_i) \mid 1 \leqslant i \leqslant n]$ which is not strongly normalizable. Then, there is an infinite sequence s of $\diamond$-steps, where $\diamond \in \{\hookrightarrow_P, \hookrightarrow_\to, \hookrightarrow_\cap, \hookrightarrow_\wedge\}$, starting from Π. By theorem 4.2.2, Π gives n LJ-deductions $\Pi^1, \ldots, \Pi^n$ in $D_{LJ}^{\to}$, where $\Pi^i : \{e(\sigma_1^i), \ldots, e(\sigma_m^i)\} \vdash_{LJ} e(\tau_i)$. If $\Pi \hookrightarrow_P \Pi'$, then, for every $i \in \{1, \ldots, n\}$, $\Pi^i \equiv (\Pi')^i$. If $\Pi \hookrightarrow_\to \Pi'$, then, for every i, $\Pi^i \hookrightarrow_{LJ_\to}^1 \ldots \hookrightarrow_{LJ_\to}^{r_i} (\Pi')^i$. If $\Pi \hookrightarrow_\cap \Pi'$, then there is an $i_0 \in \{1, \ldots, n\}$, such that: (1) $\Pi^{i_0} \hookrightarrow_{LJ_\wedge} (\Pi')^{i_0}$ and (2) for every $i \in \{1, \ldots, n\} \setminus \{i_0\}$, it is $\Pi^i \equiv (\Pi')^i$. Finally, if $\Pi \hookrightarrow_\wedge \Pi'$, then, for every i, $\Pi^i \hookrightarrow_{LJ_\wedge} (\Pi')^i$.

Case 1: There are infinitely many $\hookrightarrow_\to$-steps in s. Then, since each such step generates finitely many $\hookrightarrow_{LJ_\to}$-steps in each Π^i, we meet infinitely many $\hookrightarrow_{LJ_\to}$-steps in each Π^i, which contradicts the strong normalization of LJ.

Case 2: There are infinitely many $\hookrightarrow_\cap$-steps in s. In this case, since each such step generates a $\hookrightarrow_{LJ_\wedge}$-step in one of the Π^i, there are infinitely many $\hookrightarrow_{LJ_\wedge}$-steps to be mounted in n LJ-deductions. Consequently, there is an $i \in \{1, \ldots, n\}$, such that we meet infinitely many $\hookrightarrow_{LJ_\wedge}$-steps in Π^i, which contradicts the strong normalization of LJ.

Case 3: There are infinitely many $\hookrightarrow_\wedge$-steps in s. In this case, since each such step generates a $\hookrightarrow_{LJ_\wedge}$-step in each Π^i, we meet infinitely many $\hookrightarrow_{LJ_\wedge}$-steps in each Π^i, which contradicts the strong normalization of LJ.

Case 4: There are infinitely many $\hookrightarrow_P$-steps in s. Then, there should be infinitely many $\hookrightarrow_\cap$-steps in s, since the (P) rules initially in Π are eliminated in a finite number of $\hookrightarrow_P$-steps and so is the (P) rule generated by a single $\hookrightarrow_\cap$-step. So, this case reduces to case 2. $\dashv$

4.4.2 Sub-formula property

Sub-formulas in ISL are defined as follows.

Definition 4.4.13 *Let σ be an ISL-formula. Then:*
(i) σ is a sub-formula of σ and
(ii) if $\tau \diamond \rho$ is a sub-formula of σ, then so are τ and ρ, for $\diamond \in \{\to, \cap, \wedge\}$.

Definition 4.4.14 *Let $\Pi : \mathcal{M} \equiv [(\Gamma_i; \sigma_i) \mid 1 \leqslant i \leqslant n]$ be an ISL-deduction. We say that Π enjoys the sub-formula property, denoted $sf(\Pi)$, if every formula appearing in Π is a sub-formula of one of the formulas occuring in $\mathcal{M}$.*

Theorem 4.4.15 *Let Π be an ISL-deduction in normal form. Then $sf(\Pi)$.*

Proof: The proof is an easy extension of the same property for LJ, given theorem 4.2.2. $\dashv$

Chapter 5

Equivalence of IL and ISL

5.1 Expansion of IL

We shall expand Intersection Logic to include three connectives $\rightarrow, \cap$ and $\wedge$, as Intersection Synchronous Logic does. We remind the reader that we actually work with pIL. The following definition provides the material needed to supplement definitions 3.1.1 and 3.2.1. We note that to expand pIL we only *add* information to the system; we do not *change* any of the data given in chapter 3.

Definition 5.1.1 *(i) A* kit *is a binary tree in the language generated by the grammar:* $K ::= \sigma \mid [K, K]$*, where the leaves* σ *are now generated by the grammar:*

$$\sigma ::= \alpha \mid \sigma \rightarrow \sigma \mid \sigma \cap \sigma \mid \sigma \wedge \sigma$$

with α *belonging to a denumerable set of propositional variables.*

(ii) If $H \simeq K$*, then* $H \wedge K$ *denotes a kit that overlaps with* H, K *and is such that, for every* $p \in P_T(H)$*,* $(H \wedge K)^p \equiv H^p \wedge K^p$*.*

(iii) The deductive system pIL includes all rules presented in Figure 3.1 and rules for the introduction and elimination of conjunction.

$$\frac{\Gamma \vdash_{pIL} K_1 \qquad \Gamma \vdash_{pIL} K_2}{\Gamma \vdash_{pIL} K_1 \wedge K_2}\,(\wedge I_{pIL}) \qquad\qquad \frac{\Gamma \vdash_{pIL} K_l \wedge K_r}{\Gamma \vdash_{pIL} K_s}\,(\wedge E^s_{pIL},\ s \in \{l, r\})$$

Remark 5.1.2 The rules $(\wedge I), (\wedge E)$ are global rules, as it is the case with $(\rightarrow I)$, $(\rightarrow E)$. They act on all leaves of the kits to the right of $\vdash_{pIL}$.

5.2 From IL to ISL

In this section, we show the transition from pIL to ISL. Given a pIL-deduction which concludes by a judgement involving kits with n terminal paths, there exists an ISL-deduction proving a molecule such that each of its atoms includes formulas which are all leaves at a certain terminal path of the kits, formulas in different positions in the atom coming from different kits. We can roughly say that each terminal path of the kits generates an atom in the molecule.

Theorem 5.2.1 *Let* $\Pi : \{K_1, \ldots, K_m\} \vdash_{pIL} H$ *with* $P_T(H) = \{p_1, \ldots, p_n\}$. *Then*

$$\vdash_{ISL} [(K_1^{p_1}, \ldots, K_m^{p_1}; H^{p_1}), \ldots, (K_1^{p_n}, \ldots, K_m^{p_n}; H^{p_n})]$$

Proof: By induction on Π.

Base: Suppose $\Pi : \{K_1, \ldots, K_m\} \vdash_{pIL} K_1$ is a pIL-axiom. We have that

$$[(K_1^{p_1}; K_1^{p_1}), \ldots, (K_1^{p_n}; K_1^{p_n})]$$

is an ISL-axiom and, if we apply weakening repeatedly, we get

$$\vdash_{ISL} [(K_1^{p_1}, \ldots, K_m^{p_1}; K_1^{p_1}), \ldots, (K_1^{p_n}, \ldots, K_m^{p_n}; K_1^{p_n})]$$

Inductive step: We examine all rules of the expanded pIL.

- $$\frac{\Pi' : \{H_1, \ldots, H_m\} \vdash_{pIL} K}{\Pi : \{H_1\backslash^{ps}, \ldots, H_m\backslash^{ps}\} \vdash_{pIL} K\backslash^{ps}} \ (P_{pIL})$$

Let $H_i\backslash^{ps} \equiv K_i$ $(1 \leqslant i \leqslant m)$ and $K\backslash^{ps} \equiv H$. Suppose that

$$P_T(K) = \{q_1, \ldots, q_{n_1}, psr_1, \ldots, psr_{n_2}, ps't_1, \ldots, ps't_{n_3}\}$$

where $s, s' \in \{l, r\}$, $s \not\equiv s'$, $n_1 + n_2 = n$ and q_j, p are different paths, for $1 \leqslant j \leqslant n_1$. Then, we have:

1. $P_T(H) = \{q_1, \ldots, q_{n_1}, pr_1, \ldots, pr_{n_2}\}$
2. $H_i^{q_j} \equiv K_i^{q_j}$, $K^{q_j} \equiv H^{q_j}$ $\ (1 \le i \le m,\ 1 \le j \le n_1)$
3. $H_i^{psr_j} \equiv K_i^{pr_j}$, $K^{psr_j} \equiv H^{pr_j}$ $\ (1 \le i \le m,\ 1 \le j \le n_2)$

By the IH, we have that $\vdash_{ISL} \mathcal{M} \cup \mathcal{N}$ and, applying (P_{ISL}), we get $\vdash_{ISL} \mathcal{M}'$, where

$$\begin{array}{rcl} \mathcal{M} & \equiv & [(H_1^{q_1}, \ldots, H_m^{q_1}; K^{q_1}), \ldots, (H_1^{q_{n_1}}, \ldots, H_m^{q_{n_1}}; K^{q_{n_1}}), \\ & & (H_1^{psr_1}, \ldots, H_m^{psr_1}; K^{psr_1}), \ldots, (H_1^{psr_{n_2}}, \ldots, H_m^{psr_{n_2}}; K^{psr_{n_2}})] \\ \mathcal{N} & \equiv & [(H_1^{ps't_1}, \ldots, H_m^{ps't_1}; K^{ps't_1}), \ldots, (H_1^{ps't_{n_3}}, \ldots, H_m^{ps't_{n_3}}; K^{ps't_{n_3}})] \\ \mathcal{M}' & \equiv & [(K_1^{q_1}, \ldots, K_m^{q_1}; H^{q_1}), \ldots, (K_1^{q_{n_1}}, \ldots, K_m^{q_{n_1}}; H^{q_{n_1}}), \\ & & (K_1^{pr_1}, \ldots, K_m^{pr_1}; H^{pr_1}), \ldots, (K_1^{pr_{n_2}}, \ldots, K_m^{pr_{n_2}}; H^{pr_{n_2}})] \end{array}$$

and $\mathcal{M} \equiv \mathcal{M}'$ by 2. and 3.

- $$\frac{\Pi' : \{K_1, \ldots, K_m, L_1\} \vdash_{pIL} L_2}{\Pi : \{K_1, \ldots, K_m\} \vdash_{pIL} L_1 \to L_2 \equiv H} \ (\to I_{pIL})$$

It is $H \simeq L_2$, so $P_T(L_2) = P_T(H) = \{p_1, \ldots, p_n\}$. By the IH, we have that $\vdash_{ISL} [(K_1^{p_1}, \ldots, K_m^{p_1}, L_1^{p_1}; L_2^{p_1}), \ldots, (K_1^{p_n}, \ldots, K_m^{p_n}, L_1^{p_n}; L_2^{p_n})]$. Applying $(\to I_{ISL})$, we get $\vdash_{ISL} [(K_1^{p_1}, \ldots, K_m^{p_1}; L_1^{p_1} \to L_2^{p_1}), \ldots, (K_1^{p_n}, \ldots, K_m^{p_n}; L_1^{p_n} \to L_2^{p_n})]$. Since, for every $i \in \{1, \ldots, n\}$, $L_1^{p_i} \to L_2^{p_i} \equiv (L_1 \to L_2)^{p_i} \equiv H^{p_i}$, we have the required result.

- $$\frac{\Pi' : \{K_1, \ldots, K_m\} \vdash_{pIL} L \to H \qquad \Pi'' : \{K_1, \ldots, K_m\} \vdash_{pIL} L}{\Pi : \{K_1, \ldots, K_m\} \vdash_{pIL} H} \ (\to E_{pIL})$$

It is $L \simeq H \simeq L \to H$, so $P_T(L \to H) = P_T(L) = P_T(H) = \{p_1, \ldots, p_n\}$. By the IH, we have that $\vdash_{ISL} [(K_1^{p_1}, \ldots, K_m^{p_1}; (L \to H)^{p_1}), \ldots, (K_1^{p_n}, \ldots, K_m^{p_n}; (L \to H)^{p_n})]$

and $\vdash_{ISL} [(K_1^{p_1}, \ldots, K_m^{p_1}; L^{p_1}), \ldots, (K_1^{p_n}, \ldots, K_m^{p_n}; L^{p_n})]$. Since, for every $i \in \{1, \ldots, n\}$, $(L \to H)^{p_i} \equiv L^{p_i} \to H^{p_i}$, we can apply $(\to E_{ISL})$ on the two molecules to get the required result.

- $$\frac{\Pi' : \{H_1[p := [\sigma_1, \sigma_1]], \ldots, H_m[p := [\sigma_m, \sigma_m]]\} \vdash_{pIL} K[p := [\sigma, \tau]]}{\Pi : \{H_1[p := \sigma_1], \ldots, H_m[p := \sigma_m]\} \vdash_{pIL} K[p := \sigma \cap \tau]} (\cap I_{pIL})$$

  Let $H_i[p := [\sigma_i, \sigma_i]] \equiv K_i'$, $H_i[p := \sigma_i] \equiv K_i$ $(1 \leqslant i \leqslant m)$, $K[p := [\sigma, \tau]] \equiv H'$ and $K[p := \sigma \cap \tau] \equiv H$. Suppose that $P_T(H') = \{q_1, \ldots, q_k, pl, pr\}$, with q_j, p different paths, for $1 \leqslant j \leqslant k$. Then, we have:

  1. $P_T(H) = \{q_1, \ldots, q_k, p\}$
  2. $(K_i')^{q_j} \equiv K_i^{q_j}, \ (H')^{q_j} \equiv H^{q_j} \quad (1 \leq i \leq m, \ 1 \leq j \leq k)$
  3. $(K_i')^{ps} \equiv K_i^p \equiv \sigma_i \quad (1 \leq i \leq m, \ s \in \{l, r\})$
  4. $(H')^{pl} \equiv \sigma, \ (H')^{pr} \equiv \tau, \ H^p \equiv \sigma \cap \tau$

By the IH, we have that $\vdash_{ISL} \mathcal{M} \cup [(\sigma_1, \ldots, \sigma_m; \sigma), (\sigma_1, \ldots, \sigma_m; \tau)]$ and, applying $(\cap I_{ISL})$, we get $\vdash_{ISL} \mathcal{M}' \cup [(\sigma_1, \ldots, \sigma_m; \sigma \cap \tau)]$, where

$$\begin{array}{rcl}\mathcal{M} & \equiv & [((K_1')^{q_1}, \ldots, (K_m')^{q_1}; (H')^{q_1}), \ldots, ((K_1')^{q_k}, \ldots, (K_m')^{q_k}; (H')^{q_k})] \\ \mathcal{M}' & \equiv & [(K_1^{q_1}, \ldots, K_m^{q_1}; H^{q_1}), \ldots, (K_1^{q_k}, \ldots, K_m^{q_k}; H^{q_k})]\end{array}$$

and $\mathcal{M} \equiv \mathcal{M}'$ by 2.

- $$\frac{\Pi' : \{K_1, \ldots, K_m\} \vdash_{pIL} K[p := \sigma_l \cap \sigma_r] \equiv H'}{\Pi : \{K_1, \ldots, K_m\} \vdash_{pIL} K[p := \sigma_s] \equiv H} (\cap E_{pIL}^s)$$

  Suppose that $P_T(H') = \{p_1, \ldots, p_{n-1}, p\}$. Then:

  1. $P_T(H) = P_T(H')$
  2. $(H')^{p_j} \equiv H^{p_j} \quad (1 \leqslant j \leqslant n-1)$
  3. $(H')^p \equiv \sigma_l \cap \sigma_r, \ H^p \equiv \sigma_s$

By the IH, we have that $\vdash_{ISL} \mathcal{M} \cup [(K_1^p, \ldots, K_m^p; \sigma_l \cap \sigma_r)]$ and, applying $(\cap E_{ISL}^s)$, we get $\vdash_{ISL} \mathcal{M}' \cup [(K_1^p, \ldots, K_m^p; \sigma_s)]$, where

$$\begin{array}{rcl}\mathcal{M} & \equiv & [(K_1^{p_1}, \ldots, K_m^{p_1}; (H')^{p_1}), \ldots, (K_1^{p_{n-1}}, \ldots, K_m^{p_{n-1}}; (H')^{p_{n-1}})] \\ \mathcal{M}' & \equiv & [(K_1^{p_1}, \ldots, K_m^{p_1}; H^{p_1}), \ldots, (K_1^{p_{n-1}}, \ldots, K_m^{p_{n-1}}; H^{p_{n-1}})]\end{array}$$

and $\mathcal{M} \equiv \mathcal{M}'$ by 2.

- $$\frac{\Pi' : \{K_1, \ldots, K_m\} \vdash_{pIL} L_1 \qquad \Pi'' : \{K_1, \ldots, K_m\} \vdash_{pIL} L_2}{\Pi : \{K_1, \ldots, K_m\} \vdash_{pIL} L_1 \wedge L_2 \equiv H} (\wedge I_{pIL})$$

  It is $L_1 \simeq L_2 \simeq H$, so $P_T(L_1) = P_T(L_2) = P_T(H) = \{p_1, \ldots, p_n\}$. The IH gives

$$\vdash_{ISL} [(K_1^{p_1}, \ldots, K_m^{p_1}; L_1^{p_1}), \ldots, (K_1^{p_n}, \ldots, K_m^{p_n}; L_1^{p_n})]$$

$$\vdash_{ISL} [(K_1^{p_1}, \ldots, K_m^{p_1}; L_2^{p_1}), \ldots, (K_1^{p_n}, \ldots, K_m^{p_n}; L_2^{p_n})]$$

Applying $(\wedge I_{ISL})$, we get $\vdash_{ISL} [(K_1^{p_1},\ldots,K_m^{p_1};L_1^{p_1}\wedge L_2^{p_1}),\ldots,(K_1^{p_n},\ldots,K_m^{p_n};L_1^{p_n}\wedge L_2^{p_n})]$. Since, for every $i\in\{1,\ldots,n\}$, $L_1^{p_i}\wedge L_2^{p_i}\equiv(L_1\wedge L_2)^{p_i}\equiv H^{p_i}$, we have the required result.

- $$\frac{\Pi' : \{K_1,\ldots,K_m\}\vdash_{pIL} K_l\wedge K_r}{\Pi : \{K_1,\ldots,K_m\}\vdash_{pIL} K_s\equiv H}\ (\wedge E^s_{pIL})$$

It is $K_l\wedge K_r\simeq H$, so $P_T(K_l\wedge K_r)=P_T(H)=\{p_1,\ldots,p_n\}$. By the IH, we have that $\vdash_{ISL} [(K_1^{p_1},\ldots,K_m^{p_1};(K_l\wedge K_r)^{p_1}),\ldots,(K_1^{p_n},\ldots,K_m^{p_n};(K_l\wedge K_r)^{p_n})]$. Since, for every $i\in\{1,\ldots,n\}$, $(K_l\wedge K_r)^{p_i}\equiv K_l^{p_i}\wedge K_r^{p_i}$, we can apply $(\wedge E^s_{ISL})$ to get

$$\vdash_{ISL} [(K_1^{p_1},\ldots,K_m^{p_1};K_s^{p_1}),\ldots,(K_1^{p_n},\ldots,K_m^{p_n};K_s^{p_n})]$$

as required. $\dashv$

5.3 From ISL to IL

This section describes the transition from ISL to pIL. We first show how to represent a molecule of n atoms and context-cardinality m by a sequence of $m+1$ overlapping kits with n terminal paths each, so that the j-th kit in the sequence picks as leaves all the j-th formulas in the atoms, the leaf at path i coming from the i-th atom. We stress the fact that a molecule can have more that one kit-representations depending on the structure we chose for the representing kits and on the path enumeration we employ. Finally, we prove that, given an ISL-deduction which concludes by a certain molecule, there exists a pIL-deduction which concludes by a judgement involving the representing kit-sequence of the molecule, no matter which such sequence we consider.

Proposition 5.3.1 *Let $\mathcal{M}\equiv[(\sigma_1^1,\ldots,\sigma_m^1;\tau_1),\ldots,(\sigma_1^n,\ldots,\sigma_m^n;\tau_n)]$ be a molecule of $n\geqslant 1$ atoms, each of context-cardinality $m\geqslant 0$. Then, there exists a sequence $K_1,\ldots,K_m,H$ of $m+1$ overlapping kits with n terminal paths $p_1,\ldots,p_n$ each, such that $K_j^{p_i}\equiv\sigma_j^i$ and $H^{p_i}\equiv\tau_i$ $(1\leqslant j\leqslant m,\ 1\leqslant i\leqslant n)$.*

Proof: By induction on n with m fixed but arbitrary.

Base: For $n=1$, $\mathcal{M}\equiv[(\sigma_1^1,\ldots,\sigma_m^1;\tau_1)]$. The $m+1$ overlapping kits with one terminal path—the empty path ϵ—are the single-node kits $\sigma_1^1,\ldots,\sigma_m^1,\tau_1$.

Inductive step: Let $n>1$, $\mathcal{M}\equiv[(\sigma_1^1,\ldots,\sigma_m^1;\tau_1),\ldots,(\sigma_1^n,\ldots,\sigma_m^n;\tau_n)]$ and suppose the proposition holds for any molecule of $n-1$ atoms, each of context-cardinality m. Consider $\mathcal{M}'\equiv[(\sigma_1^1,\ldots,\sigma_m^1;\tau_1),\ldots,(\sigma_1^{n-1},\ldots,\sigma_m^{n-1};\tau_{n-1})]$, which includes all atoms of $\mathcal{M}$ except $(\sigma_1^n,\ldots,\sigma_m^n;\tau_n)$. By the IH, there exists a sequence $H_1,\ldots,H_m,K$ of $m+1$ overlapping kits with $n-1$ terminal paths $q_1,\ldots,q_{n-1}$ each, such that $H_j^{q_i}\equiv\sigma_j^i$ and $K^{q_i}\equiv\tau_i$ $(1\leqslant j\leqslant m,\ 1\leqslant i\leqslant n-1)$. Let us now consider the $m+1$ kits

$$H_1[q_{n-1}:=[\sigma_1^{n-1},\sigma_1^n]],\ldots,H_m[q_{n-1}:=[\sigma_m^{n-1},\sigma_m^n]],K[q_{n-1}:=[\tau_{n-1},\tau_n]]\quad(*)$$

They are overlapping (since $H_1,\ldots,H_m,K$ are), they have n terminal paths

$$p_1\equiv q_1,\ldots,p_{n-2}\equiv q_{n-2},p_{n-1}\equiv q_{n-1}l,p_n\equiv q_{n-1}r$$

and they are such that:

1. $(H_j[q_{n-1}:=[\sigma_j^{n-1},\sigma_j^n]])^{p_i}\equiv H_j^{q_i}\equiv\sigma_j^i$, $(K[q_{n-1}:=[\tau_{n-1},\tau_n]])^{p_i}\equiv K^{q_i}\equiv\tau_i$ $(1\leqslant j\leqslant m,\ 1\leqslant i\leqslant n-2)$

2. $(H_j[q_{n-1} := [\sigma_j^{n-1}, \sigma_j^n]])^{p_{n-1}} \equiv \sigma_j^{n-1}$, $(K[q_{n-1} := [\tau_{n-1}, \tau_n]])^{p_{n-1}} \equiv \tau_{n-1}$ $(1 \leqslant j \leqslant m)$

3. $(H_j[q_{n-1} := [\sigma_j^{n-1}, \sigma_j^n]])^{p_n} \equiv \sigma_j^n$, $(K[q_{n-1} := [\tau_{n-1}, \tau_n]])^{p_n} \equiv \tau_n$ $(1 \leqslant j \leqslant m)$

Consequently, $(*)$ is the required kit-sequence for $\mathcal{M}$. $\dashv$

Definition 5.3.2 *The sequence $K_1, \ldots, K_m, H$ of proposition 5.3.1 will be called a* kit-representation *of $\mathcal{M}$.*

Remark 5.3.3 The kit-representation of a molecule $\mathcal{M}$ is not unique. Different kit-representations of $\mathcal{M}$ may have different kit-structures or the same structure but different path enumerations. For example, consider the molecule

$$\mathcal{M} \equiv [(\sigma_1^1, \sigma_2^1, \sigma_3^1; \tau_1), (\sigma_1^2, \sigma_2^2, \sigma_3^2; \tau_2), (\sigma_1^3, \sigma_2^3, \sigma_3^3; \tau_3)]$$

The following kit-sequences are kit-representations of $\mathcal{M}$.

$$[[\sigma_1^1, \sigma_1^2], \sigma_1^3], [[\sigma_2^1, \sigma_2^2], \sigma_2^3], [[\sigma_3^1, \sigma_3^2], \sigma_3^3], [[\tau_1, \tau_2], \tau_3] \quad \text{(a)}$$

$$[[\sigma_1^3, \sigma_1^2], \sigma_1^1], [[\sigma_2^3, \sigma_2^2], \sigma_2^1], [[\sigma_3^3, \sigma_3^2], \sigma_3^1], [[\tau_3, \tau_2], \tau_1] \quad \text{(b)}$$

$$[\sigma_1^1, [\sigma_1^2, \sigma_1^3]], [\sigma_2^1, [\sigma_2^2, \sigma_2^3]], [\sigma_3^1, [\sigma_3^2, \sigma_3^3]], [\tau_1, [\tau_2, \tau_3]] \quad \text{(c)}$$

Sequences (a) and (b) employ the same kit-structure but different path enumerations, while sequence (c) displays a different kit-structure from that of (a) and (b).

Theorem 5.3.4 *If $\Pi : \mathcal{M} \equiv [(\sigma_1^1, \ldots, \sigma_m^1; \tau_1), \ldots, (\sigma_1^n, \ldots, \sigma_m^n; \tau_n)]$ $(n \geqslant 1,\ m \geqslant 0)$, then, for every kit-representation $K_1, \ldots, K_m, H$ of $\mathcal{M}$, $\{K_1, \ldots, K_m\} \vdash_{pIL} H$.*

Proof: By induction on Π.

Base: Suppose $\Pi : \mathcal{M} \equiv [(\sigma_1^1; \tau_1), \ldots, (\sigma_1^n; \tau_n)]$, where $\sigma_1^i \equiv \tau_i$, for $i \in \{1, \ldots, n\}$, is an ISL axiom and let K_1, H be a kit-representation of $\mathcal{M}$. Then, if $P_T(H) = \{p_1, \ldots, p_n\}$, we have $K_1^{p_i} \equiv \sigma_1^i \equiv \tau_i \equiv H^{p_i}$ $(1 \leqslant i \leqslant n)$, so $K_1 \equiv H$. Hence, $\{K_1\} \vdash_{pIL} H$ is a pIL-axiom.

Inductive step: We check all ISL-rules.

- $$\frac{\Pi' : \mathcal{M} \cup \mathcal{N}}{\Pi : \mathcal{M}}\ (P_{ISL})$$

Take $\mathcal{M} \equiv [(\sigma_1^i, \ldots, \sigma_m^i; \tau_i) \mid 1 \leqslant i \leqslant n]$, $\mathcal{N} \equiv [(\sigma_1^i, \ldots, \sigma_m^i; \tau_i) \mid n+1 \leqslant i \leqslant n+k]$ and let $K_1, \ldots, K_m, H$ be a kit-representation of $\mathcal{M}$ and $H_1, \ldots, H_m, K$ a kit-representation of $\mathcal{N}$. Then, the sequence $[K_1, H_1], \ldots, [K_m, H_m], [H, K]$ is a kit-representation of $\mathcal{M} \cup \mathcal{N}$. By the IH, we have $\{[K_1, H_1], \ldots, [K_m, H_m]\} \vdash_{pIL} [H, K]$. Applying (P_{pIL}), we get $\{[K_1, H_1]\backslash^l \equiv K_1, \ldots, [K_m, H_m]\backslash^l \equiv K_m\} \vdash_{pIL} [H, K]\backslash^l \equiv H$.

- $$\frac{\Pi' : \mathcal{M}' \equiv [(\sigma_1^i, \ldots, \sigma_m^i; \tau_i) \mid 1 \leqslant i \leqslant n]}{\Pi : \mathcal{M} \equiv [(\sigma_1^i, \ldots, \sigma_m^i, \sigma_{m+1}^i; \tau_i) \mid 1 \leqslant i \leqslant n]}\ (W_{ISL})$$

Let $K_1, \ldots, K_m, K_{m+1}, H$ be a kit-representation of $\mathcal{M}$. Then, $K_1, \ldots, K_m, H$ is a kit-representation of $\mathcal{M}'$. By the IH, we have $\{K_1, \ldots, K_m\} \vdash_{pIL} H$ and, since

K_{m+1} overlaps with $K_1, \ldots, K_m, H$, by the weakening property for pIL, we get $\{K_1, \ldots, K_m, K_{m+1}\} \vdash_{pIL} H$.

- $$\frac{\Pi' : \mathcal{M}' \equiv [(\sigma_1^i, \ldots, \sigma_m^i, \upsilon_i, \rho_i, \sigma_{m+1}^i, \ldots, \sigma_{m+l}^i; \tau_i) \mid 1 \leqslant i \leqslant n]}{\Pi : \mathcal{M} \equiv [(\sigma_1^i, \ldots, \sigma_m^i, \rho_i, \upsilon_i, \sigma_{m+1}^i, \ldots, \sigma_{m+l}^i; \tau_i) \mid 1 \leqslant i \leqslant n]} \ (X_{ISL})$$

Let $K_1, \ldots, K_m, R, Y, K_{m+1}, \ldots, K_{m+l}, H$ be a kit-representation of $\mathcal{M}$. Then, the sequence $K_1, \ldots, K_m, Y, R, K_{m+1}, \ldots, K_{m+l}, H$ is a kit-representation of $\mathcal{M}'$ and, by the IH, we have $\{K_1, \ldots, K_m, Y, R, K_{m+1}, \ldots, K_{m+l}\} \vdash_{pIL} H$. But this is the required result, since $\{K_1, \ldots, K_m, Y, R, K_{m+1}, \ldots, K_{m+l}\} = \{K_1, \ldots, K_m, R, Y, K_{m+1}, \ldots, K_{m+l}\}$.

- $$\frac{\Pi' : \mathcal{M}' \equiv [(\sigma_1^i, \ldots, \sigma_m^i, \rho_i; \upsilon_i) \mid 1 \leqslant i \leqslant n]}{\Pi : \mathcal{M} \equiv [(\sigma_1^i, \ldots, \sigma_m^i; \rho_i \to \upsilon_i) \mid 1 \leqslant i \leqslant n]} \ (\to I_{ISL})$$

Let $K_1, \ldots, K_m, H$ be a kit-representation of $\mathcal{M}$ with $P_T(H) = \{p_1, \ldots, p_n\}$. Then, the sequence $K_1, \ldots, K_m, H[p_i := \rho_i \mid 1 \leqslant i \leqslant n], H[p_i := \upsilon_i \mid 1 \leqslant i \leqslant n]$ is a kit-representation of $\mathcal{M}'$. By the IH, we have

$$\{K_1, \ldots, K_m, H[p_i := \rho_i \mid 1 \leqslant i \leqslant n]\} \vdash_{pIL} H[p_i := \upsilon_i \mid 1 \leqslant i \leqslant n]$$

Applying $(\to I_{pIL})$, we get

$$\{K_1, \ldots, K_m\} \vdash_{pIL} H[p_i := \rho_i \mid 1 \leqslant i \leqslant n] \to H[p_i := \upsilon_i \mid 1 \leqslant i \leqslant n] \equiv H$$

- $$\frac{\Pi' : \mathcal{M}' \equiv [(\Gamma_i; \rho_i \to \tau_i) \mid 1 \leqslant i \leqslant n] \qquad \Pi'' : \mathcal{M}'' \equiv [(\Gamma_i; \rho_i) \mid 1 \leqslant i \leqslant n]}{\Pi : \mathcal{M} \equiv [(\Gamma_i; \tau_i) \mid 1 \leqslant i \leqslant n]} \ (\to E_{ISL})$$

Suppose $\Gamma_i \equiv \sigma_1^i, \ldots, \sigma_m^i$ and let $K_1, \ldots, K_m, H$ be a kit-representation of $\mathcal{M}$ with $P_T(H) = \{p_1, \ldots, p_n\}$. If $H' \equiv H[p_i := \rho_i \mid 1 \leqslant i \leqslant n]$, the sequence $K_1, \ldots, K_m, H' \to H$ is a kit-representation of $\mathcal{M}'$ and $K_1, \ldots, K_m, H'$ is a kit-representation of $\mathcal{M}''$. By the IH, we have

$$\{K_1, \ldots, K_m\} \vdash_{pIL} H' \to H, \quad \{K_1, \ldots, K_m\} \vdash_{pIL} H'$$

Applying $(\to E_{pIL})$, we get $\{K_1, \ldots, K_m\} \vdash_{pIL} H$.

- $$\frac{\Pi' : \mathcal{M}' \equiv [(\sigma_1^i, \ldots, \sigma_m^i; \tau_i) \mid 1 \leqslant i \leqslant n-1] \cup [(\sigma_1^n, \ldots, \sigma_m^n; \rho), (\sigma_1^n, \ldots, \sigma_m^n; \upsilon)]}{\Pi : \mathcal{M} \equiv [(\sigma_1^i, \ldots, \sigma_m^i; \tau_i) \mid 1 \leqslant i \leqslant n-1] \cup [(\sigma_1^n, \ldots, \sigma_m^n; \rho \cap \upsilon)]} \ (\cap I_{ISL})$$

Let $K_1, \ldots, K_m, H$ be a kit-representation of $\mathcal{M}$ with $P_T(H) = \{p_1, \ldots, p_n\}$. Then, the sequence $K_1[p_n := [\sigma_1^n, \sigma_1^n]], \ldots, K_m[p_n := [\sigma_m^n, \sigma_m^n]], H[p_n := [\rho, \upsilon]]$ is a kit-representation of $\mathcal{M}'$. By the IH, we have

$$\{K_1[p_n := [\sigma_1^n, \sigma_1^n]], \ldots, K_m[p_n := [\sigma_m^n, \sigma_m^n]]\} \vdash_{pIL} H[p_n := [\rho, \upsilon]]$$

Applying $(\cap I_{pIL})$, we get

$$\{K_1[p_n := \sigma_1^n] \equiv K_1, \ldots, K_m[p_n := \sigma_m^n] \equiv K_m\} \vdash_{pIL} H[p_n := \rho \cap \upsilon] \equiv H$$

- $$\dfrac{\Pi' : \mathcal{M}' \equiv [(\sigma_1^i, \ldots, \sigma_m^i; \tau_i) \mid 1 \leqslant i \leqslant n-1] \cup [(\sigma_1^n, \ldots, \sigma_m^n; \rho_l \cap \rho_r)]}{\Pi : \mathcal{M} \equiv [(\sigma_1^i, \ldots, \sigma_m^i; \tau_i) \mid 1 \leqslant i \leqslant n-1] \cup [(\sigma_1^n, \ldots, \sigma_m^n; \rho_s)]} \, (\cap E^s_{ISL})$$

Let $K_1, \ldots, K_m, H$ be a kit-representation of $\mathcal{M}$ with $P_T(H) = \{p_1, \ldots, p_n\}$. Then, the sequence $K_1, \ldots, K_m, H[p_n := \rho_l \cap \rho_r]$ is a kit-representation of $\mathcal{M}'$. By the IH, we have $\{K_1, \ldots, K_m\} \vdash_{pIL} H[p_n := \rho_l \cap \rho_r]$ and, by $(\cap E^s_{pIL})$, we get

$$\{K_1, \ldots, K_m\} \vdash_{pIL} H[p_n := \rho_s] \equiv H$$

- $$\dfrac{\Pi' : \mathcal{M}' \equiv [(\Gamma_i; \rho_i) \mid 1 \leqslant i \leqslant n] \qquad \Pi'' : \mathcal{M}'' \equiv [(\Gamma_i; \upsilon_i) \mid 1 \leqslant i \leqslant n]}{\Pi : \mathcal{M} \equiv [(\Gamma_i; \rho_i \wedge \upsilon_i) \mid 1 \leqslant i \leqslant n]} \, (\wedge I_{ISL})$$

Suppose $\Gamma_i \equiv \sigma_1^i, \ldots, \sigma_m^i$ and let $K_1, \ldots, K_m, H$ be a kit-representation of $\mathcal{M}$ with $P_T(H) = \{p_1, \ldots, p_n\}$. Then, $K_1, \ldots, K_m, H[p_i := \rho_i \mid 1 \leqslant i \leqslant n]$ is a kit-representation of $\mathcal{M}'$ and $K_1, \ldots, K_m, H[p_i := \upsilon_i \mid 1 \leqslant i \leqslant n]$ a kit-representation of $\mathcal{M}''$. By the IH, we have

$$\{K_1, \ldots, K_m\} \vdash_{pIL} H[p_i := \rho_i \mid 1 \leqslant i \leqslant n], \quad \{K_1, \ldots, K_m\} \vdash_{pIL} H[p_i := \upsilon_i \mid 1 \leqslant i \leqslant n]$$

Applying $(\wedge I_{pIL})$, we get

$$\{K_1, \ldots, K_m\} \vdash_{pIL} H[p_i := \rho_i \mid 1 \leqslant i \leqslant n] \wedge H[p_i := \upsilon_i \mid 1 \leqslant i \leqslant n] \equiv H$$

- $$\dfrac{\Pi' : \mathcal{M}' \equiv [(\sigma_1^i, \ldots, \sigma_m^i; \tau_i \wedge \rho_i) \mid 1 \leqslant i \leqslant n]}{\Pi : \mathcal{M} \equiv [(\sigma_1^i, \ldots, \sigma_m^i; \tau_i) \mid 1 \leqslant i \leqslant n]} \, (\wedge E^l_{ISL})$$

Let $K_1, \ldots, K_m, H$ be a kit-representation of $\mathcal{M}$ with $P_T(H) = \{p_1, \ldots, p_n\}$. If $H' \equiv H[p_i := \rho_i \mid 1 \leqslant i \leqslant n]$, then $K_1, \ldots, K_m, H \wedge H'$ is a kit-representation of $\mathcal{M}'$. By the IH, we have $\{K_1, \ldots, K_m\} \vdash_{pIL} H \wedge H'$ and, applying $(\wedge E^l_{pIL})$, we get $\{K_1, \ldots, K_m\} \vdash_{pIL} H$. $\dashv$

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