Sphere-cut Decompositions and Dominating Sets in Planar Graphs

Michalis Samaris
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Scientific committee:
Dimitrios M. Thilikos,
Professor,
Dep. of Mathematics, National and Kapodistrian University of Athens.

Stavros G. Kolliopoulos,
Associate Professor,
Dep. of Informatics and Telecommunications, National and Kapodistrian University of Athens.

Lefteris M. Kirousis,
Professor,
Dep. of Mathematics, National and Kapodistrian University of Athens.

Supervisor:
Dimitrios M. Thilikos,
Professor,
Dep. of Mathematics, National and Kapodistrian University of Athens.

Αποσυνθέσεις σφαιρικών τομών και σύνολα κυριαρχίας σε επίπεδα γραφήματα

Μιχάλης Σάμαρης
Α.Μ. 201314

Τριμελής Επιτροπή:

Δημήτριος Μ. Θηλυκός,
Καθηγητής, Τμ. Μαθηματικών, Ε.Κ.Π.Α.

Σταύρος Γ. Κολλιόπουλος,
Αναπληρωτής Καθηγητής, Τμ. Πληροφορικής και Τηλ/νιών, Ε.Κ.Π.Α.

Λευτέρης Μ. Κυρούσης,
Καθηγητής, Τμ. Μαθηματικών, Ε.Κ.Π.Α.

Επιβλέπων:

Δημήτριος Μ. Θηλυκός,

Καθηγητής του Τμήματος Μαθηματικών του Πανεπιστημίου Αθηνών

Περίληψη

Ένα σημαντικό αποτέλεσμα στη Θεωρία Γραφημάτων αποτελεί η απόδειξη της εικασίας του Wagner από τους Neil Robertson και Paul D. Seymour. στη σειρά εργασιών 'Ελλάσσονα Γραφήματα' απο το 1983 εώς το 2011. Η εικασία αυτή λέει ότι στην κλάση των γραφημάτων δεν υπάρχει άπειρη αντιαλυσίδα ώς προς τη σχέση των ελλασόνων γραφημάτων. Η Θεωρία που αναπτύχθηκε για την απόδειξη αυτής της εικασίας είχε και έχει ακόμα σημαντικό αντίκτυπο τόσο στην δομική όσο και στην αλγοριθμική Θεωρία Γραφημάτων, άλλα και σε άλλα πεδία όπως η Παραμετρική Πολυπλοκότητα.

Στα πλάισια της απόδειξης οι συγγραφείς εισήγαγαν και νέες παραμέτρους πλάτους. Σε αυτές ήταν η κλαδοαποσύνθεση και το κλαδοπλάτος ενός γραφήματος. Η παράμετρος αυτή χρησιμοποιήθηκε ιδιαίτερα στο σχεδιασμό αλγορίθμων και στην χρήση της τεχνικής 'διαίρει και βασίλευε'. Επιπλεόν εισήχθησαν νέες παρεμφερείς έννοιες όπως οι αποσυνθέσεις σφαιρικών τομών που είναι κλαδοαποσυνθέσεις στην κλαση των επίπεδων γραφημάτων που έχουν κάποιες επιπλέον ιδιότητες.

Στην εξέλιξη της έρευνας υπήρξαν σημαντικά αποτελέσμα σχετικά με το κλαδοπλάτος στην κλάση των επίπεδων γραφημάτων. Οι Fedor V. Fomin και Δημήτριος Μ. Θηλυκός απέδειξαν ότι το κλαδοπλάτος ενός επίπεδου γραφήματος με n κορυφές είναι το πολύ $\sqrt{4.5 \cdot n}$. Βασιζόμενος σε αυτό το αποτέλεσμα, ο Δημήτριος Μ. Θηλυκός συσχέτισχε το κλαδοπλάτος με μια άλλη παράμετρο σε επίπεδα γραφήματα, το r-ακτινικό σύνολο κυριαρχίας. Απέδειξε ότι αν ένα εμβαπτισμένο επίπεδο γράφημα έχει r- ακτινικό σύνολο κυριαρχίας το πολύ k, τότε το κλαδοπλάτος του γραφήματος θα είναι το πολύ $r \cdot \sqrt{4.5 \cdot k}$.

Η παρούσα διπλωματική εργασία κάνει μια ποιοτική επέκταση του αποτελέσματος αυτού. Αποδεικνύουμε ότι το παραπάνω όριο μπορεί να αναζητηθεί σε ένα δάσος που είναι υπογράφημα του δέντρου μιας αποσύνθεσης σφαιρικών τομών του γραφήματος, όπου το μέγεθος του είναι γραμμικό ως προς το k.

Abstract

An important result in Graph Theory is the proof of Wagner's Conjecture by Neil Robertson and Paul D. Seymour in *Graph Minor Series* from 1983 until 2011. This conjecture state that there is no infinite anti-chain in the class of graphs under the minor relation. The theory that was built for the proof of this conjecture had, and continues to have, an important impact not only in structural and algorithmic Graph Theory, but also in other fields such as Parameterized Complexity.

In the context of this proof, the authors have introduced some new width parameters. Within these were branchwidth and branch decompositions. This parameter was used for algorithm design via the "divide and conquer" technique. Moreover, the authors have introduced, similar to branch decompositions, concepts such as sphere-cut decompositions which are a special type of branch decompositions in planar graphs that have some additional properties.

In the course of the research there was a lot of important results about branchwidth in the class of planar graphs. Fedor V. Fomin and Dimitrios M. Thilikos proved that the branchwidth of a n-vertex planar graph is at most $\sqrt{4.5 \cdot n}$. Based on this result Dimitrios M. Thilikos connected the branchwidth with r-radial dominating set which is another parameter in plane graphs. He proved that if a plane graph has an r-radial dominating set of size at most k, then the branchwidth of the graph is at most $r \cdot \sqrt{4.5 \cdot k}$.

The purpose of this thesis is to provide a qualitative extension of this result. What we show is that this upper bound is attained by a number of edges of a sphere-cut decomposition, that is a linear function of k.

Πρόλογος

Η παρούσα διπλωματική εργασία εκπονήθηκε στα πλαίσια της ολοκλήρωση των σπουδών μου στο *Μεταπτυχιακό Πρόγραμμα Λογικής, Αλγορίθμων και Θεωρία Υπολογισμού (Μ.Π.Λ.Α.)*.

Ευχαριστώ ιδιαίτερα τον επιβλέποντα της παρούσας εργασίας, Καθηγητή κ. Δημήτριο Μ. Θηλυκό, για την στήριξη που μου προσέφερε, για το γεγονός οτι με εισήγαγε στο ιδιόμορφο πεδίο της έρευνας αλλά και για τις ενδιαφέρουσες αντιπαραθέσεις σε μη μαθηματικές συζητήσεις.

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Θα μπορούσα να γράψω πολλά λόγια για πολλούς ακόμα ανθρώπους, επειδή όμως τα πραγματικά ευχαριστώ είναι πράξεις και όχι οι λίγες γραμμές ενός προλόγου θα προτιμήσω να είμαι συνοπτικός. Θεωρώ όμως αναγκαίες και τις παρακάτω αναφορές.

Κατά τη διάρκεια της φοίτησης μου υπήρξε το ζήτημα της ανανέωσης του μεταπτυχιακού προγράμματος του Μ.Π.Λ.Α. που τελικά δεν κατέστη εφικτό. Για την αγωνία που μοιραστήκαμε, και τις ανεπαρκείς, όπως αποδείχθεικαν, προσπάθειες να αποτρέψουμε το γεγονός αυτό, θα ήθελα να ευχαριστήσω τους συμφοιτητές μου.

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Chapter 1

Introduction

1.1 About graphs

One of the most common entertaining riddle is the following:

Can you draw a given figure (for example, the left-most figure in Figure 1.1.) without picking up your pen and overlapping lines?

or

Can you draw a given figure (for example, the right-most figure in Figure 1.1.) without picking up your pen, overlapping lines, and by beginning and ending at the same point?

The solution of this riddle is the first result in the history of Graph Theory and was given by L. Euler in 1736. This result is a theorem stating when a figure

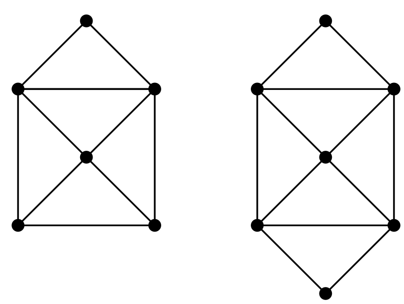

Figure 1.1: The drawing riddle

can de drawn without picking up your pen and overlapping lines (i.e. when it has an Euler path) and when a figure can be drawn without picking up your pen or overlapping lines and by beginning and ending at the same point (i.e. when it has an Euler cycle.)

Science is developed and utilized in order to fulfill certain needs (real or fictitious). In Mathematics this connection is not apparent at all. Graph Theory as a field of Mathematics, and more specifically of Discrete Mathematics, is no exception. Let consider that a graph G is a pair (V, E) where V is a finite set of elements, called vertices and E is a set of subsets of V such that the cardinality of each subset is two. The elements of E are called edges. If someone though questioned where he could find Graph theory, the answer would be Facebook. Facebook can be expressed as a graph, whose vertices are people that have signed in. For every friend-connection of two people there is an edge between their vertices. This example help us to understand some of the reasons that Graph Theory developed. Using graphs can model a lot of problems from several areas of Discrete Mathematics.

We focus our attention in Structural Graph Theory. This field of Graph Theory deals with establishing results that characterize various properties of graphs. An important result in this field is Kuratowski-Pontryagin Theorem (1930), stating that a graph is planar iff has no K_5 and $K_{3,3}$ as minor [42]. The most well-known open problem in Structural Graph Theory is Hadwinger Conjecture [37], stating that for every $k \geq 1$, every graph with chromatic number at least k contains the complete graph K_k as a minor. For $k = 1, 2, 3$, this is easy to prove, and for $k = 4$ Hadwiger [37] proved it. The conjecture also holds for $k = 5$ [6, 7, 52, 61] and $k = 6$ [54]. So far, the conjecture is open for every $k \geq 7$.

There is no field of Mathematics, and generally of science, that is developed without being influenced from others. The development of Graph Theory is directly connected with Algorithms, Complexity, and Logic. For example it is easy to consider that a lot of Graph problems such as Independent Set, Vertex Cover and Dominating Set are NP-complete. Klaus Wagner conjectured that for every infinite set of finite graphs, one of its members is a minor of another. Graph Minor Series, that is a series of 23 papers of Neil Robertson and Paul D. Seymour, was mainly dedicated to the proof of this conjecture (now Graph Minor Theorem, [55]). Two interesting surveys that describe Graph Minor Theorem and its consequences are in [46] and [39].

The theory that was developed in order to prove Wagner's conjecture has a significant impact in Algorithmic Graph Theory, Parameterized Complexity, Logic, and Computational Complexity and upgraded the importance of Graph Minor Theory. A key issue to this connection is width-parameters. Width-parameters express topological or geometrical properties of a graph. For example treewidth express how a graph is "tree-like". Typically, every width-parameter is related to a respective graph decomposition. Graph decompositions are used for algorithm design as a guide to apply the "divide and conquer" method. Moreover, width-parameters have a particular importance for Parameterized Complexity. A well-known theorem that connect width-parameters with Logic and Parameterized Complexity is Courcelle's theorem stating that every graph property definable in the monadic second-order logic of graphs can be decided in linear time on graphs of bounded treewidth [16].

At this thesis branchwidth is a width-parameter that has a central role in our results. There are a lot of interesting results related to branchwidth. One of them is that every n-vertex plane graph has branchwidth at most $\sqrt{4.5 \cdot n}$ [30]. This result was extended in [59] where it was proven that if a plane graph has an r-radial dominating set of k vertices, then branchwidth of G is at most $r \cdot \sqrt{4.5 \cdot k}$. The purpose of this thesis is to provide a qualitative extension of this result. What we show is that the upper bound proved in [59] is attained by a number of edges of the branch decomposition that is a linear function of k.

1.2 Main description of our result

We say that a tree is *ternary* if all its vertices that are non-leaves have degree 3. Given a graph G we denote by $V(G)$ and $E(G)$ the set of its vertices and edges respectively.

Carvings. Let S be a finite set and let τ be a bijection mapping the leaves of T to the elements of S. Notice that each edge e of T defines a partition $\mathcal{P}_e = \{S_1, S_2\}$ of S as follows: if T_1 and T_2 are the connected components of $T \setminus e$, then S_i contains the preimages, via τ, of the leaves of T that are leaves of T_i, for $i = 1, 2$. We define the *carving of S generated by the pair* (T, τ) as the collection of partitions

$$\mathbf{carv}(S, T, \tau) = \{\mathcal{P}_e \mid e \in E(T)\}$$

A *carving* of V is any collection of partitions of S that is generated by some pair (T, τ).

Sphere-cut decompositions. Let G be a planar graph and we consider it as being embedded in the 2-dimensional sphere $\mathbb{S}_0 = \{(x, y) \mid x^2 + y^2 = 1\}$. In that way we denote by G not only the graph itself but also some particular embedding of it. We refer to such an embedded graph as a *plane graph*. A *sphere-cut* of G is a Jordan curve N of $\mathbb{S}_0$ that does not meet the edges of G and where each of the two open disks that N defines (i.e., the connected components of $\mathbb{S}_0 \setminus N$) contains some of the edges of G. We define the *cost* of N as the number of vertices that it meets and denote as $V(N)$ the set of these vertices. Notice that each sphere-cut N defines a partition $\mathcal{P}_N$ of the edges of G into two sets: one contains the edges that are subsets of one of the connected components of $\mathbb{S}_0 \setminus N$ and the other contains the rest. A collection $\mathcal{N}$ of sphere-cuts of G is a *sphere-cut decomposition of* G if the set

$$\{\mathcal{P}_N \mid N \in \mathcal{N}\}$$

is a carving of $E(G)$. This means that there exists a pair (T, τ) where T is a ternary tree and τ is a bijection from the leaves of T to $E(G)$ such that

$$\mathbf{carv}(S, T, \tau) = \{\mathcal{P}_N \mid N \in \mathcal{N}\}$$

The *cost* of a sphere-cut decomposition is the maximum cost of its sphere-cuts. From now each sphere-cut decomposition will be denoted by the pair (T, τ) that certifies it.

Example 1.1. *A graph G and a sphere-cut decomposition of it in Figure 1.2 and a noose of an edge in Figure 1.3.*

A sphere-cut decomposition of a plane graph represents a way to recursively decompose it by recursively cutting along the sphere where it is embedded without touching its edges. If all of these cuts meet a small number of vertices, then the cost of such a decomposition is also equally small. As each cut represents a way to separate the graph, sphere-cut decompositions can be used in algorithm design as a "guide" for a divide and conquer approach. In [22] algorithms based on sphere-cut decomposition are used by Dorn et al. to solve Planar Hamiltonial Cycle , Planar TSP and Planar k-Cycle in subexponential time. Another use of them is in [21] where a subexponential algorithm is given for Hamiltonial Cycle problem on graphs of bounded genus. Moreover, the concept of sphere-cut decomposition was extended for graphs on surfaces and were introduced new types of branch decompositions. These types were used for dynamic programming [56, 57].

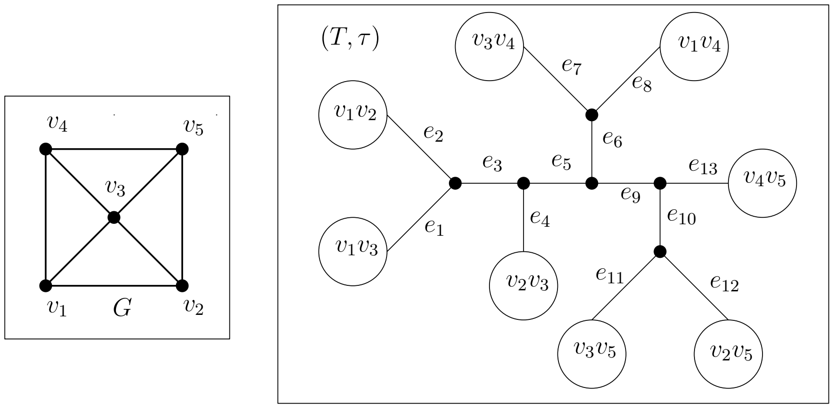

Figure 1.2: A graph G and a sphere-cut decomposition of it. Given that N_{e_i} is the sphere-cut obtained by the edge e_i of T, we denote by V_i the set of vertices that it meets and c_i the cost of the corresponding sphere-cut. Therefore, we have that $V_1 = \{v_1, v_3\}, c_1 = 2, V_2 = \{v_1, v_2\}, c_2 = 2, V_3 = \{v_1, v_2, v_3\}, c_3 = 3, V_4 = \{v_2, v_3\}, c_4 = 2, V_5 = \{v_1, v_2, v_3\}, c_5 = 3, V_6 = \{v_1, v_3, v_4\}, c_6 = 3, V_7 = \{v_3, v_3\}, c_7 = 2, V_8 = \{v_1, v_4\}, c_8 = 2, V_9 = \{v_2, v_3, v_4\}, c_9 = 3, V_{10} = \{v_2, v_3, v_5\}, c_{10} = 3, V_{11} = \{v_3, v_5\}, c_{11} = 2, V_{12} = \{v_2, v_5\}, c_{12} = 2, V_{13} = \{v_4, v_5\}, c_{13} = 2$. The cost of (T, τ) is 3.

In [30], it was proven that every n-vertex plane graph admits a sphere-cut decomposition of cost at most $\sqrt{4.5 \cdot n}$. This result improves the bounds of subexponential algorithms for a lot of problems in planar graphs. But why are those algorithms subexponential? Recall that by making use of the well-known approach of Lipton and Tarjan [44] based on the celebrated planar separator theorem [43] one can obtain algorithms with time complexity $c^{O(\sqrt{n})}$ for many problems on planar graphs. Graph decompositions is a similar approach. One can use a decomposition of a small width instead of graph separators and dynamic programming instead of the "divide and conquer" technique. The main idea is very simple: Let a problem that we are able to solve for every n-vertex graph G, which has a graph decomposition of width at most l, in time $2^{O(l(G))}n^{O(1)}$. Since the width in a class of graphs, such as plane graphs in the aforementioned result, is $O(\sqrt{n})$ the problem is solvable is subexponential time.

However, the constants "hidden" in $O(\sqrt{n})$ can be crucial for practical implementations. During the last few years, a lot of techniques have been developed to compute and improve the "hidden" constants. Sphere-cut decompositions, that

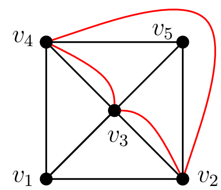

Figure 1.3: N_{e_9} is drawn with red.

we deal with, is one of them. Some other techniques are : Alber et al. use graph separators theorems in combination with linear problem kernels [4], Deineko et al. use cyclic separators of triangulations [18], Demaine and Hajiaghayi use layers of k-outerplanar graphs [20]. Other similar work is in [2, 19, 24, 40, 41].

Radial dominating sets. Let G be a plane graph and let x, y be a vertex or face of G. The *radial distance* between x and y is one less than the minimum length of a sequence $p_1, \ldots, p_q$ of alternating faces and vertices of G such that each two consecutive elements are incident to each other (see Figure 1.4).

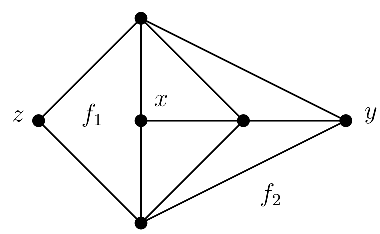

Figure 1.4: The radial distance between x and y is 4. One of the respective sequences is x, f_1, z, f_2, y.

Let G be a plane graph and let $S \subseteq V(G)$ and $r \in \mathbb{Z}_+$. We say that S is an r*-radial dominating set* of G if every face or vertex of G is in radial distance at most r from some vertex in S (see Figure 1.5).

The main result of [30] was extended in [59] where it was proven that if a plane graph has an r-radial dominating set of k vertices, then G has a sphere-cut decomposition of cost $r \cdot \sqrt{4.5 \cdot k}$. Notice that when $r = 1$, then this relation yields the main result of [30]. The purpose of this thesis is to provide a qualitative extension of this result. What we show is that the upper bound proved in [59] is attained by a number of sphere-cuts that is a linear function of k. As in most

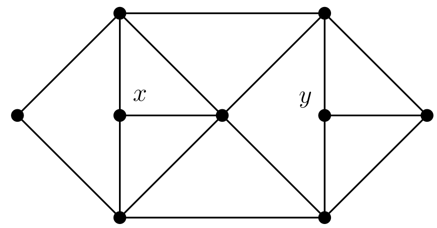

Figure 1.5: The vertices x and y are a 3-radial dominating set in this graph.

applications k is typically much smaller than the size of the graph, this implies that the "essential cost" of an optimal sphere-cut decomposition can be located in a small (linear on k) part of the graph. Interestingly, this locality phenomenon implies that the algorithmic complexity of many problems on planar graphs with small dominating sets is concentrated to restricted (linear on k) parts of the input graph [10, 27].

q-cores and their weight. To formalize the above landscape we need to introduce first the notion of a q-core of a sphere-cut decomposition. Let (T, τ) be a sphere-cut decomposition and $q \in \mathbb{Z}_{\geq 2}$. We say that a subgraph Y of T is a *q-core* of (T, τ) if every sphere-cut of G of cost greater than q corresponds to an edge of Y and none of the leaves of T is a vertex of Y. Notice that a q-core Y is not necessarily a connected subgraph of T. If Z is a connected component of Y, then we define its *weight* as the number of edges of T that contain exactly one endpoint in Z. Moreover, the *weight* of the q-core Y is the sum of the weights of its connected components. Intuitively, the edges of a q-core represent sphere-cuts that might correspond to hight-cost sphere-cuts and the weight of a q-core bounds their number.

Example 1.2. *The weight of the q-core in the Figure 1.6 is 21.*

Using this terminology, we can summarize the main result of this thesis with the following result.

Theorem 1.3. *Every plane graph with an r-radial dominating set of size at most k has a sphere-cut decomposition of cost $\leq r \cdot \sqrt{4.5 \cdot k}$ that additionally contains a $2r$-core of weight at most $3k - 6$.*

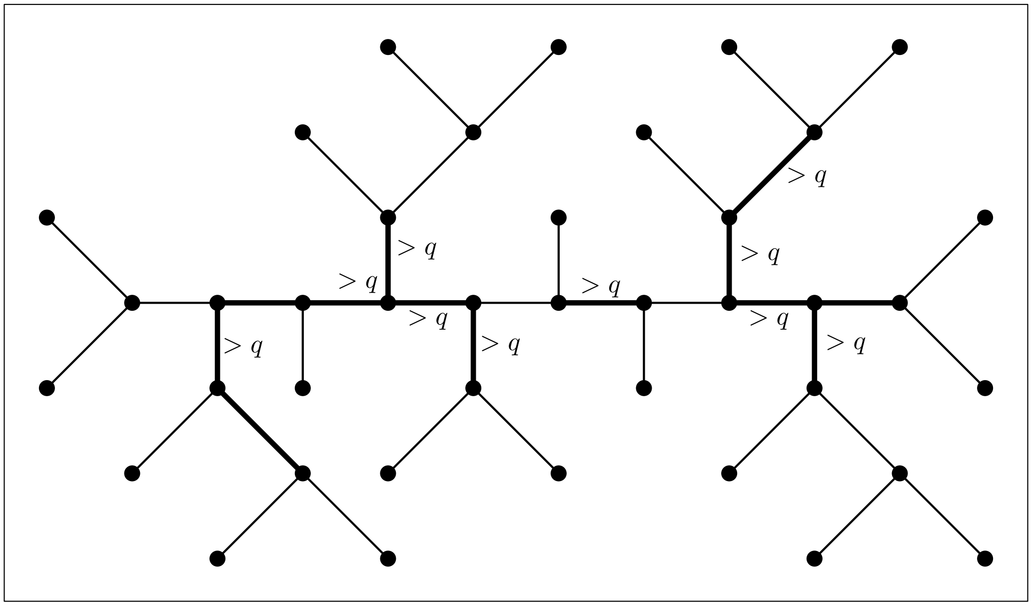

Figure 1.6: The edges with $> q$ correspond to sphere-cuts with cost $> q$. The edges that have been drawn fat are the edges of the q-core that we have choose. This q-core has 3 connected components with weight 10,4,7 respectively. The weight of the q-core is 21.

1.3 Structure of the thesis

The rest of this thesis is dedicated to the proof of Theorem 1.3. To this aim, we need to introduce several combinatorial concepts. In Chapter 2 we give some basic definitions and results about Graph Theory that are necessary for the proof of the Theorem. We refer to concepts such as connectivity, planarity, duality, etc.

In Chapter 3 we deal with partially ordered sets. A lot of binary relations can be defined in the class of graphs. Some of the most important that gather the most interest of the researchers are subgraphs, topological minors, and minors. These are the relations we need at this thesis. Graph parameters that are closed under topological minors, is a key issue for the proof of Theorem 1.3.

The subject of Chapter 4 is width-parameters and Decompositions. We introduce the reader to the most well-known width-parameter, which is treewidth, but we mainly deal with its "twin" parameter. This parameter is branchwidth and was first defined by Robertson and Seymour in [53]. We also formulate the main results associated with branchwidth that are necessary for the proof of Theorem 1.3.

Moreover, we present sphere-cut decompositions, which were mentioned in the previous section, as a special type of branch decompositions in planar graphs.

In Chapter 5, we concentrate on radial dominating sets and properties of this graph parameter. Particular reference is deserved for results linking radial dominating sets with branchwidth.

In this thesis the most research interest is in Chapters 6 and 7. We define formally the concepts of q-weight and (q, k)-capacity and prove the necessary lemmas (Chapter 6). Particularly Chapter 6 is dedicated to minor results of great importance for the proof of the main theorem, which is presented in Chapter 7. In conclusion we deal with the applications of our results.

Chapter 2

Basic definitions and results

2.1 Basic definitions

Graphs. A *graph* G is a pair (V, E) where V is a finite set of elements, called *vertices* and E is a set of subsets of V such that the cardinality of each subset is two. The elements of E are called *edges*. For a given graph G we use $V(G)$ to denote its vertex set and $E(G)$ to denote its edge set. For an edge $e = \{x, y\} \in E(G)$, the vertices x and y are called the *endpoints* of e. Two vertices x, y are called *adjacent* if $\{x, y\} \in E(G)$. We use the notation $\mathcal{G}$ for the set of all graphs.

Neighbourhood and degree. The *neighbourhood* of a vertex v in a graph G is the set $N_G(v) = \{u \in V(G) \mid \{u, v\} \in E(G)\}$. The *Degree* of a vertex v in G is $\deg_G(v) = |N_G(v)|$, the minimum degree of G is $\delta(G) = \min\{\deg_G(v) \mid v \in V(G)\}$ and maximum degree of G is $\Delta(G) = \max\{\deg_G(v) \mid v \in V(G)\}$. Similarly for $S \subseteq V(G)$, $N_G(S) = \{v \in V(G) \setminus S \mid \exists u \in S$ such that $\{u, v\} \in E(G)$. For a vertex v, (or a vertex set $S \subseteq V(G)$) $N_G[v] = N_G(v) \cup v$ ($N_G[S] = N_G(S) \cup S$ respectively). We say that a vertex v is *isolated* if $deg_G(v) = 0$.

Common graphs. We now give the definitions of some special graphs that are used frequently:

- *Path* of length k is the graph $P_k = (\{v_1, v_2, \ldots, v_{k+1}\}, \{\{v_1, v_2\}, \ldots, \{v_k, v_{k+1}\}\})$. The vertices v_1 and v_{k+1} are the endpoints of P_k
- *Cycle* of size k is the graph $C_k = (\{v_1, v_2, \ldots, v_k\}, \{\{v_1, v_2\}, \ldots, \{v_{k-1}, v_k, \{v_k, v_1\}\})$.

- *Clique* of size k is the graph $K_k = (\{v_1, v_2, \ldots, v_k\}, \{\{v_i, v_j\} \mid 1 \leq i \leq j \leq k\})$.
- $K_{k,l} = (A \cup B, \{\{a, b\} \mid a \in A, b \in B\})$ where $|A| = k, |B| = l$.

See also Figure 2.1.

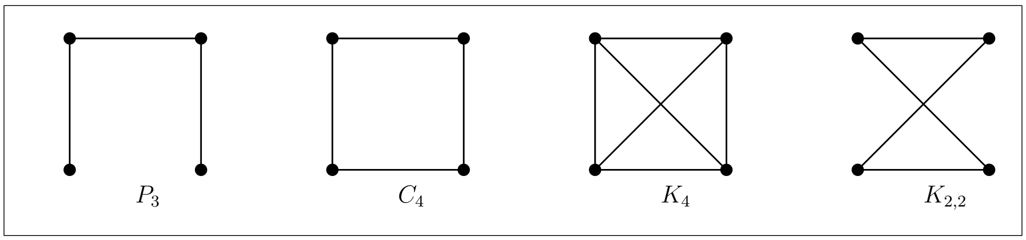

Figure 2.1: Examples of graphs that are used frequently.

The graph $G' = (V', E')$ is called *subgraph* of $G = (V, E)$, if $V' \subseteq V$ and $E' \subseteq E$. For a subset $S \subseteq V(G)$, the graph that is *induced* by the vertices of S is $G[S] = (S, \{\{u, v\} \in E(G) \mid u, v \in S\})$. Similarly for a subset $D \subseteq E(G)$, the graph that induced by the edges of D is

$$G[D] = (\{x \mid x \text{ is an endpoint of an edge } e \text{ such that } e \in D\}, D).$$

Let $S \subseteq V(G)$, the graph $G \setminus S = (V(G) \setminus S, \{\{x, y\} \in E(G) \mid \{x, y\} \cap S = \emptyset\})$ is the subgraph of G that obtained by removing the vertices of S. For $D \subseteq E(G)$, the graph $G \backslash D = (V(G), E(G) \backslash D)$ is the graph obtained by removing the edges of D.

For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *disjoint union* of these graphs is the graph $G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2)$

Connectivity. A graph G is *connected* if every two of its vertices are linked by a path in G. A maximal connected subgraph of G is a *connected component* of G. A graph G is k-connected (for $k \in \mathbb{N}$) if $V(G) > k$ and $G \setminus S$ is connected for every set $S \subseteq V(G)$ with $|S| < k$. Similarly a graph G is k-edge connected if $G \setminus D$ is connected for every set $D \subseteq E(G)$ with $|D| < k$.

A graph T is called *tree* if it is connected and contains no cycles as subgraph. A *leaf* in a tree is a vertex that has degree one. Given a tree T, we denote by $L(T)$ the set of its leaves. Also, we denote by $M(T)$ the set of all edges of T that have an endpoint in $L(T)$. The set $V(T) \setminus L(T)$ is the set of internal vertices of the tree. A *rooted tree* is a tree in which one vertex has been designated as the root, otherwise the tree is *unrooted*.

Planarity. We use the term *arc* for any subset of the plane homomorphic to the closed interval $[0, 1]$. A graph can be drawn in the plane by locating each vertex at one point of the plane and an arc from one point to another is drawn between the points corresponding to vertices connected by an edge. A graph is *planar* if it can be drawn in the plane without crossings. A *planar embedding* of a planar graph is the function that locates vertices to points and edges to curves. A graph that is embedded in the plane without crossings is called *plane* graph. Whitney proved that any two planar embeddings of a 3-connected graph are equivalent [62]. For simplicity, we do not distinguish between a vertex of a plane graph and the point of the plane used in the drawing to represent the vertex or between an edge and the open line segment representing it.

The connected components of $\mathbb{R}^2 \setminus G$, that are open subsets of the plane are called *faces*. We denote by $F(G)$ the set of the faces of G. We use the notation $A(G)$ for the set $V(G) \cup F(G)$ and we say that $A(G)$ contains the *elements* of G. If $a_i, i = 1, 2$ is an edge or an element of G, we say that a_1 is *incident* to a_2 if $a_1 \subseteq \overline{a}_2$ or $a_2 \subseteq \overline{a}_1$, where $\overline{x}$ is the closure of the set x. For every face $f \in F(G)$, we denote by $\mathbf{bor}(f)$ the *boundary* of f, i.e., the set $\overline{f} \setminus f$ where $\overline{f}$ is the closure of f.

A graph G is called *outerplanar* if there exists a face f of G such that $V(G) \subseteq \overline{f}$.

Proposition 2.1. *Every planar graph with n vertices has at most $3n - 6$ edges.*

Graph parameters. A *graph parameter* is a function mapping a plane graph to $\mathbb{N}$. There are a lot of parameters in graphs, some of them we have already seen such as $|V(G)|, |E(G)|$, or $\Delta(G)$.

Multigraphs. A *multigraph* is a graph that can have multiple edges (i.e., more than one edge between a pair of vertices) and loops, i.e., edges that connect a vertex to itself (see Figure 2.2). We say that we simplify a multigraph when we remove the loops and multiple edges and add an edge for every pair of vertices that was connected with a multiple edge. If G is a multigraph, we denote the graph obtained by the simplification as $\overline{G}$.

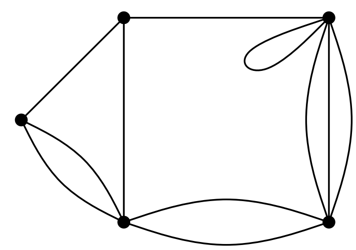

Figure 2.2: A multigraph with one loop and three multiple edges.

2.2 Basic concepts

Separators. A *separator* of a graph G is a set of vertices $S \subseteq V(G)$, where removing them from G, the number of the connected components of the remaining graph is greater than the number of connected components of G. For two vertices s, t which are in the same connected component of G, a subset $S \subseteq V(G)$ is called (s,t)-*separator* if S is a separator of G and s, t are not in the same connected component of $G \setminus S$. An (s,t)-separator S is *minimal* if there is no other (s,t)-separator S' such that S' is a proper subgraph of S. A separator S is a *minimal separator* of G if it is a minimal (s,t)-separator for some $s, t \in V(G)$.

Clique Sum. We denote $G \in G_1 \oplus_h G_2, h \in \mathbb{N}$ if G can be obtained from the disjoint union of G_1 and G_2 by identifying pairs of vertices of a clique of size h of G_1 and G_2 to form a single shared clique of size at most h, and then possibly deleting some of the clique edges.

Triconnected components. Let G be a connected graph, let $S \subseteq V(G)$, and let $V_1, \ldots, V_q$ be the vertex sets of the connected components of $G \setminus S$. We define $\mathcal{C}(G,S) = \{G_1, \ldots, G_q\}$ where G_i is the graph obtained from $G[V_i \cup S]$ if we add all edges between vertices in S.

Given a graph G, the set $\mathcal{Q}(G)$ of its *triconnected components* is recursively defined as follows:

- If G is 3-connected or a clique of size ≤ 3, then $\mathcal{Q}(G) = \{G\}$.
- If G contains a minimal separator S where $|S| \leq 2$, then we define $\mathcal{Q}(G) = \bigcup_{H \in \mathcal{C}(G,S)} \mathcal{Q}(H)$ (see Figure 2.3 for an example).

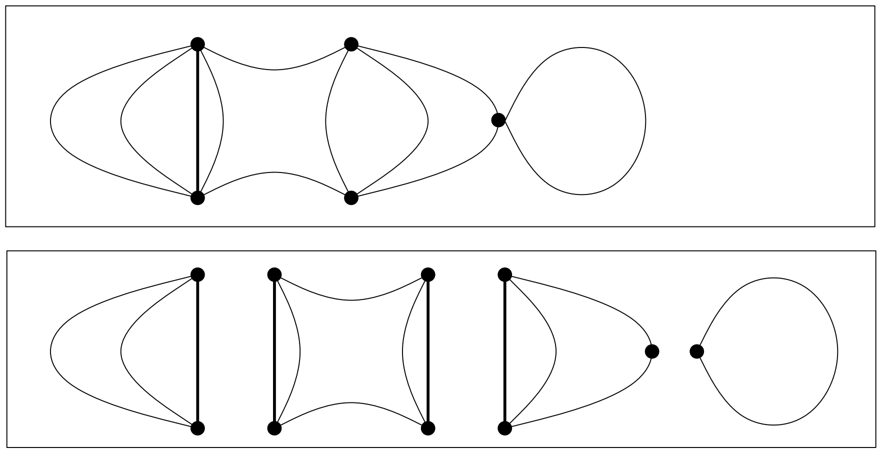

Figure 2.3: A graph and its triconnected components.

A *triangulation* H of a plane graph G is a plane graph H where $V(H) = V(G)$, $E(G) \subseteq E(H)$, and where H is triangulated, i.e., every face of H has exactly three edges incident upon it (see Figure 2.4).

Two paths P and P' with s, t as endpoints are *vertex internally disjoint* if $V(P) \cap V(P') = \{s, t\}$.

Theorem 2.2 (Menger, [48])**.** *Let G be a graph and let $s, t \in V(G)$ be distinct, non-adjacent vertices. The maximum number of vertex internally disjoint paths with s, t as endpoints equals to the minimum size of an (s,t)-separator.*

As corollary of the theorem can be obtained the next proposition.

Corollary 2.3. *A graph G is k-connected if and only if for every two non-adjacent vertices s and t there are k vertex internally disjoint paths with endpoints s and t.*

A subset $I \subseteq V(G)$ is *radially independent set* of a plane graph G if for every $x, y \in I$, there is no face f of G such that $x \subseteq \overline{f}$ and $y \subseteq \overline{f}$.

Lemma 2.4. *If G is a 3-connected plane graph, and $I \subseteq V(G)$ is a radially independent set of G, then $G \setminus I$ is a 2-connected graph.*

Proof. Let s, t be two arbitrary non-adjacent vertices of $G \setminus I$ and P_1, P_2, P_3 vertex internally disjoint paths of G with endpoints s and t. Let $v \in I$ and suppose that F_v is a subset of $V(G)$ such that

$$F_v = \{u \in V(G) \mid \exists f \in F(G) \text{ such that } u \subseteq \overline{f} \text{ and } v \subseteq \overline{f}\}.$$

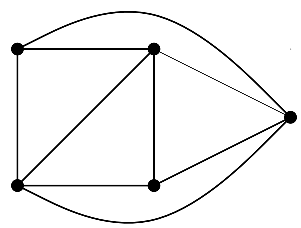

Figure 2.4: An example of a triangulated graph.

Because of 3-connectivity of G the graph $G[F_v]$ is a cycle. Of course if $(V(P_1) \cup V(P_2) \cup V(P_2)) \cap I = \emptyset$ the graph $G \setminus I$ remains 3-connected. Suppose that $v \in I$ belongs to the vertex set of P_1, then there are two vertices v_1, v_2 such that $V(P_1) \cap N_G(v) = \{v_1, v_2\}$ and P_1 is in form $s \dots v_1 v v_2 \dots t$. Let R_1, R_2 be the two paths with endpoints v_1 and v_2 in $G[F_v]$. We observe that it may be true at most one of $V(P_i) \cap V(R_1) \neq \emptyset$ and $V(P_i) \cap V(R_2) \neq \emptyset$ for $i = 2, 3$. Assume that for $i = 2$ both of them are true and $x_1 \in V(P_2) \cap V(R_1), x_2 \in V(P_2) \cap V(R_2)$. Let also that P' is the subpath of P_2 with endpoints x_1 and x_2 and R' the path of $G[F_v]$ with endpoints x_1 and x_2 that contains v_1. The disjoint union of P' and R' is a cycle C. Let Δ_1, Δ_2 be the two open discs bounded by C and assume that $v \in \Delta_1$. If $t \in \Delta_2$, then P_1 and P_2 cannot be vertex internally disjoint paths. Moreover, if $t \in \Delta_1$, then it is contradiction both of the pairs P_1, P_3 and P_2, P_3 to be vertex internally disjoint paths . □

2.3 Duality

Given a plane graph $G = (V, E)$, the dual graph of G denoted by $G^* = (V^*, E^*)$ is the graph that satisfies the following conditions:

- Every vertex of G^* is a point of a face of G.
- For every face f of G, $V^* \cap f$ contains exactly one point of the plane
- If $f_1, f_2 \in F(G)$ and $v_1, v_2 \in V^*$ such that $v_i = V^* \cap f_i$ for $i = 1, 2$, for every edge $e \in E(G)$ where $e = \overline{f_1} \cap \overline{f_2}$ there is an edge $e^* \in E^*$ with endpoints v_1 and v_2 (see also Figure 2.5).

We observe (as also shown in the example) that the dual graph of a simple graph G can be a multigraph. Moreover, if G is a simple graph and $\Delta(G) \geq 2$, then its dual has no loops.

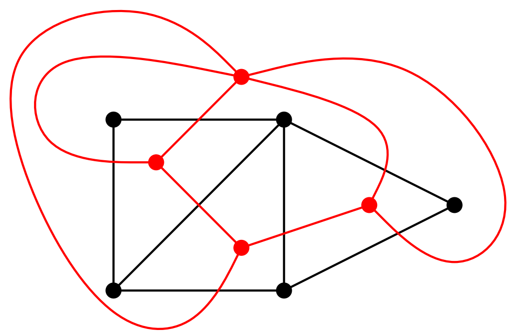

Figure 2.5: A graph drawn with black and its dual with red.

Let $G = (V, E)$ be a plane graph. The radial graph of G (we denote it as R_G) is the graph $R_G = (V(G) \cup F(G), \{\{v, f\} \mid v \in V(G), f \in F(G) \text{ and } v \subseteq \overline{f}\})$. We observe that G and its dual have the same radial graph (see Figure 2.6).

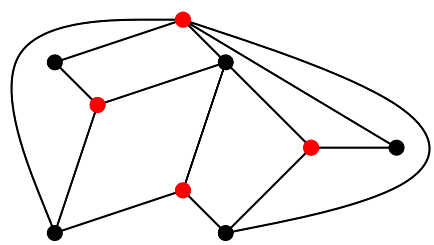

Figure 2.6: The radial graph of graphs in Figure 2.5. The black vertices are the vertices of G. The red verticesare the vertices of G^*. Observe that all the edges of R_G have one black and one red endpoint.

A graph parameter $\mathbf{p}$ on plane graphs is said to be *self-dual* if $\mathbf{p}(G) = \mathbf{p}(G^*)$.

Example 2.5. *The number of edges in a graph G is self-dual parameter, $E(G) = E(G^*)$.*

Chapter 3

Partially Ordered Sets

Posets. A partially ordered set(poset) is a set P and a binary realtion $\leq$ such that for all $x, y, z \in P$

1. $x \leq x$ (reflexivity).
2. $x \leq y$ and $y \leq z$ implies that $x \leq z$ (transitivity).
3. $x \leq y$ and $y \leq x$ implies that $x = y$ (anti-symmetry).

A pair of $x, y \in P$ are comparable if $x \leq y$ or $y \leq x$. Otherwise they are incomparble. We write $x < y$ if $x \leq y$ and $x \neq y$. A chain is a sequence $x_1 < x_2 < x_3 < \ldots < x_n$. A set A is anti-chain if every pair of elements in A are incomparable.

Posets in graphs. In the set of $\mathcal{G}$ there are a lot of interesting partial orderings. Some of them are subgraph, topological minor and minor.

As we have mentioned in the introduction, the graph $H = (V', E')$ is called subgraph of $G = (V, E)$, if $V' \subseteq V$ and $E' \subseteq E$. We denote it by $H \leq_{\mathrm{s}} G$.

A subdivision of a graph H is any graph that can be obtained from H if we apply a sequence of subdivisions to some (possibly none) of its edges (a subdivision of an edge is the operation of replacing an edge $e = \{x, y\}$ by a path with x and y as endpoints of length two. We say that a graph H is topological minor of a graph G (we denote it by $H \leq_{\mathrm{t}} G$) if some subdivision of H is subgraph of G.

A contraction of an edge $e = \{x, y\}$ is the graph $G' = (V', E')$ where $V' = V(G) \setminus \{x, y\} \cup v_{xy}$ and $E' = E(G) \setminus \{E_G(x) \cup E_G(y)\} \cup \{\{v_{xy}, v\} \mid v \in (N_G(x) \cup N_G(y)) \setminus \{x, y\}\}$. A graph H obtained by a sequence of edge contractions is said to be a contraction of G. A graph H is a minor of G (we denote it by $H \leq_{\mathrm{m}} G$) if

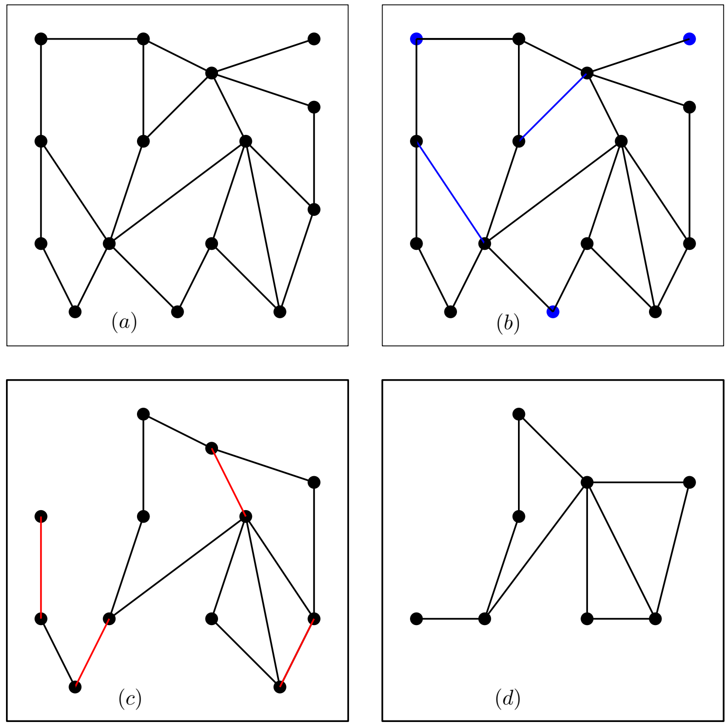

Figure 3.1: The graph in (d) is minor of the graph in (a) that is obtained by deleting the blue vertices and edges in (b) and contracting the red edges in (c).

H is a subgraph of some contraction of G (see Figure 3.1). A well-known result, that Robertson and Seymour proved in the Graph Minors series, known as Graph Minor Theorem is the following:

Theorem 3.1 (N. Robertson, P. D. Seymour [55])**.** *There is no infinite anti-chain in the class $\mathcal{G}$ under the minor relation.*

Definition 3.2. *Let $\leq$ be a relation on graphs such as $\leq_{\mathrm{s}}, \leq_{\mathrm{t}}$ or $\leq_{\mathrm{m}}$. A graph class $\mathcal{C}$ is closed under $\leq$ if for every graph $G \in \mathcal{C}$, if $H \leq G$ for a graph H, then $H \in \mathcal{C}$.*

Definition 3.3. *The obstruction set of a graph class $\mathcal{C}$ which is closed under $\leq$,* **obs**$(\mathcal{C})$ *is the minimal set of graphs $\mathcal{H}$ satisfying the following property*

$$G \in \mathcal{C} \Leftrightarrow \text{ for every } H \in \mathcal{H}, H \not\leq G$$

Example 3.4. *The obstruction set in the class of trees $\mathcal{T}$, for $\leq \in \{\leq_{\mathrm{s}}, \leq_{\mathrm{t}}, \leq_{\mathrm{m}}\}$, is $\mathbf{obs}(\mathcal{T}) = K_3$.*

For the obstruction set in the class of planar graphs for $\leq \in \{\leq_{\mathrm{m}}\}$ the next result that was proven by L. Pontryagin around 1927, however he never published his proof. Independently K. Kuratowski published his proof in 1930 [42]. Now the result is known as Kuratowski's theorem or Kuratowski–Pontryagin theorem.

Theorem 3.5. *A graph G is planar if and only if has none of K_5 and $K_{3,3}$ as minor.*

The following result which we will use later is a consequence of the previous theorem.

Proposition 3.6. *A graph G is outerplanar if and only if has none of K_4 and $K_{2,3}$ as minor.*

Proof. It is easy to observe that K_4 and $K_{2,3}$ are not outerplanar graphs. Also, the class of outerplanar graphs is closed under minors, so if a graph G has K_4 or $K_{2,3}$ as minor is not outerplanar. Now let G is non-outerplanar graph and set $G^+ = (V(G) \cup v, E(G) \cup \{v, x\} \mid x \in V(G))$. We claim that G^+ is a non-planar graph. Assume that G^+ is planar and is given to us with an embedding of it. Then v belong to a face of G that its boundary contains all the vertices of G. This is a contradiction. From Theorem 3.5, G^+ has no K_5 and $K_{3,3}$ as minor. We observe that removing a vertex can affect either a vertex or an edge of a subdivision of a graph. In any case that means that G has either K_4 or $K_{2,3}$ as minor. $\square$

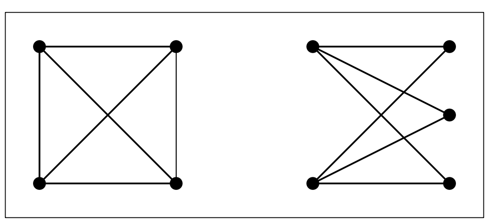

Figure 3.2: The obstruction set of outerplanar graphs, K_4 and $K_{2,3}$.

A graph parameter $\mathbf{p}$ is said to be closed under a partial ordering of P if the following holds: if $H, G \in P$ and $H \leq G$, then $\mathbf{p}(H) \leq \mathbf{p}(G)$.

Example 3.7. *If $H \leq_t G$, then $\Delta(H) \leq \Delta(G)$.*

At the next chapters we will mainly refer to graph parameters which are closed under topological minors.

Chapter 4

Width-parameters and Decompositions

In this chapter we deal with width-parameters. Width-parameters have a lot of applications in several areas such as graph searching, structural graph theory, parameterized algorithms, and others. Moreover, it is easier to understand the structure of a graph if we know a width-parameter of it. For example a well-known parameter, which has a lot of applications is treewidth.

Treewidth. A tree decomposition of a graph G is a pair $\mathcal{T} = (T, \{X_t\}_t \in V(T))$ where T is a tree whose every node t is assigned a vertex subset $X_t \subseteq V(G)$ called a bag, such that the following three conditions hold:

1. $\bigcup_{t\in V(T)} X_t = V(G)$. In other words, every vertex of G is in at least one bag.
2. For every edge $\{u, v\} \in V(G)$ there exists a node t of T such that bag X_t contains both u and v.
3. For every $u \in V(G)$, the set $Tu = t \in V(T) : u \in X_t$, i.e., the set of nodes whose corresponding bags contain u, induces a connected subtree of T.

The width of a tree decomposition $\mathcal{T} = (T, \{X_t\}_t \in V(T))$ is equal to

$$\max_{t\in V(T)} |X_t| - 1.$$

The treewidth of a graph G, denoted by $\mathbf{tw}(G)$, is the minimum possible width of a tree decomposition of G. Intuitively treewidth shows how similar is a graph with a tree.

Example 4.1. *The following propositions hold:*

1. *The treewidth of trees is 1.*

2. *The treewidth of K_k is $k-1$.*

3. *The treewidth of the graph in Figure 4.1 is 2.*

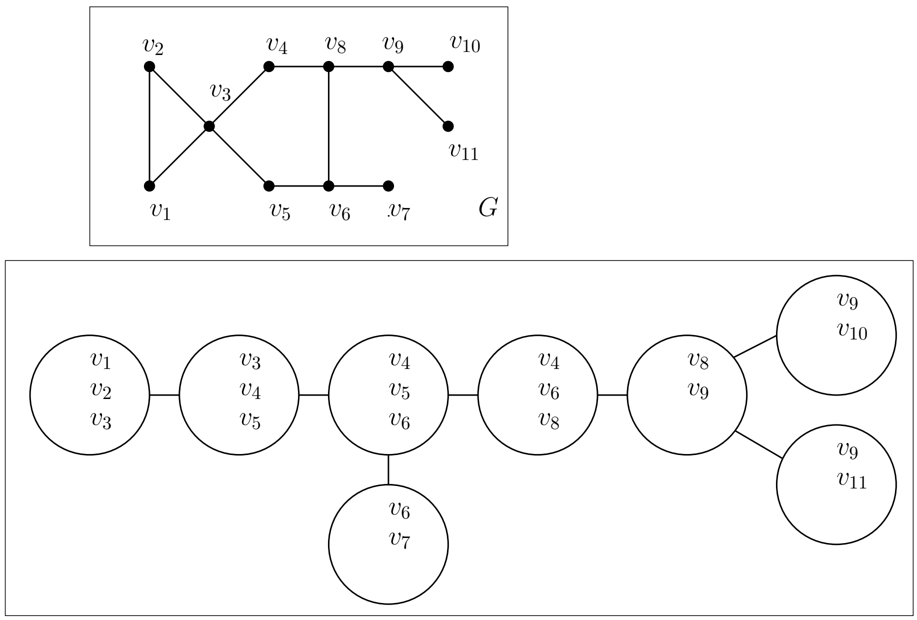

Figure 4.1: A graph G with a tree decomposition of it. As easily can be seen from the decomposition, every bag has at most 3 vertices, so $\mathbf{tw}(G) = 2$.

By the definition of tree decomposition, if t_1, t_2 are connected vertices in T, then $X_{t_1} \cap X_{t_2}$ is a separator of the graph. This property of tree decompositions is very useful to deal with algorithmic problems on graphs [8,9]. In what follows, we mainly deal with its "twin" parameter, which is a constant factor approximation of treewidth. This parameter is branchwidth and was first defined by Robertson and Seymour in [53]. There branchhwidth has been introduced as an alternative to the parameter of treewidth, as it appeared to be easier to handle for the purposes of their proofs. We stress, that especially for our study, branchwidth is much more suitable for dealing and exposing the concepts related to our main result.

4.1 Branchwidth

Let G be a graph on n vertices. A *branch decomposition* (T, τ) of a graph G consists of an unrooted ternary tree T (i.e., all internal vertices are of degree three) and a bijection $\tau : L \to E(G)$ from the set L of leaves of T to the edge set of G. We define for every edge e of T the *middle set* $\omega(e) \subseteq V(G)$, as follows: Let T_1^e and T_2^e be the two connected components of $T \setminus e$. Then, let G_i^e be the graph induced by the edge set $\{\tau(f) : f \in L \cap V(T_i^e)\}$ for $i \in \{1, 2\}$.

The *middle set* is the intersection of the vertex sets of G_1^e and G_2^e, i.e., $\omega(e) = V(G_1^e) \cap V(G_2^e)$. We denote $\mathbf{mid}_G(e) = |\omega(e)|$. The *width* of (T, τ) is the maximum order of the middle sets over all edges of T (in case T has no edges, then the width of (T, τ) is equal to 0).

The *branchwidth*, denoted by $\mathbf{bw}(G)$, of G is the minimum width over all branch decompositions of G.

Example 4.2. *The following propositions hold:*

1. *The branchwidth of a graph is at most 1 iff it does not contain a path on 4 vertices as a subgaph.*
2. *The branchwidth of cycles is 2.*
3. *The branchwidth of K_k is at most $\lceil 2k/3 \rceil$.*
4. *The graph G of Figure 4.2 has $\mathbf{bw}(G) = 2$.*

We can observe that in a branch decomposition (T, τ), for every edge $e \in E(T)$ the vertices of G that belongs to $\omega(e)$ is a separator of G. The relation between branchwidth and treewidth is given by the following result

Proposition 4.3 (P. D. Seymour and R. Thomas [53]). *If G is a graph, then* $\mathbf{bw}(G) \leq \mathbf{tw}(G) + 1 \leq \frac{3}{2}\mathbf{bw}(G)$

Now let us see some other results related on branchwidth.

Proposition 4.4 (P. D. Seymour and R. Thomas [53]). *A graph G has branchwidth ≤ 2 if and only if has no K_4 as minor.*

Proposition 4.5 (F. V. Fomin, D. M. Thilikos [30]). *If G is a n-vertex planar graph, then* $\mathbf{bw}(G) \leq \sqrt{4.5 \cdot n}$.

Proposition 4.6 (F. V. Fomin, D. M. Thilikos [28]). *If G is a graph that contains a cycle, then* $\mathbf{bw}(G) = \max\{\mathbf{bw}(H) \mid H \in \mathcal{Q}(G)\}$.

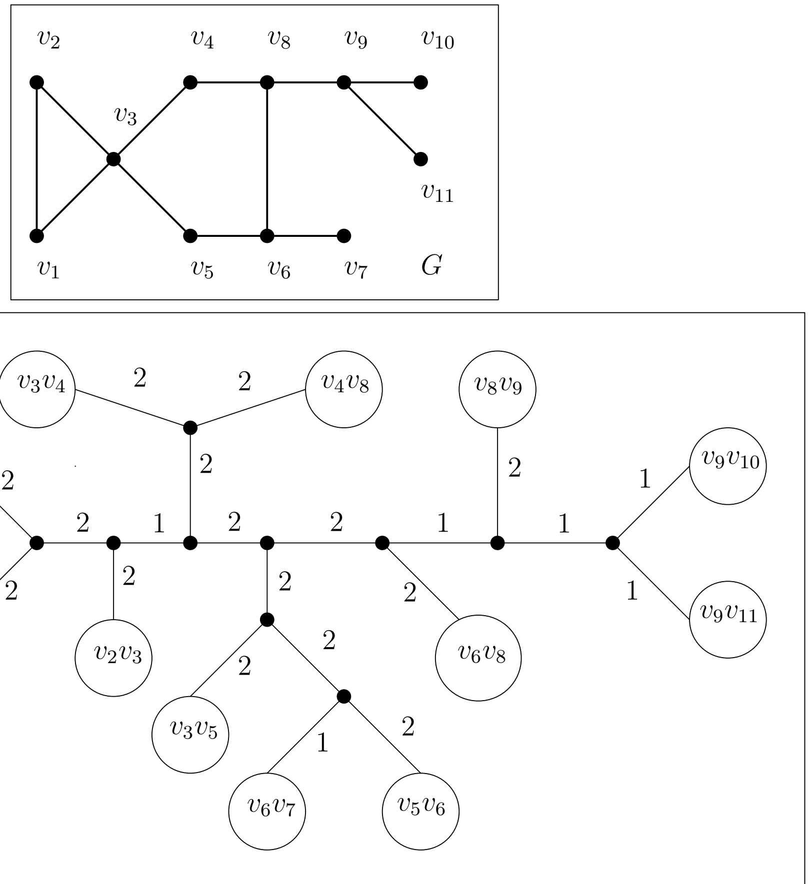

Figure 4.2: A graph G and a branch decomposition of it.

Proposition 4.7 (P. D. Seymour and R. Thomas [53])**.** *The parameter* **bw** *is closed under topological minors, i.e., if $H \leq_{\mathrm{t}} G$, then* $\mathbf{bw}(H) \leq \mathbf{bw}(G)$.

Proposition 4.8 ([47, 58])**.** *If G is a plane graph with a cycle, then* $\mathbf{bw}(G) = \mathbf{bw}^*(G)$.

Lemma 4.9. *If G is a multigraph and $\overline{G}$ is a graph with a cycle, then* $\mathbf{bw}(\overline{G}) = \mathbf{bw}(G)$.

Proof. Let $D = (T, \tau)$ be a branch decomposition of $\overline{G}$ that achieves the minimum width. If x and y are vertices of G that are connected with $l \geq 2$ edges, let E_{xy} be the set of these edges. Then suppose that T' is a rooted binary tree with l leaves. Let r be the root of the tree and f a function such as $f : E_{xy} \to L(T')$. Let z be the only neighbour of $\tau(\{x,y\})$ in T, then we delete the edge $\{z, \tau(\{x,y\})$ from T and add T' in T by identifying z with r and extend τ with f. Similarly for the loops of a vertex x and $\{x,y\}$ an arbitary edge of $\overline{G}$, we divide the edge of T (let z be the new vertex) that has $\tau(\{x,y\})$ as an endpoint and identify z with r where r is the root of the binary tree with the loops of x. With this process for every multiple edge and all the loops of G a branch decomposition of $\overline{G}$ that its width is equal to the width of D is obtained. □

4.2 Sphere-cut decompositions.

In the introduction we defined sphere-cut decompositions using carvings. Now we give another equivalent definition which is more suitable for our proofs. This definition examine sphere-cut decompositions as a special type of branch decompositions in plane graphs.

Given a plane graph G, an arc I that does not intersect the edges of G (i.e., $I \cap G \subseteq V(G)$) is called *normal.* The *length* $|I|$ of a normal arc I is equal to the number of elements of $A(G)$ that it intersects minus one. If x and y are the elements of $A(G)$ intersected by the extreme points a normal arc I, then we also call I *normal* (x,y)*-arc*. A *noose* of the plane, where G is embedded, is a Jordan curve that does not intersect the edges of G. We also denote by $V(N)$ the set of vertices of G met by N, i.e., $V(N) = V(G) \cap N$.

The *length* $|N|$ of a noose N is $|V(N)|$, i.e., is the number of the vertices it meets.

Let G be a plane graph. A branch decomposition (T,τ) of G is called a *sphere-cut decomposition* if for every edge e of T there exists a noose N_e, such that

(a) $\omega(e) = V(N_e)$,

(b) $G_i^e \subseteq \Delta_i \cup N_e$ for $i = 1,2$, where Δ_i is the open disc bounded by N_e, and

(c) for every face f of G, $N_e \cap f$ is either empty or connected (i.e., if the noose traverses a face, then it traverses it once).

We denote by $\mathcal{SC}_k(G)$ the set of all sphere-cut decompositions of G with width at most k.

Proposition 4.10 (P. D. Seymour, R. Thomas [58])**.** *Let G be a planar graph where $\delta(G) \geq 2$ and with branchwidth at most k embedded on a sphere. Then there exists a sphere-cut decomposition of G of width at most k.*

Chapter 5

Radial Dominating Set

Let G be a plane graph and let r be a non-negative integer. Given two elements $x, y \in A(G)$, we say that they are *within radial distance at most* r if there is a normal (x, y)-arc of the plane of length at most r. We denote this fact by $\mathbf{rdist}_G(x, y) \leq r$

Given a vertex set $S \subseteq V(G)$ and a non-negative integer r, we denote by $\mathbf{R}_G^r(S)$ the set of all elements of G that are within radial distance at most r from some vertex in S. We say that a set $S \subseteq V(G)$ is an *r-radial dominating set* of G (or, alternatively we say that *S r-radially dominates G*) if $\mathbf{R}_G^r(S) = A(G)$. We define

$$\mathbf{rds}_r(G) = \min\{k \mid \text{ } G \text{ contains an } r\text{-radial dominating set of size at most } k\}.$$

Observation 5.1. *The parameter* **rds** *is closed under topological minors. In other words, if H, G are graphs, $r \in \mathbb{N}$, and $H \leq_{\mathrm{t}} G$, then $\mathbf{rds}_r(H) \leq \mathbf{rds}_r(G)$.*

Observation 5.2. *If G is a multigraph, then $\mathbf{rds}_r(G) = \mathbf{rds}_r(\overline{G})$.*

Let G be a plane graph, $y \in \mathbb{N}$, and $S \subseteq V(G)$. We say that S is *y-radially scattered* if for any $a_1, a_2 \in S$, $\mathbf{rdist}_G(a_1, a_2) \geq y$. We say that S is *r-radially extremal in G* if S is an r-radial dominating set of G and S is $2r$-radially scattered in G.

The relation between radial domination and radially extremal set is provided by the following result.

Proposition 5.3 (D. M. Thilikos [59])**.** *Let G be a 3-connected plane graph and S be an r-radial dominating set of G. Then G is the topological minor of a triangulated 3-connected plane graph H where S is r-radially extremal in H.*

The Proposition 5.3 served as the main tool of the proof of the main theorem of [59] (combined with Proposition 4.5 as induction basis) which is the following.

Proposition 5.4 (D. M. Thilikos [59])**.** *Let r be a positive integer and let G be a plane graph. Then* $\mathbf{bw}(G) \leq r \cdot \sqrt{4.5 \cdot \mathbf{rds}_r(G)}$*.*

Based on the same induction rule with the previous theorem, we prove the following lemma.

Lemma 5.5. *For every positive integer r, if G is a plane graph with* $\mathbf{rds}_r(G) \leq 2$*, then* $\mathbf{bw}(G) \leq 2r$*.*

Proof. We use induction on $r \geq 2$. For $r = 1$ the result is trivial but cannot be used as induction basis because there are no edges in the branch decomposition. For the induction basis, if $r = 2$ we examine the cases where $\mathbf{rds}_2(G) = 1$ or $\mathbf{rds}_2(G) = 2$. If $\mathbf{rds}_2(G) = 1$, that means that G is outerplanar graph. By Proposition 3.6, G has no K_4 as minor and follows by Proposition 4.4 that branchwidth of G is at most 2. If $\mathbf{rds}_2(G) = 2$ let s_1, s_2 be the 2-radial dominating set of G. If we remove s_1, s_2 from G the remaining graph will be a cycle. The branchwidth of a cycle is 2, so by adding s_1, s_2 branchwidth of G will be at most 4. From these two cases it is clear that if $\mathbf{rds}_2(G) \leq 2$, then $\mathbf{bw}(G) \leq 4$.

Assume now that the lemma holds for values smaller than r and we will prove that it also holds for r where $r > 2$. Using Proposition 4.6, we choose $H \in \mathcal{Q}(G)$ such that $\mathbf{bw}(H) = \mathbf{bw}(G)$ (we may assume that G contains a cycle, otherwise the result follows trivially). By Observation 5.1, $\mathbf{rds}_r(H) \leq \mathbf{rds}_r(G)$. Let S be a r-radial dominating set of H where $|S| \leq 2$. From Theorem 5.3, H is the topological minor of a 3-connected plane graph H_1 where S is r-radially extremal.

Let H_2 be the graph obtained if we remove from H_1 the vertices of S. Because of 3-connectivity of H_1 for any $v \in S$ the graph $H[N_{H_1}(v)]$ is a cycle and each such cycle is the boundary of the face of H_2. We denote by F the set of these faces and observe that F^* is a $(r-1)$-radial dominating set of H_2^* (we denote by F^* the vertices of H_2^* that are duals of the faces of F in H_2). Moreover, the fact that S is a $2r$-scattered dominating set in H_1 implies that F^* is a $2(r-1)$-scattered dominating set in H_2^*. From the induction hypothesis and the fact that $|S| = |F^*|$, we obtain that $\mathbf{bw}(H_2^*) \leq 2(r-1)$. This fact along with Proposition 4.8 implies that $\mathbf{bw}(H_2) \leq 2(r-1)$.

In graph H_2, for any face $f_i \in F$, let $(x_0^i, \ldots, x_{m-1}^i)$ be the cyclic order of the vertices in its boundary cycle. We also denote by x^i the vertex in H_1 that was removed in order f_i to appear in H_2. Let (T, τ) be a branch decomposition of

H_2 of width at most $2(r-1)$. By Proposition 4.10, we may assume that (T,τ) is a sphere cut decomposition. We use (T,τ) in order to construct a branch decomposition of H_1 by adding new leaves in T and mapping them to the edges $E(H_1)/E(H_2) = \cup_{i=1,\dots,|F|}\{\{x^i, x^i_h\} \mid h = 0,1,\dots m-1\}$ in the following way: for every $i = 1,\dots,|F|$ and $h = 0,1,\dots m-1$ we set $t^i_h = \tau^{-1}(\{x^i_h, x^i_{h+1 \mod m_i}\})$ and let $e^i_h = \{y^i_h, t^i_h\}$ be the unique edge of T that is incident to t^i_h. We subdivide e^i_h and we call the subdivision vertex s^i_h. We also add a new vertex z^i_h and make it adjacent to s^i_h. Finally, we extend the mapping of τ by mapping the vertex z^i_h to the edge $\{x^i, x^i_h\}$ and we use the notation (T',τ') for the resulting branch decomposition of H_1. We claim that the width of (T',τ') is at most $2r$.

To prove this, we use the functions ω and ω' to denote the middle sets of (T,τ) and (T',τ') respectively. Let e be an edge of T'. If e is not an edge of T, then $|\omega'(e)| \leq 3$. If e is also an edge of T, let N_e be the noose of H_2 meeting the vertices of $\omega(e)$. Because $|F|$ is at most 2, N_e meets at most all the faces of F, then the vertices in $\omega'(e)$ of a noose N'_e of H_1 meeting all vertices of $\omega(e)$ plus at most 2, x^1, x^2. Then $|\omega'(e)| \leq |\omega(e)| + 2 \leq 2(r-1) + 2 = 2r$. Therefore the width of (T',τ') is at most $2r$.

We just proved that $\mathbf{bw}(H_1) \leq 2r$. As H is topological minor of H_1, from Proposition 4.7, $\mathbf{bw}(H) \leq 2r$ and also from Proposition 4.6, $\mathbf{bw}(H) = \mathbf{bw}(G) \leq 2r$. □

At the next lemma we give a relation between **rds** of a graph G which is an 1- or 2-clique sum of two other graph with the **rds** of these graphs.

Lemma 5.6. *Let G, G_1, and G_2 be plane connected graphs such that $\mathbf{rds}_r(G) \leq k$ and $G \in G_1 \oplus_h G_2$ for some $h \in \{1,2\}$. Then $\mathbf{rds}_r(G_1) + \mathbf{rds}_r(G_2) \leq k+2$.*

Proof. Let $S \subset V(G)$ be the vertices of G that are in G_1 and also in G_2, k_1 (k_2 respectively) is the number of vertices of $\mathbf{rds}_r(G)$ that are in $G_1 \setminus S$ ($G_2 \setminus S$ respectively). Let l be the number of vertices of $\mathbf{rds}_r(G)$ that are in S, so $k_1 + k_2 + l = k$. If $l = 0$, it is clear that if a vertex or a face x in G_1 is dominated in G by a vertex of $\mathbf{rds}_r(G)$, that is also in G_2, then if in $\mathbf{rds}_r(G_1)$ we add a vertex s of S, x will be dominated by s. So if $l = 0$, $\mathbf{rds}_r(G_1) \leq k_1 + 1$ and $\mathbf{rds}_r(G_2) \leq k_1 + 1$, then $\mathbf{rds}_r(G_1) + \mathbf{rds}_r(G_2) \leq k+2$. If $l = 1$ or $l = 2$, then there is no need to add any vertex to dominate vertices or faces of G_1 or G_2, so $\mathbf{rds}_r(G_1) + \mathbf{rds}_r(G_2) \leq k+2$. □

Chapter 6

Weight and Capacity

6.1 q-weight

Let G be a graph and let $D = (T, \tau)$ be a branch decomposition of G. Given a $q \geq 2$ and a subgraph Y of T, we say that Y is a *q-core* of D if

- $L(T) \cap V(Y) = \emptyset$
- $\forall e \in E(T)$ if $\mathbf{mid}_G(e) > q$, then $e \in E(Y)$.
- There are no isolated vertices in Y.

We denote by $\mathcal{C}(Y)$ the connected components of Y. For each $Z \in \mathcal{C}(Y)$ we define its extension $\hat{Z}$ as the subtree of T obtained if in Z we add all edges with one endpoint in $V(Z)$, we also call these edges *boundary* of Z and we denote it by ∂Z. The *q-weight* of Y in D is defined as the quantity

$$\mathbf{w}(Y) = \sum_{Z \in \mathcal{C}(Y)} |\partial Z|$$

Given a graph G a $q \geq 2$ and a $k \geq 2$, we define (q,k)-*capacity* of G as follows:

$$\begin{aligned}\mathbf{cap}_{q,k}(G) \quad = \quad & \min\{s \mid \exists D \in \mathcal{SC}_k(G), \text{ there exists a } q\text{-core } Y \\ & \text{of } D \text{ such that } \mathbf{w}(Y) \leq s\} \cup \{0\}.\end{aligned} \tag{6.1}$$

6.2 Before the main proof

Observation 6.1. *If G is a multigraph and $\overline{G}$ is a graph with a cycle, then* $\mathbf{cap}_{q,k}(G) = \mathbf{cap}_{q,k}(\overline{G})$.

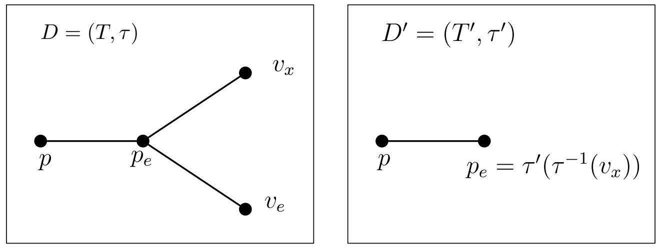

Figure 6.1: D and D' are branch decompositions of G_2 and G_1 respectively, where $G_1 = G_2 \setminus e$. If $\{p_e, p\} \in E(Y)$ in D, then $Y' = Y \setminus e\{p_e, p\}$, otherwise $Y' = Y$.

The proof does as in Lemma 4.9, by choosing the same q-core for the branch decomposition of G as the branch decomposition of $\overline{G}$.

Lemma 6.2. *Let $q, k \geq 2$ and let G_1 and G_2 be graphs where G_1 is a topological minor of G_2. Then $\mathbf{cap}_{q,k}(G_1) \leq \mathbf{cap}_{q,k}(G_2)$.*

Proof. Let $D = (T, \tau)$ be the branch decomposition of G_2 and let Y is a q-core of D such that Y achieves the minimum q-weight $c_{q,k}(G_2)$. Let G_1 be the graph that obtained if we remove an edge e from G_2 and let v_e be the vertex of T such that $\tau(e) = v_e$. Let also p_e be the parent node of v_e and v_x be the other vertex of $L(T)$ that is connected with p_e in T. Let $D' = (T, \tau')$ be the branch decomposition of G_1 obtained from D if we remove the edges $\{p_e, v_e\}$ and $\{p_e, v_x\}$ and $\tau'(\tau^{-1}(v_x)) = p_e$. We choose Y' such as follows: let p be the third neighbour of p_e in T. If $\{p_e, p\} \in E(Y)$, then Y' be the graph induced by $E(Y) \setminus \{p_e, p\}$. If $\{p_e, p\} \notin E(Y)$, then $Y' = Y$. If $e \in E(T) \cap E(T')$, $\mathbf{mid}_{G_1}(e) \leq \mathbf{mid}_{G_2}(e)$ so every edge of $E(T) \cap E(Y)$ can also be in $E(Y')$. Then the boundary of Y' is at most equal with the boundary of Y so $\mathbf{c}_{q,k}(G_1) \leq \mathbf{c}_{q,k}(G_2)$ (see Figure 6.1).

Removing a vertex is the same as removing all the edges that have this vertex as an endpoint.

Now let G_1 be obtained if we contract a vertex v of degree 2 in G_2. Let v_1, v_2 be the two neighbours of v in G_2 and $\{v_1, v_2\}$ the edge after the contraction of v in G_1. The branch decomposition $D' = (T', \tau')$ obtained by removing the vertex $\tau(\{v_1, v\})$ in T and do the same as before with p_e and also $\tau'(\{v_1, v_2\}) = \tau(\{v_2, v\})$. We choose Y' as before. Moreover, if $e \in E(T) \cap E(T')$, $\mathbf{mid}_{G_1}(e) \leq \mathbf{mid}_{G_2}(e)$ so every edge of $E(T) \cap E(Y)$ can also be in $E(Y')$. Then the boundary of Y' is at most equal with the boundary of Y so $\mathbf{c}_{q,k}(G_1) \leq \mathbf{c}_{q,k}(G_2)$ (see Figure 6.2).

□

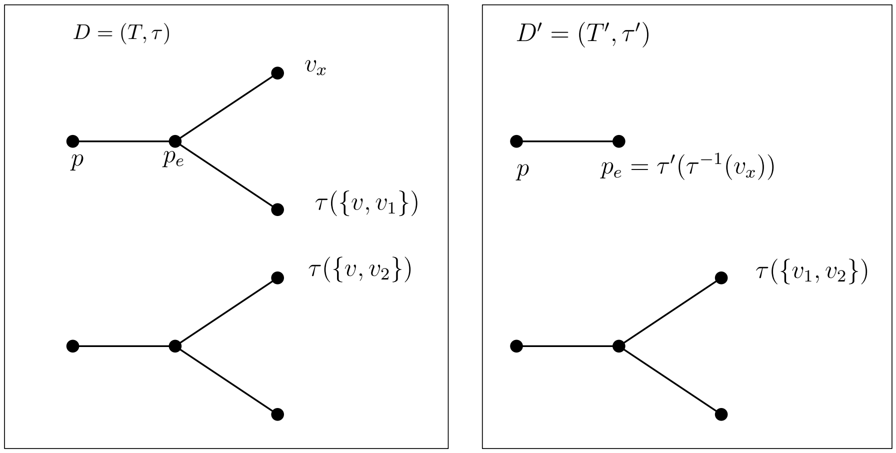

Figure 6.2: D and D' are branch decompositions of G_2 and G_1 respectively, where G_1 is obtained by contracting a vertex v of degree 2 with neighbours v_1, v_2. If $\{p_e, p\} \in E(Y)$ in D, then $Y' = Y \setminus e\{p_e, p\}$, otherwise $Y' = Y$.

Lemma 6.3. *Let G be a 2-edge connected plane graph and let G^* be its dual and $q, k \geq 2$.* $\mathbf{cap}_{q,k}(G) = \mathbf{cap}_{q,k}(G^*)$

Proof. Let G has a sphere-cut decomposition $D = (T, \tau)$ with width at most k and D be a q-core Y. We define the branch decomposition $D^* = (T^*, \tau^*)$ of G^* where $T^* = T$ and for each $e^* \in E(T^*)$, $\tau^*(e^*) = \tau(e)$. We also set $Y^* = Y$. Notice that D^* is a sphere-cut decomposition.
Claim : For every $e \in E(T)$, $\mathbf{mid}_{G^*}(e) = \mathbf{mid}_G(e)$.
Proof of the claim. Let R_G be the radial graph of G that is the same for G^*. Assume also that every noose N_e meets only points that are in edges and vertices of R_G and not in the faces. Let $V_R = N_e \cap V(R_G)$ and that means $V_R = (V_R \cap V(G)) \cup (V_R \cap V(G^*))$. It is clear that in N_e there is an alternative sequence between the vertices of $(V(R) \cap V(G))$ and the vertices of $(V(R) \cap V(G^*))$, so $\mathbf{mid}_{G^*}(e) = \mathbf{mid}_G(e)$ (see Figure 6.3).

This implies that D has width at most k and that, $\mathbf{w}(Y^*) = \mathbf{w}(Y)$. □

Lemma 6.4. *Let G, G_1, and G_2 be plane connected graphs such that $G \in G_1 \oplus_h G_2$ for some $h \in \{1, 2\}$. Let also $D_i = (T_i, \tau_i)$ be a branch decomposition of G_i, Y_i is a q-core of D_i, $i \in \{1, 2\}$. Then there is a branch decomposition $D = (T, \tau)$ of G and Y is a q-core of D such that* $\mathbf{w}(Y) \leq \mathbf{w}(Y_1) + \mathbf{w}(Y_2)$ *and the width of D is the maximum width of D_1, D_2*

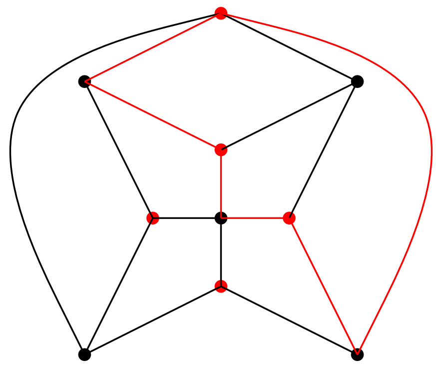

Figure 6.3: The radial graph of the graph G of the Figure 1.2. The black vertices are the vertices of G and the red are the vertices of G^*. The red edges consistute the noose of the edge e_9 in Figure 1.3 of T of the sphere-cut decomposition of Figure 1.2.

Proof. For the proof of this lemma we construct a branch decomposition of G, $D = (T, \tau)$ based on $D_i = (T_i, \tau_i)$ of $G_i, i \in \{1, 2\}$. Denote by ω the function indicating the middle set of the edges of T and by ω_1, ω_2 indicating the middle sets of the edges of T_1, T_2 respectively. There are three cases:

1. $G \in G_1 \oplus_2 G_2$. Let a, b be the vertices of G such that $V(G_1) \cap V(G_2) = \{a, b\}$ and additionally assume that, $\{a, b\}$ is not an edge of G,

2. Like above, but now $\{a, b\}$ is an edge of G and

3. $G \in G_1 \oplus_1 G_2$.

In the first case, we remove in T_1 the edge e'_1 that has an endpoint which corresponds to $e = \{a, b\}$ in G_1 and we call p_1 the parent node of this endpoint. Similarly remove e'_2 and call p_2 in T_2. To construct the tree T of the branch decomposition of G we add the edge $\{p_1, p_2\}$. We define $\tau : E(G) \to L(T)$ such as follows: if $e \in E(G) \cap E(G_i), i \in \{1, 2\}$, then $\tau(e) = \tau_i(e)$. From the construction of T if $e \in E(T) \cap E(T_i), i \in \{1, 2\}$, then $\omega(e) = \omega_i(e)$ and $\omega(\{p_1, p_2\}) = \{a, b\}$. We choose Y such as $Y = Y_1 \cup Y_2$ and it satisfies the properties of the lemma. That means that for $e \in E(T) \cap E(T_i), i \in \{1, 2\}$ if $e \in Y_i$, then $e \in Y$ and also if e belongs to the boundary of Y_i, then e belongs to the boundary Y. If e'_i is in the boundary of Y_i, then $\{p_1, p_2\}$ is in the boundary of Y. If this is true for e'_1 and e'_2, then we count $\{p_1, p_2)$ twice in the q-weight of Y. Now it is obvious that for this case there is a branch decomposition $D = (T, \tau)$ of G and Y a q-core of D such that $\mathbf{w}(Y) \leq \mathbf{w}(Y_1) + \mathbf{w}(Y_2)$ (see also Figure 6.4).

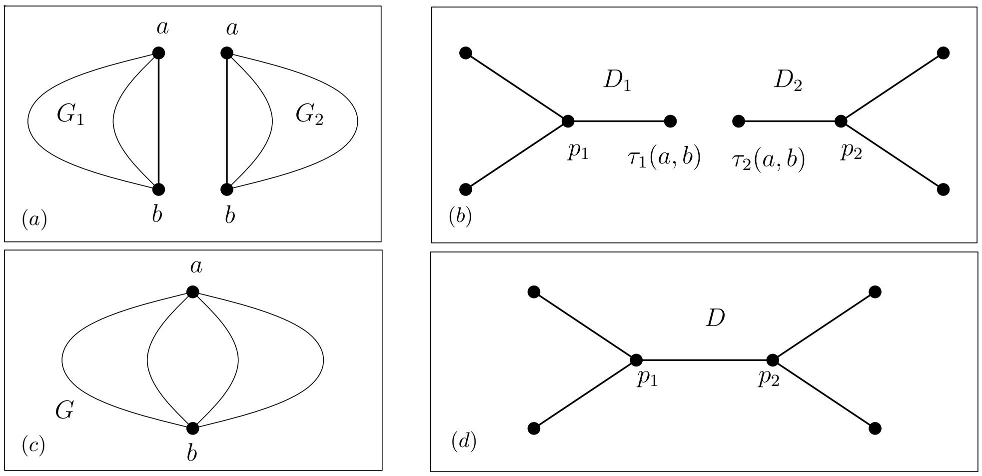

Figure 6.4: In (a) are depicted two graphs G_1,G_2 that both have labelled the adjacent vertices a and b. In (b) are the the parts of their branch decompositions D_1 and D_2 that contain $\tau(\{a, b\})$. The graph G in (c) is a 2-clique sum of G_1 and G_2, where have been identified the common labelled vertices and deleted the edge $\{a, b\}$. In (d) is shown the way we construct from D_1 and D_2 a branch decomposition of G where $\mathbf{w}(Y) \leq \mathbf{w}(Y_1) + \mathbf{w}(Y_2)$.

In the second case we similarly add the edge $\{p_1, p_2\}$ but also subdivide it and call s_1 the subdivison vertex. To construct T, we add a leaf v_{ab} and connect it with s_1. We define $\tau : E(G) \to L(T)$ as follows: if $e \in E(G) \cap E(G_i), i \in \{1, 2\}$, then $\tau(e) = \tau_i(e)$ and $\tau(\{a, b\}) = v_{ab}$. The only difference from the previous case is that, if e'_i is in the boundary of Y_i, then $\{p_i, s_1\}$ is in the boundary of Y (see also Figure 6.5).

In case $G \in G_1 \oplus_1 G_2$, let v be the vertex of G that is in G_1 and G_2. Let e_1, e_2 be two edges of G_1, G_2 respectively that have v as an endpoint. Let ${e_1}', {e_2}'$ be the edges of T_1, T_2 that have the vertex that corresponds to e_1, e_2 as an endpoint and call p_1, p_2 the other endpoint of ${e_1}', {e_2}'$ respectively. We subdivide ${e_1}', {e_2}'$ and we denote the subdivisions vertices by v_1, v_2. In order to construct T, we add the edge $\{v_1, v_2\}$. We define $\tau : E(G) \to L(T)$ as follows: if $e \in E(G) \cap E(G_i), i \in \{1, 2\}$, then $\tau(e) = \tau_i(e)$. From the construction of T, if $e \in E(T) \cap E(T_i), i \in \{1, 2\}$, then $\omega(e) = \omega_i(e)$. As previously, we choose Y so that $Y = Y_1 \cup Y_2$ and notice that

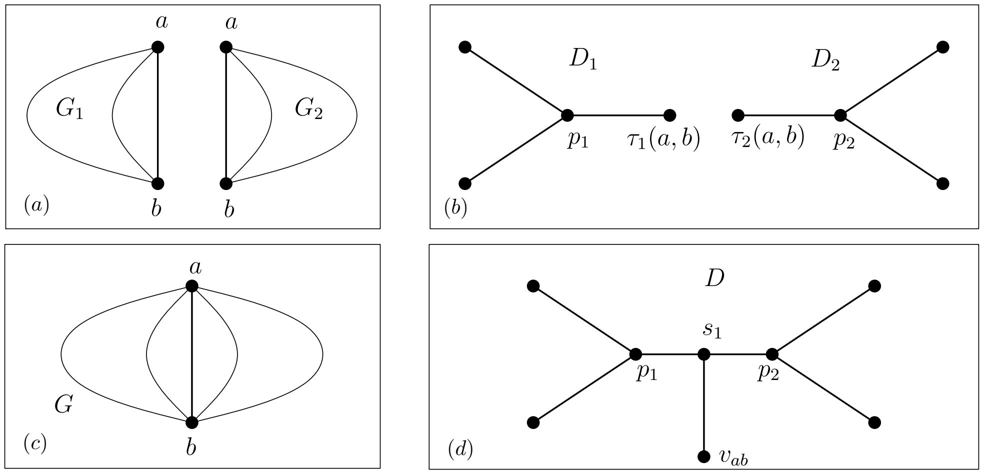

Figure 6.5: In (a) are depicted two graphs G_1,G_2 that both have labelled the adjacent vertices a and b. In (b) are the the parts of their branch decompositions D_1 and D_2 that contain $\tau(\{a, b\})$. The graph G in (c) is a 2-clique sum of G_1 and G_2, where have been identified the common labelled vertices. In (d) is shown the way we construct from D_1 and D_2 a branch decomposition D of G where $\mathbf{w}(Y) \leq \mathbf{w}(Y_1) + \mathbf{w}(Y_2)$.

Y satisfies the properties of the lemma. Also, for $e \in E(T) \cap E(T_i), i \in \{1, 2\}$, if $e \in Y_i$, then $e \in Y$. Also, if e belongs to the boundary of Y_i, then e belongs to the boundary Y. If e'_i is in the boundary of Y_i, then $\{p_i, v_i\}$ is in the boundary of Y. We just constructed the branch decomposition $D = (T, \tau)$ of G. Moreover, Y is a q-core of D such that $\mathbf{w}(Y) \leq \mathbf{w}(Y_1) + \mathbf{w}(Y_2)$ (see Figure 6.6).

In any case, it follows that the width of D is the maximum width of D_1, D_2 because for every $e \in E(T) \cap E(T_i), i \in \{1, 2\}$, then $\omega(e) = \omega_i(e)$. □

Observation 6.5. *If the branch decompositions D_1 and D_2 of the previous lemma are sphere-cut decompositions of G_1 and G_2 respectively, then D is also a sphere-cut decomposition of G.*

Lemma 6.6. *If G is a 3-connected plane graph where $\mathbf{rds}_1(G) \leq k$ and $D = (T, \tau)$ is a branch decomposition of G, then there exists Y which is 2-core of D such that $\mathbf{w}(Y) \leq 3k - 6$.*

Proof. Let $L \subseteq V(T)$ be the leaves of T and $E_L \subseteq E(T)$ be the edges of T that have one endpoint in L. Beacause G is a plane graph, from Proposition 2.1,

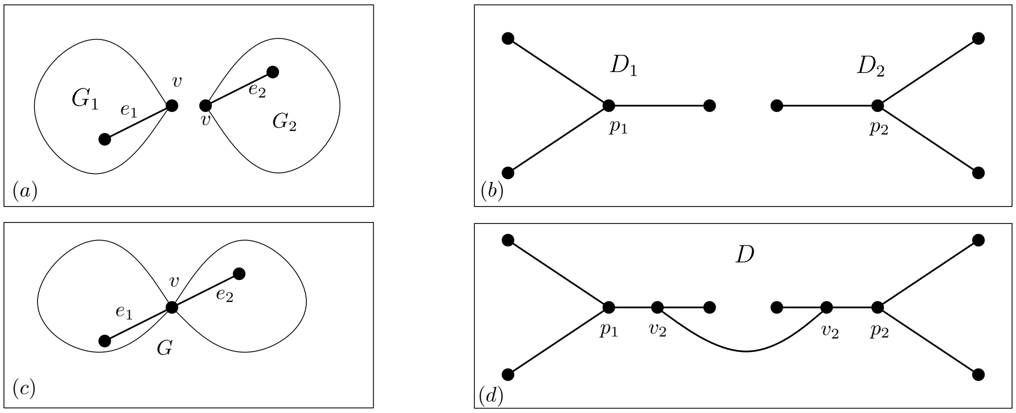

Figure 6.6: In (a) are depicted two graphs G_1,G_2 that both have a labelled vertex v and an edge which has v as an endpoint, e_1, e_2 in G_1, G_2 respectively. In (b) are the parts of their branch decompositions D_1 and D_2 that contains $\tau(e_1)$ and $\tau(e_2)$. The graph G in (c) is a 1-clique sum of G_1 and G_2 where the vertex v has been identified. In (d) is shown the way we construct from D_1 and D_2 a branch decomposition D of G where $\mathbf{w}(Y) \leq \mathbf{w}(Y_1) + \mathbf{w}(Y_2)$.

obtained that $|E(G)| = |L| = |E_L| \leq 3k - 6$. For the middle sets of edges of T we note that if $e \in E_L$, then $\mathbf{mid}_G(e) = 2$ and if $e \notin E_L$ $\mathbf{mid}_G(e) > 2$ because of the 3-connectivity of G. We choose Y as the subgraph induced by $E(T)/E_L$ and E_L is the boundary of Y. Recall that the 2-weight of Y in D is $|E_L|$ which is at most $3k - 6$. $\square$

Chapter 7

Main Result

7.1 Weakly dominated and strong dominated triconnected components

Given a graph G and a positive integer r, the set of weakly dominated components $\mathcal{W}_r(G)$ and the set of strong dominated triconnected components $\mathcal{S}_r(G)$ are recursively defined as follows:

- If $\mathbf{rds}_r(G) \leq 2$, then $\mathcal{W}_r(G) = \{G\}$ and $\mathcal{S}_r(G) = \{\emptyset\}$.
- If $\mathbf{rds}_r(G) \geq 3$ and G is 3-connected, then $\mathcal{S}_r(G) = \{G\}$ and $\mathcal{W}_r(G) = \{\emptyset\}$.
- If $G \in G_1 \oplus_h G_2$ for some $h \in \{1, 2\}$, then $\mathcal{S}_r(G) = \mathcal{S}_r(G_1) \cup \mathcal{S}_r(G_2)$ and $\mathcal{W}_r(G) = \mathcal{W}_r(G_1) \cup \mathcal{W}_r(G_2)$.

By this recursive definition we construct a tuple (T_G^r, z), where T_G^r is a rooted tree and z is a function such that $z : V(T_G^r) \to \mathcal{G}$. The construction based on the following conditions:

- The restriction of z in the leaves is a bijection from the leaves to $\mathcal{W}_r(G) \cup \mathcal{S}_r(G)$.
- If G' is a graph such that $G' \in z(v_1) \oplus_h z(v_2)$ for some $h \in \{1, 2\}$, then $z(v) = G'$ where v is the parent node of v_1 and v_2 in T_G^r.

Observe that if R is the root of T_G^r, then $z(R) = G$.

Notice that all graphs in $\mathcal{S}_r(G)$ and $\mathcal{W}_r(G)$ are topological minors of G. It is also clear from Proposition 4.6 that if $G \in G_1 \oplus_h G_2$ for some $h \in \{1, 2\}$, then $\mathbf{bw}(G) = \max\{\mathbf{bw}(G_1), \mathbf{bw}(G_2)\}$

7.2 Main Proof

Now we are in position to prove Theorem 1.3. But before that we restate the theorem more formally in a way related on the concepts of the previous chapters.

Theorem 7.1. *Let G be plane graph and r a positive integer. If $\mathbf{rds}_r(G) \leq k$, then $\mathbf{bw}(G) \leq r \cdot \sqrt{4.5 \cdot k}$ and $\mathbf{cap}_{2r, r\cdot\sqrt{4.5\cdot k}}(G) \leq 3k - 6$.*

Proof. Assume that there are no vertices of degree one in G. We use induction on r. If $r = 1$, for every graph the minimum 1-radial dominating set is equal to the number of its vertices.

Now by induction on T^1_G, we will prove that there is a sphere-cut decomposition D of G and a Y which is a 2-core of D such that $w(Y) \leq 3k - 6$, where $k = \mathbf{rds}_1(G) = |V(G)|$.

Let $G' \in \mathcal{S}_1(G)$ and $D_1 = (T', \tau')$ be the branch decomposition of G' of minimum width. By 4.10 we can assume that is a sphere-cut decomposition. Lemma 6.6 implies that there exists a Y' which is a 2-core of D_1 and such that $\mathbf{w}(Y') \leq 3k' - 6$, where $k' = \mathbf{rds}_1(G') = |V(G')|$. We observe that $\mathcal{S}_1(G) = \emptyset$ because there are no vertices of degree one.

Let R be the root of T^1_G and assume now that for every $v \in V(T^1_G) \setminus R$ there is a sphere-cut decomposition D_v of minimum width of $z(v)$ and a a Y_v which is a 2-core of D_v such that $w(Y_v) \leq 3k_v - 6$, where $k_v = \mathbf{rds}_1(z(v)) = |V(z(v))|$. This propositions hold also for $G = z(R)$. Let v_1, v_2 be the children of R in T^1_G, so $G \in z(v_1) \oplus_h z(v_2)$ for some $h \in \{1, 2\}$. From the induction hypothesis, there exist a branch decompositions D_i of $z(v_i)$ and a Y_i which is a 2-core of D_i such that $w(Y_i) \leq 3k_i - 6$, where $k_i = \mathbf{rds}_1(z(v_i)) = |V(z(v_i))|$ for $i \in \{1, 2\}$.

Using Lemma 6.4 and Observation 6.5 there exists a sphere-cut decomposition D of G and a Y which is a 2-core of D such that $w(Y) \leq \mathbf{w}(Y_1) + \mathbf{w}(Y_2) = 3k_1 - 6 + 3k_2 - 6 = 3(k_1 + k_2) - 12$. Lemma 5.6 implies that $\mathbf{w}(Y) \leq 3(k+2) - 12 = 3k - 6$ and the width of D is $\mathbf{bw}(G)$ because equals to $\max\{\mathbf{bw}(G_1), \mathbf{bw}(G_2)\}$.

Assume now that the theorem holds for values smaller than r and we will prove that it also holds for $r \geq 2$. Let $H \in \mathcal{S}_r(G)$ and S is a radial dominating set of H. H is topological minor of G. By Observation 5.1 $k_H = \mathbf{rds}_r(H) \leq \mathbf{rds}_r(G) = k$. By Proposition 5.3 H is the topological minor of a 3-connected plane graph H_1 where S is r-radially extremal.

Let H_2 be the graph obtained if we remove from H_1 the vertices of S. Because of 3-connectivity of H_1, for any $v \in S$ the graph $H[N_{H_1}(v)]$ is a cycle and each such cycle is the boundary of the face of H_2. We denote by F the set of these faces and observe that F^* is a $(r-1)$-radial dominating set of $\overline{H_2^*}$ (we denote by F^* the vertices of $\overline{H_2^*}$ that are duals of the faces of F in H_2). Moreover, the fact that S is a $2r$-scattered dominating set in H_1 implies that F^* is a $2(r-1)$-scattered dominating set in $\overline{H_2^*}$.

From the induction hypothesis and the fact that $|F^*| = |S|$, we obtain that

$$\mathbf{cap}_{2(r-1),(r-1)\cdot\sqrt{4.5\cdot k}}(\overline{H_2^*}) \leq 3k_H - 6.$$

From Observation 6.1 and Proposition 4.9,

$$\mathbf{cap}_{2(r-1),(r-1)\cdot\sqrt{4.5\cdot k}}(H_2^*) \leq 3k_H - 6.$$

The graph H_2 is 2-connected because of Lemma 2.4. This fact along with Lemma 6.3 implies that

$$\mathbf{cap}_{2(r-1),(r-1)\cdot\sqrt{4.5\cdot k}}(H_2) \leq 3k_H - 6.$$

In graph H_2, for any face $f_i \in F$, let $(x_0^i, \ldots x_m - 1^i)$ be the cyclic order of the vertices in its boundary cycle. We also denote by x^i the vertex in H_1 that was removed in order for f_i to appear in H_2. Let $D = (T, \tau)$ be a sphere-cut decomposition of H_2 of width at most $(r-1)\cdot\sqrt{4.5\cdot k}$ and a $2(r-1)$-core Y of D such that $\mathbf{w}(Y) \leq 3k_H - 6$. We use (T, τ) in order to construct a sphere-cut decomposition of H_1 by adding new leaves in T and mapping them to the edges $E(H_1)/E(H_2) = \cup_{i=1,\ldots,|F|}\{\{x^i, x_h^i\} \mid h = 0, 1, \ldots m-1\}$ in the following way: for every $i = 1, \ldots, |F|$ and $h = 0, 1, \ldots m-1$ we set $t_h^i = \tau^{-1}(\{x_h^i, x_{h+1 \mod m_i}^i\})$ and let $e_h^i = \{y_h^i, t_h^i\}$ be the unique edge of T that is incident to t_h^i. We subdivide e_h^i and we call the subdivision vertex s_h^i. We also add a new vertex z_h^i and make it adjacent to s_h^i. Finally, we extend the mapping of τ by mapping the vertex z_h^i to the edge $\{x^i, x_h^i\}$ and we use the notation $D' = (T', \tau')$ for the resulting sphere-cut decomposition of H_1.

By Proposition 5.4 the width of D' is at most $r\cdot\sqrt{4.5\cdot k}$. Let $e \in E(T)/E(Y)$ and N_e be the noose of e in D. By the definition of q-core, $\mathbf{mid}_{H_2}(e) \leq 2(r-1)$. Because H_1 is r-radially extremal N_e meets at most 2 faces of F, $\mathbf{mid}_{H_1}(e) \leq 2r$. We choose $Y' = Y$ as $2r$-core of D'. Now it is clear that $w(Y') \leq 3k_H - 6$ and also $\mathbf{cap}_{2r,r\cdot\sqrt{4.5\cdot k}}(H_1) \leq 3k_H - 6$.

Lemma 6.2 implies that $\mathbf{cap}_{2r,r\cdot\sqrt{4.5\cdot k}}(H) \leq 3k_H - 6$.

This result can be applied for every $H \in \mathcal{S}_r(G)$.

We have proven that if $G_i \in \mathcal{S}_r(G)$, then $\mathbf{cap}_{2r,r\cdot\sqrt{4.5\cdot k}}(G_i) \leq 3k_{G_i} - 6$. Moreover, by the definition of capacity and Proposition 7.1 if $G_i \in \mathcal{W}_r(G)$, then

$\mathbf{cap}_{2r,l}(G_i) = 0$ for every $l > 0$. In this case there exists a branch decomposition of G_i, by Lemma 7.1 the width of it is at most $2r$, and as $2r$-core we choose the empty graph. By Proposition 4.10 we assume that it is a sphere-cut decomposition.

Let R be the root of T_G^r and assume now that for every $v \in V(T_G^r) \setminus R$ there is a sphere-cut decomposition D_v of minimum width of $z(v)$ and a a Y_v which is a $2r$-core of D_v such that $w(Y_v) \leq 3k_v - 6$, where $k_v = \mathbf{rds}_r(z(v)) = |V(z(v))|$. This propositions hold also for $G = z(R)$. Let v_1, v_2 be the children of R in T_G^r, so $G \in z(v_1) \oplus_h z(v_2)$ for some $h \in \{1, 2\}$. From the induction hypothesis, there exists a branch decompositions D_i of $z(v_i)$ and a Y_i which is a $2r$-core of D_i such that $w(Y_i) \leq 3k_i - 6$, where $k_i = \mathbf{rds}_r(z(v_i)) = |V(z(v_i))|$ for $i \in \{1, 2\}$.

Using Lemma 6.4 there exist a branch decomposition D of G and a Y which is a $2r$-core of F such that $w(Y) \leq \mathbf{w}(Y_1)+\mathbf{w}(Y_2) = 3k_1-6+3k_2-6 = 3(k_1+k_2)-12$. Lemma 5.6 implies that $\mathbf{w}(Y) \leq 3(k + 2) - 12 = 3k - 6$ and the width of D is at most $r \cdot \sqrt{4.5 \cdot k}$. We observe that if for exactly one of Y_1 and Y_2 is the empty graph the result is the same because if for example Y_2 is the empty graph, then $w(Y) \leq \mathbf{w}(Y_1) + \mathbf{w}(Y_2) = 3k_1 - 6 \leq 3k - 6$. Both of Y_1 and Y_2 are the empty graph iff $\mathcal{S}_r(G) = \emptyset$ and at this case the capacity of G is zero.

Now let us remember that we have assumed that in G there are no vertices of degree one. To complete the proof of the theorem we will add these vertices, if there exist, and the related edges in the decomposition without any change in the result.

Let v be a vertex of degree one and u the only neighbour of it. Let e' be an edge of G that has u as an endpoint but not v. In T we subdivide the edge that has $\tau(e')$ as an endpoint and call p the vertex obtained by the subdivision. We add also a new vertex z and connect it with p. We extend τ such as $\tau(\{u, v\}) = z$ and hold the same $2r$-core. For every $e \in E(T) \setminus \{\{p, z\}, \{p, \tau(e')\}\}$ we can easily extend the corresponding noose N_e such that $\{p, z\}$ and $\{p, \tau(e')\}$ be in the same open disc without $v \in V(N_e)$ at any case. The width of the decomposition and the capacity of G do not change, so we complete the proof of the theorem. □

7.3 Conclusion and further work

In this master-thesis we have introduced some new concepts, q-weight and (q, k)-capacity, which are related to branch decompositions. We have proven that in plane graphs, given an optimal sphere-cut decomposition and an r-radial dominating set of it, which has size at most k, the edge of the tree T with the maximum middle set can be found in a subgraph of T which is a forest. The number of leaves of this forest is at most $3k - 6$.

At first glance, it seems our results cannot be applied directly to other related problems of Graph Theory, Parameterized Algorithms, Parameterized Complexity, etc. We believe that a closer look can disprove this assessment. Let us now give some definitions regarding Parameterized problems, Kernelization, etc, and then we will explain what we think is the real impact of our results. Of course these extensions need a lot of further work and cannot be obtained directly by our results.

Parameterized Graph Problems. A parameterized graph problem Π in general can be seen as a subset of $\Sigma^* \times \mathbb{Z}^+$ where, in each instance (x, k) of Π, x encodes a graph and k is the parameter.

Kernelization. A parameterized problem is said to admit a *polynomial kernel* if there is a polynomial time algorithm (the degree of polynomial is independent of k), called a *kernelization* algorithm, that reduces the input instance down to an instance with size bounded by a polynomial $p(k)$ in k, while preserving the answer. This reduced instance is called a $p(k)$ *kernel* for the problem. If $p(k) = O(k)$, then we call it a *linear kernel*.

A more formal definition of kernelization follows.

Definition 7.2. *Let* $\Pi \subseteq \Sigma^* \times \mathbb{Z}$ *be a parameterized problem and* g *be a computable function. We say that* Π admits a kernel of size g *if there exists an algorithm* $\mathcal{K}$, *called* kernelization algorithm*, or, in short, a* kernelization*, that given* $(x, k) \in \Sigma^* \times \mathbb{Z}^+$*, outputs, in time polynomial in* $|x| + k$*, a pair* $(x', k') \in \Sigma^* \times \mathbb{Z}^+$ *such that*

(a) $(x, k) \in \Pi$ *if and only if* $(x', k') \in \Pi$*, and*

(b) $\max\{|x'|, k'\} \leq g(k)$.

When $g(k) = k^{O(1)}$ *or* $g(k) = O(k)$*, then we say that* Π admits a polynomial *or* linear kernel *respectively.*

Kernelization has been extensively studied in the realm of parameterized complexity, resulting in polynomial kernels for a variety of problems. Some examples of kernelization in well-known problems of parameterized complexity are the following. VERTEX COVER has a $2k$-sized vertex kernel [15], PLANAR DOMINATING SET has $67k$-sized vertex kernel [13], and FEDDBACK VERTEX SET has an $O(k^2)$ kernel parameterized by the solution size [60].

For a variety of parameterized graph problems linear kernels can be obtained for their planar version. Some of these problems are Connected Vertex Cover, Minimum Edge Dominating Set, Maximum Triangle Packing, Efficient Edge Dominating Set, Induced Matching, Full-Degree Spanning Tree, Feedback Vertex Set, Cycle Packing, and Connected Dominating Set [1,3,11,12,14,35,36,38,45,49]. In particular, a specific kernelization approach for the Dominating Set has led to polynomial kernels for some graph classes other than planar graphs. For example there is a linear kernel for graphs with bounded genus [29], and a polynomial kernel for graphs excluding a fixed graph H as a minor and for d-degenerated graphs [5,51]. A detailed survey about the area of kernelization is in [34].

Moreover, in this area there are some meta-results, meaning if a parameterized graph problem Π has some properties and Π is restricted in a graph class, then it has poylnomial (or linear) kernel. In [10] Bodlaender et al. proved a meta-result for graphs of bounded genus, in [27] Fomin et al. for H-minor-free graphs, and in [31] Gajarský et al. for sparse graph classes. Recently another meta-result was given by Kim et al. for graphs that are H-topological-minor-free [23].

Protrusion Decompositions. Protrusion decompositions have been defined in [10] and were used in the same paper to prove that a lot of problems in planar graphs admit polynomial or linear kernels. First we will give the definitions of t-protrusion and (α,β)-Protrusion decomposition and then we will explain how our results may be related to them.

Definition 7.3. *Given a graph G, we say that a set $X \subseteq V$ is a t-*protrusion *of G if $|N_G(X)| \leq t$ and $\mathbf{tw}(G[X]) \leq t$.*

Definition 7.4. [(α,β)**-Protrusion decomposition**] *An* (α,β)-protrusion decomposition *of a graph G is a partition $\mathcal{P} = \{R_0, R_1, \ldots, R_\rho\}$ of $V(G)$ such that*

- $\max\{\rho, |R_0|\} \leq \alpha$,
- *each $R_i^+ = N_G[R_i]$, $i \in \{1,\ldots,\rho\}$, is a β-protrusion of G, and*
- *for every $i \in \{1,\ldots,\rho\}$, $N_G(R_i) \subseteq R_0$.*

We call the sets R_i^+, $i \in \{1,\ldots,\rho\}$, the protrusions *of $\mathcal{P}$.*

The next proposition is a lemma that was also proven in [10].

Proposition 7.5 (H. L. Bodlaender et al. [10], Lemma 6.2)**.** *Let r be a positive integer and let $G = (V, E)$ be a graph embedded in a surface Φ of Euler genus g that contains a set S of vertices, $|S| \leq k$, such that $\mathbf{R}^r_G(S) = V$. Then G has an $(\alpha k, \beta)$-protrusion decomposition for some constants α and β that depend only on r and g.*

In parameterized complexity, protrusion decompositions are mainly applicable in algorithm design and kernelization. Some work in algorithm design via protrusion decompositions appears in problems such as Planar $\mathcal{F}$-Deletion, l-Pseudoforest Deletion, etc. [26, 50]. Furthermore, in kernelization protrusion decompositions were used for problems such as r-Dominating Set, r-Scattered Set, Planar $\mathcal{F}$-Deletion, Connected Dominating Set, etc. [17,25,32,33,50]. Of course in the meta-results that we have mentioned protrusion decompositions are a main tool for the proofs. [23, 27, 31].

In the landscape that we have defined, all graphs are embedded in the the 2-dimensional sphere $\mathbb{S}_0 = \{(x, y) \mid x^2 + y^2 = 1\}$, so its Euler genus is 2. Our aim is to make exact the constant α of Proposition 7.5 at this landscape. Our approach will use protrusions of branchwidth instead of treewidth and will be based on the concepts q-weight and (q, k)-capacity that we have defined. We will also try to improve the bound for the constant β.

Achieving this goal is likely to have a significant impact in a lot of kernelization problems by improving or finding explicit bounds for linear kernels of several problems on graphs.

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