Asymptotic analysis of outerplanar graphs with subgraph obstructions

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Msc Thesis

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Abstract

In this thesis we study techniques for combinatorial specification and asymptotic enumeration in the context of *analytic combinatorics*. Our guide through this variety of techniques is the enumeration of certain types of planar graphs, under subgraph exclusion constraints. In particular, we examine outerplanar graphs where the constraint is the exclusion of cycles of certain length. Our starting point is [1], where general outerplanar graphs were firstly specified and analysed asymptotically. We then build specifications for the 2-connected components that exclude certain cycles in order to obtain asymptotics for the general constrainded outerplanar class. The challenges here are combinatorial as well as computational, as the specifications become more involved when the length of the excluded cycle grows and the generating functions obtained are in implicit form. The combinatorial language that we use is the so-called *Symbolic Method* that comes in hand with corresponding analytic techniques, as was first suggested as a whole in [9]. Furthermore, we study certain parameters of general outerplanar graphs, namely the number of triangles and quadrangles. We obtain Gaussian limiting distibutions and extract explicit constants for the mean and variance.

Preface

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Chapter 1

Introduction

Analytic combinatorics is a relatively new field at its own right, whose major landmark was the appearance of [9] in 2009. It has applications in many and diverse scientific areas, like the analysis of algorithms, probability theory, statistical physics, logic, computational biology and others. The body of this theory lies in the intersection of combinatorics and analysis: A combinatorial class $\mathcal{A}$ can be encoded by a generating function

$$\sum_{n\geq 0} a_n z^n$$

which constitutes an algebraic object, as well an analytic one. Allowing ourselves to view generating functions as analytic objets provides rich information about the exact or asymptotic growth of the series' coefficients, and thus about our combinatorial problem.

The first step of the analysis is typically the combinatorial modelling of the problem in terms of what is called the *symbolic method*. The symbolic method that will be presented in the following chapter relies on specifying a combinatorial class by using in a recursive way simpler structures that bind together through familiar operations such as union, product, sequence, cycle, etc and then translate to operations between generating functions.

The next step is the asymptotic analysis that comes in hand with a variety of techniques for each category of generating functions and certain fundamental theorems called *transfer theorems* that rely crucially on the existence and nature of the generating function's singularities, in order to extract information for their coefficients. It must be noted that we can also talk about properties of a combinatorial class (for example the number of nodes of degree 2 on a tree) and extract information about their mean, variance, and limiting behaviour as random variables on the objects of size n, as n tends to infinity.

About this thesis

In this thesis we explore a part of these techniques in the context of enumeration of certain planar graphs under constraints, namely outerplanar graphs with restricted cycles of some length. We rely on the work done in [8] and [1], where dissections and general outerplanar graphs were analysed respectively, in order to build our specifications. Our challenges are both combinatorial and computational as our specifications are more involved and our generating functions are in purely implicit form (except for the case of excluding triangles), as solutions of polynomial systems. In particular, we build explicit specifications for the case of excluding $3, 4, 5$ and $6-$cycles and perform the corresponding asymptotic analysis. Our main tools are the symbolic method, singularity analysis, and properties of algebraic functions, which we will present in the next chapter. Furthermore, we obtain limiting laws for the number of triangles and quadrangles in unristricted dissections, with explicit and computable to any degree of accuracy constants. In more detail, the thesis has the following structure:

Chapter 1 consists of three sections and deals with all the necessary background and definitions. In the first section the symbolic method is introduced for unlabelled as well as labelled constructions and we also take notice of multivariate generating functions that correspond to combinatorial structures with parameters. Elementary examples are given for each case.

In the second section we present the basic analytic techniques that are used. We first recall basic results from complex analysis. Afterwards, we present the case of inverse functions and algebraic functions, and highlight known results for their singular expansions. All this ends up to basic theorems of singularity analysis that exploit this kind of expansions in order to extract information for the asymptotic behaviour of the original series' coefficients. In the end, we state a version of the *Quasi-Powers* theorem that helps us obtain limit laws for parameters of combinatorial structures, under some assumptions.

The third section introduces the class of outerplanar graphs and illustrates with the necessary detail the way they were specified combinatorially and analysed asymptotically in [1].

All the above are found usefull in chapter 2, where we specify the class of outerplanar graphs excluding cycles of some length. The analysis is done for the cases of $3, 4, 5$, and $6-$cycles, where we can find combinatorial and computational challenges that seperate them from the general case that is presented in chapter 1. The specifications are more involved, as different cases have to be taken in mind, while computations are not straightforward since we only work with implicit representations of the generating functions. The asymptotics follow the pattern $g \cdot n^{-\frac{5}{2}} \cdot \rho^{-n} \cdot n!$ and the explicit constants are computed as shown in the following table, up to five digits of accuracy:

Class	ρ	g	ρ_{2con}	g_{2con}
3-cycles	0.20836	0.01578	0.29336	0.02330
4-cycles	0.18919	0.01462	0.26488	0.02177
5-cycles	0.18054	0.01804	0.25383	0.02217
6-cycles	0.17516	0.01870	0.24835	0.02321

Table 1.1: The computed constants for general restricted and 2-connected restricted outerplanar graphs.

The first pair of columns demonstrates the constants for the general restricted outerplanar graphs, while the second pair demonstrates the constants for the 2-connected counterparts that follow the schema $g \cdot n^{-\frac{3}{2}} \cdot \rho^{-n} \cdot n!$.

Chapter 3 is directed to bivariate generating functions. A parameter is being introduced in general outerplanar graphs, namely the number of 3 and $4-$cycles. The singular expansions of the univariate case are established to lift in a convenient way when the second variable is introduced, which allows us to use the *Quasi-powers* theorem and obtain gaussian limit laws with linear mean and variance. In particular, the constants are the ones demonstrated in Table 1.2:

Parameter	α	β
triangles	0.34793	0.40737
quadrangles	0.33705	0.36145

Table 1.2: The constants for the mean and variance.

Chapter 2

The main methods

2.1 The symbolic method

The symbolic approach to combinatorial enumeration constitutes an attempt to express combinatorial structures with some kind of recursive specification in a uniform way, starting from elementary classes and constructions and building up more complex ones, in the spirit of a formal language.[1] The insentive for this kind of expressions is that, if they are based on *admissible* constructions, then they translate immediately to generating functions. All the struggle is thus suppressed to finding a proper specification for the combinatorial class that is under examination.

Let $\mathcal{A}$ be a *combinatorial class*, meaning a set that is at most denumerable, along with a size function $|\cdot| \in (\mathcal{A} \to \mathbb{Z}_{\geq 0})$ such that the inverse image of any integer is finite. Then the corresponding ordinary generating function (OGF) of $\mathcal{A}$ is defined, as[2]

$$A(z) = \sum_{n=0}^{\infty} A_n z^n = \sum_{\alpha \in \mathcal{A}} z^{|\alpha|}.$$

Two elementary combinatorial classes are considered, namely the *neutral class* $\mathcal{E}$ and the *atomic class* $\mathcal{Z}$. The neutral class consists of a single object of zero size ϵ, called a neutral object, and the atomic class consists of one object of size equal to one (represented by a circle $\bullet$ or $\circ$), called an *atom*. Their generating functions are respectively $E(z) = 1$ and $Z(z) = z$, and of course we may take as many discrete copies of these classes as needed each time, using symbols such as $\epsilon_1, \epsilon_2, ...$ for the first case, and $\bullet, \circ, \star$ etc for the second.

An $m-$ary construction Φ such that $\mathcal{A} = \Phi[\mathcal{B}^{(1)}, ..., \mathcal{B}^{(m)}]$ is called *admissible* if the counting sequence A_n of $\mathcal{A}$ only depends on the counting sequences of the $\mathcal{B}^{(i)}$, and thus corresponds to a well defined operator Ψ such that $A(z) = \Psi[\mathcal{B}^{(1)}, ..., \mathcal{B}^{(m)}]$. Table 1.1 summarizes the basic admissible constructions, along with their respective operators in the realm of the generating

[1] The exposition in this chapter follows the 1st part of [9] and all the omitted proofs can be found there.

[2] Through this exposition, the symbol $\mathcal{A}$ will denote a combinatorial class, $\mathcal{A}_n$ the restriction of $\mathcal{A}$ to objects of size n, and A_n their respective cardinalities.

The ordinary constructions [3]		
Sum:	$\mathcal{A} = \mathcal{B} \cup \mathcal{C}$	$A(z) = B(z) + C(z)$
Cartesian product:	$\mathcal{A} = \mathcal{B} \times \mathcal{C}$	$A(z) = B(z) \cdot C(z)$
Sequence:	$\mathcal{A} = SEQ(\mathcal{B})$	$A(z) = \frac{1}{1-B(z)}$
Powerset:	$\mathcal{A} = PSET(\mathcal{B})$	$A(z) = \begin{cases} \prod_{n\geq 1}(1+z^n)^{B_n} \\ \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} B(z^k)\right) \end{cases}$
Multiset:	$\mathcal{A} = MSET(\mathcal{B})$	$A(z) = \begin{cases} \prod_{n\geq 1}(1+z^n)^{-B_n} \\ \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} B(z^k)\right) \end{cases}$
Cycle:	$\mathcal{A} = CYC(\mathcal{B})$	$A(z) = \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \log \frac{1}{1-B(z^k)}$
Pointing:	$\mathcal{A} = \Theta\mathcal{B} := \sum_{n\geq 0} \mathcal{B}_n \times \{\epsilon_1, ...\epsilon_n\}$	$A(z) = z\partial_z B(z)$
Substitution:	$\mathcal{A} = \mathcal{B} \circ \mathcal{C} := \sum_{k\geq 0} \sum_{\beta \in \mathcal{B}} \epsilon_\beta \times SEQ_k(\mathcal{C})$	$A(z) = B(C(z))$

Table 2.1: Here are the the basic admissible constructions for unlabelled structures, along with their translation to generating functions.

functions, where for the sequence, powerset, multiset and cycle constructions it is assumed that $B_0 \neq \emptyset$. Most of these classes have their obvious meaning: the sum of two classes corresponds to the disjoint sum of these classes with size being the induced one from the class each object belongs. The cartesian product has the usual meaning, with size being the sum of the components. Size is defined likewise for sequences, powersets, multsets and cycles, where sequences of a class $\mathcal{B}$ comprise the elements of $1 \cup \mathcal{B} \cup \mathcal{B}^2 \cup \mathcal{B}^3 \cup ...$, multisets are sequences taken up to any shift of the components, powersets are multisets that involve no repetitions, and cycles are powersets taken up to circular shifts. Pointing to a class $\mathcal{B}$ means chosing for each object of the class a particular atom (for example, we might consider rooted trees out of unrooted ones), while substitution suggests that each atom of an object in $\mathcal{B}$ is being substituted by an object of the class $\mathcal{C}$. It must be noted that the restricted counterparts of all these constructions have similar translations (for example $\mathcal{A} = SEQ_k = \mathcal{B}^k$ yields $A(z) = B(z)^k$).

Here follow two elementary examples of the above, for the case of compositions and partitions. Firstly, define the class $\mathcal{I} = \{1, 2, 3, ...\} \cong \{\bullet, \bullet\bullet, \bullet\bullet\bullet, ...\} = SEQ_{\geq 1}\{\bullet\}$. Then, in order to deal with natural numbers we can use $\mathcal{I}$ with $I(z) = \frac{z}{1-z}$, which gives us:

$$\mathcal{C} = SEQ(\mathcal{I}) \Rightarrow C(z) = \frac{1}{1-\mathcal{I}(z)} = \frac{1}{1-\frac{1}{1-z}} = \frac{z}{1-2z}.$$

For partitions:

$$\mathcal{P} = MSET(\mathcal{I}) \Rightarrow P(z) = \prod_{m=1}^{\infty} \frac{1}{1-z^m}.$$

Now, there is also the possibility to define a combinatorial class implicitly (or *recursively*) We

[3] The function ϕ is the Euler totient function $\phi(k) = p_1^{a_1-1}(p_1-1)\cdots p_r^{a_r-1}(p_r-1)$ for the prime number decomposition $k = p_1^{a_1}\cdots p_r^{a_r}$.

call a specification for an r-tuple $\vec{\mathcal{A}} = (\mathcal{A}^{(1)}, ..., \mathcal{A}^{(r)})$ of classes a collection of r equations

$$\begin{cases} \mathcal{A}^{(1)} = \Phi_1(\mathcal{A}^{(1)}, ..., \mathcal{A}^{(r)}) \\ \mathcal{A}^{(2)} = \Phi_1(\mathcal{A}^{(1)}, ..., \mathcal{A}^{(r)}) \\ ... \\ \mathcal{A}^{(r)} = \Phi_1(\mathcal{A}^{(1)}, ..., \mathcal{A}^{(r)}) \end{cases}$$

where each Φ_i denotes a term built from the $\mathcal{A}^{(j)}$ using the constructions of disjoint union, cartesian product, sequence, powerset, multiset, and cycle, as well as the intial neutral and atomic classes. This kind of representation does not necessary lead to explicit generating functions, but, even so, asymptotic results can be extracted for the coefficients using techniques from analytic combinatorics (some of them are presented in 1.2).

An elementary example for the above, using only one equation, is the specification of general plane trees[4]:

$$\mathcal{G} = \mathcal{Z} \times SEQ(\mathcal{G}). \tag{2.1}$$

Note that, begining by setting as $\mathcal{G}$ the empty class and iterating in the above relation, we can recreate the whole class of such trees, where on each iteration j all trees up to hight j will be produced.

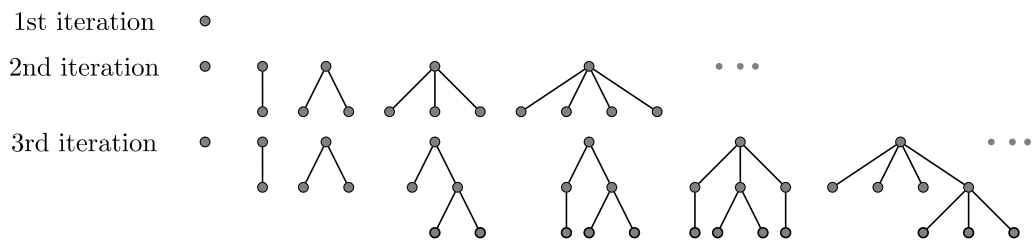

Figure 2.1: The structure of plane trees at each iteration.

Now, equation (2.1) translates to

$$G = \frac{z}{1-G} \quad \text{and} \quad G - G^2 - z = 0,$$

which can be solved by the quadratic formula that gives in the end:

$$G(z) = \frac{1 - \sqrt{1-4z}}{2}.$$

Using Newton's binomial expansion, we have then $g_n = \frac{1}{n}\binom{2n-2}{n-1}$

[4]Plane meaning that the order by which the subtress are embedded under the root matters.

The labelled constructions		
Sum:	$\mathcal{A} = \mathcal{B} \cup \mathcal{C}$	$A(z) = B(z) + C(z)$
Labelled product:	$\mathcal{A} = \mathcal{B} \star \mathcal{C}$	$A(z) = B(z) \cdot C(z)$
Sequence:	$\mathcal{A} = SEQ(\mathcal{B})$	$A(z) = \frac{1}{1-B(z)}$
Set:	$\mathcal{A} = SET(\mathcal{B})$	$A(z) = \exp(B(z))$
Cycle:	$\mathcal{A} = CYC(\mathcal{B})$	$A(z) = \log \frac{1}{1-B(z)}$
Pointing:	$\mathcal{A} = \Theta\mathcal{B} := \sum_{n\geq 0} \mathcal{B}_n \times \{\epsilon_1, ...\epsilon_n\}$	$A(z) = z\partial_z B(z)$
Substitution:	$\mathcal{A} = \mathcal{B} \circ \mathcal{C} := \sum_{k\geq 0} \sum_{\beta\in\mathcal{B}} \epsilon_\beta \times SET_k(\mathcal{C})$	$A(z) = B(C(z))$

Table 2.2: Here are the the basic admissible constructions for labelled structures, along with their translation to exponential generating functions.

Labelled structures

Labelled classes can also be handled, using exponential generating functions (EGFs). Labelled classes are combinatorial classes where each object of size n bears n different labels on its atoms from the set $[n] = \{1, 2, ..., n\}$.

We once again consider the neutral class $\mathcal{E}$ that consists of the neutral object ϵ with size 0 and bearing no label at all and the atomic class $\mathcal{Z}$ which is formed by a unique object of size one and label **1**. Now all the basic constructions from the unlabelled case (except for the multiset) have their labelled counterparts (see table 2.2): Sum follows the same reasoning. The cartesian product must be seen in a different way, since concantenating two objects from different classes will create double labels. So we suppose that for an object of size n being the concatenation of two different objects we can split the labels in all different ways and then place them in an order-preserving way. This corresponds exactly to the product of two exponential generating functions, and gives way to construct of the so-called labelled product, $\mathcal{A} = \mathcal{B} \star \mathcal{C}$. We can use the same reasoning for labelled sequences, sets and cycles (note that multisets is the only constuction with no labelled counterpart, since then the labels would be repeated). Pointing and substitution also have their labelled counterparts. Pointing translates the same way as in the unlabelled case, while substitution is slightly different: for an object in $\mathcal{B}$ we take a $k-$set of objects in $\mathcal{C}$ that is ordered naturally by the greatest labell of each object. Then each object of rank j goes to the spot of the object in $\mathcal{B}$ that bears the label j.

Two characteristic elementary examples are the cases of surjections and set partitions: Surjections of the form $[k] \to [n]$ correspond to a sequence of n non empty urns, filled with k numbered balls. To create the class of all non-empty urns, we can take $SET_{\geq 1}(z)$. Then, taking all possible sequences of non empty urns we have:

$$\mathcal{R} \cong SEQ(SET_{\geq 1}(\mathcal{Z})) \Rightarrow R(z) = \frac{1}{2 - e^z}.$$

The partitions are equivalent to taking sets of non-empty urns, which gives:

$$\mathcal{S} \cong SET(SET_{\geq 1}(\mathcal{Z})) \Rightarrow S(z) = e^{e^z - 1}.$$

Multivariate generating functions

Given a combinatorial class $\mathcal{A}$, we can consider a certain parameter $\chi : \mathcal{A} \to \mathbb{Z}_{\geq 0}$ that we would like to study. For example, if $\mathcal{A}$ is a graph class, then a parameter could be the number of edges of the graph. This gives way to bivariate generating functions (BGFs) of the form

$$\sum_{n,k} a_{n,k} z^n u^k \quad \text{for OGFs} \qquad \sum_{n,k} a_{n,k} \frac{z^n}{n!} u^k \quad \text{for EGFs}$$

where $a_{n,k}$ is the number of objects $a \in \mathcal{A}_n$ such that $\chi(a) = k$. The definition can be generalised for any number of parameters that are considered together as $\chi = (\chi_1, \chi_2, ..., \chi_k)$ and then there would be as many u_i variables as the dimension of the parameter χ. Nevertheless, in the present context the bivariate formulation is enough.

Considering the uniform probability distribution over each $\mathcal{A}_n$, a parameter χ defines a discrete random variable over them, as in $\mathbb{P}_{\mathcal{A}_n}(\chi = k) = \frac{A_{n,k}}{A_n}$. Recalling the definition of a probability generating function $p(u) = \sum_k \mathbb{P}(X = k)u^k$, we can directly express it in terms of BGFs, as:

$$\sum_k \mathbb{P}_{\mathcal{A}_n}(\chi = k)u^k = \frac{[z^n]A(z,u)}{[z^n]A(z,1)}.$$

Recall now the definition of the expectation of $f(X)$ for a discete random variable X:

$$\mathbb{E}(f(X)) = \sum_k \mathbb{P}(X = k)f(k).$$

It can be observed that

$$\mathbb{E}_{\mathcal{A}_n}(\chi(\chi-1)...(\chi-r+1)) = \frac{[z^n]\partial_u^r A(z,u)|_{u=1}}{[z^n]A(z,1)},$$

and then we can recover all power moments $\mathbb{E}(Z^r)$ as linear combinations of factorial moments of the above type. In particular, for the usual expectation of χ, we can say that

$$\mathbb{E}_{\mathcal{A}_n}(\chi) = \frac{[z^n]\partial_u A(z,u)|_{u=1}}{[z^n]A(z,1)},$$

and having also

$$\mathbb{E}_{\mathcal{A}_n}(\chi^2) = \frac{[z^n]\partial_u^2 A(z,u)|_{u=1}}{[z^n]A(z,1)} + \frac{[z^n]\partial_u A(z,u)|_{u=1}}{[z^n]A(z,1)}$$

we can expresse the variance $\mathbb{V}(\chi) = \mathbb{E}(\chi^2) - \mathbb{E}(\chi)^2$ as well.

A last important thing about BGFs is the following: operations between combinatorial classes $\mathcal{A}^{(j)}(|\cdot|, \chi)$ that compose a new combinatorial class $\mathcal{A}(|\cdot|, \chi)$ can be translated into bivariate generating functions with exactly the same way as in the previous cases, if the parameters obey two conditions: the parameters must be *inherited* and *compatible*:

> If $\mathcal{A} = \mathcal{A}^{(1)} \cup \mathcal{A}^{(2)}$, a parameter χ is *inherited* from χ_1 and χ_2 if its value is the same as the one the object had in its original class $\mathcal{A}^{(i)}$. Otherwise, if $\mathcal{A} = \mathcal{A}^{(1)} \times \mathcal{A}^{(2)}$, or $\mathcal{A}$ is the SEQ, $MSET$, $PSET$, CYC of another combinatorial class, then the parameter is inherited if it is the sum of the parameters of the object's components. This definition applies for labelled classes as well.
>
> A parameter for labelled objects is called *compatible* if it is the same for all the relabellings of the object.

In order to give a simple demonstration of the above, we can consider again the specification of integer compositions, where the parameter is the number of components. Then each number of I has one component and $I(z,u) = \frac{zu}{1-z}$, while because the parameter is inherited we can say immediately that

$$C(z,u) = \frac{1}{1 - I(z,u)} = \frac{1}{1 - \frac{zu}{1-z}} = \frac{1-z}{1 - z(u+1)}.$$

Then $\partial_u C(z,u)|_{u=1} = \frac{z(1-z)}{(1-2z)^2}$ and thus

$$\mathbb{E}(\chi) = \frac{1}{2^{n-1}} [z^n] \frac{z(1-z)}{(1-2z)^2} = \frac{1}{2}(n+1),$$

while the variance can be found to be $\frac{1}{2}\sqrt{n-1}$.

2.2 Analytic techniques

Recall that a function f is called analytic at a point z_0 if it is representable as a sum

$$f(z) = \sum_n a_n (z - z_0)^n$$

in an open disk around some $z_0 \in \Omega$. Then f is analytic in a region[5] $\Omega \in \mathbb{C}$ if it is analytic for every $z_0 \in \Omega$. We can then view the generating functions of section 1.1 as analytic functions around zero (when they are convergent). It is a basic theorem of complex analysis that complex analyticity is equivalent to differentiability of any degree and that the derivatives can be obtained through term by term differentiation of the power series.

We say that a function defined in a region Ω is analytically continuable to a point z_0 on its boundary if there exists a function F analytic in a region Ω_1 that contains z_0, such that the restriction of F in $\Omega \cap \Omega_1$ is equal to f. Analytic continuation in the complex field is unique in the sense that if there is a third function F' that obeys the same premises as F, then $F' = F$ in the region $\Omega_1 \cap \Omega_2$.

A point z_0 is called a singularity of f, if f can't be analytically continuable there. For our domain of generating functions, Pringsheim's theorem states that every function that is analytic around zero with non-negative coefficients and radius of convergence ρ has a real singularity on ρ. This comes very useful from the following reasoning: By the Cauchy-Hadamard formula for the radius of convergence of a power series, we know that this is equal to $\frac{1}{\limsup |a_n|^{1/n}} = \rho$, and then: for any $\epsilon > 0$, a_n exceeds $(\rho^{-n} + \epsilon)$ for only a finite number of values, while it exceeds $(\rho^{-n} - \epsilon)$ for infinitely many of them. In other words, a_n has an exponential growth of the order ρ^{-n}. This is illustrated by the definition

$$a_n \bowtie \rho^{-n} \qquad \text{iff} \qquad \limsup |A_n|^{1/n} = \rho^{-1},$$

and suggests the first principle for coefficient asymptotics, namely that

1. *The location of singularities determines the exponential growth of the coefficients.*

Now, the above discussion indicates that $a_n = \rho^{-n}\theta(n)$ for some function θ, called the subexponential factor, such that $\limsup |\theta(n)|^{1/n} = 1$. The second principle of coefficient asymptotics ammounts to the following:

2. *The nature of the singularities determines the associate subexponential factor $\theta(n)$.*

In our context, two types of singularities will be prominent: singularities emanating from inversion and algebraic singularities, both of which will be presented shortly.

Inverse functions

Let us at first recall the Analytic Inversion theorem:

[5] A region is an open and connected set.

Theorem 1. *(Analytic Inversion). Let $\psi(y)$ be analytic at y_0, with $\psi(y_0) = z_0$. Assume that $\psi'(y_0) \neq 0$. Then, for z in some small neighbourhood Ω_0 of z_0, there exists an analytic function $y(z)$ that solves the equation $\psi(y) = z$ and is such that $y(z_0) = y_0$.*

In particular, a function can be analytically inverted if and only if the above holds: otherwise, the point z_0 consists a branch point, meaning there are multiple inverses, all of which can only be defined in a slit neighbourhood of z_0.

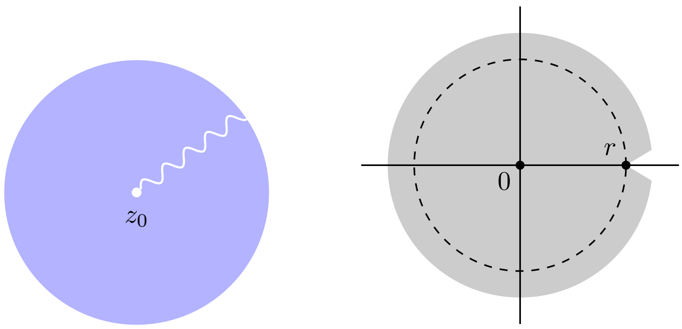

Figure 2.2: A slit neighbourhood of z_0 and a dented or $\Delta-$domain on r.

For completeness, we will also state a generalization of the above, the *implicit function theorem*:

Theorem 2. *(Multivariate Implicit Functions). Let $f_i(x_1, ..., x_m; z_1, ..., z_p)$, with $i = 1, .., m$, be analytic functions in the neighbourhood of a point $x_j = \alpha_j$, $z_k = c_k$. Assume that the* Jacobian determinant *defined as*

$$J := det\Big(\frac{\partial f_i}{\partial x_j}\Big)$$

is non-zero at the point considered. Then the equations (in the x_j) $y_i = f_i(x_1, ..., x_m; z_1, ..., z_p)$, $i = 1, ..., m$, admit a solution with the x_j near to the a_j, when the z_k are sufficiently near to the c_k and the y_i near to the $b_i := f_i(a_1, ..., a_m; c_1, ..., c_p)$: one has

$$x_j = g_j(y_1, ..., y_m; z_1, ..., z_p),$$

where each g_j is analytic in a neighbourhood of the point $(b_1, .., b_m; c_1, ..., c_p)$.

Now consider the case where a generating function $y(z)$ is defined by an equation of the form

$$y(z) = z\phi(y(z))$$

for some nonlinear ϕ with non-negative coefficients, analytic on $u = 0$ and with $\phi(0) \neq 0$. If there is a $\tau > 0$ within the radius of convergence of ϕ such that

$$\phi(\tau) - \tau\phi'(\tau) = 0,$$

then the Singular Inversion theorem applies:

Theorem 3. *(Singular Inversion). Let $y(z)$ be the solution of an equation $y(z) = z\phi(y(z))$. If ϕ fulfills the above conditions, then the quantity $\rho = \tau/\phi(\tau)$ is the radius of convergence of $y(z)$ at 0, and the singular expansion of $y(z)$ near ρ is of the form*

$$y(z) = \tau - d_1\sqrt{1 - z/\rho} + \sum_{j\geq 2}(-1)^j d_j(1 - z/\rho)^{j/2} \quad \text{with} \quad d_1 := \sqrt{\frac{2\phi(\tau)}{\phi''(\tau)}}$$

and d_j some computable constants.

Algebraic power series

A formal power series $f(z)$ is called *algebraic* if there exists a polynomial $p(z, x)$ such that $p(z, f(z)) = 0$. In many cases, we can establish the existence and uniqueness of such a power series given a bivariate polynomial through the implicit function theorem, a simple version of which is stated below:

Theorem 4. *(Analytic Implicit Functions). Let F be bivariate analytic near $(0, 0)$. Assume that $F(0, 0) = 0$ and $F'_x(0, 0) \neq 0$. Then there exists a unique function $f(z)$ analytic in a neighbourhood $|z| < \rho$ of 0 such that $f(0) = 0$ and*

$$F(z, f(z)) = 0, \quad |z| < \rho.$$

Note that for any algebraic power series $f(z)$ there is a unique monic irreducible polynomial $p(x, z)$ that vanishes for $x = f(z)$, called its minimal polynomial (for more, see Ch.6 of [11]). Let

$$p(x, z) = p_0(z)x^d + p_1(z)x^{d-1} + ... + p_d(z) \in \mathbb{C}(z)[x]. \tag{2.2}$$

Now fix some z_0 on the radius of convergence of $f(z)$. Then the above polynomial in x has usually d solutions, except for the case where there is a multiple solution or the coefficient $p_0(z_0)$ vanishes. If none of this happens on z_0, then by the implicit function theorem there are d distinct analytic solutions $x_1, ..., x_d$ on each point (z_0, x_i) (called *branches* of the algebraic curve $p(z, x) = 0$) and necessarily one of them constitutes an analytic continuation of $f(z)$. Noting that the problematic points z_0 are finite, we can deduce that $f(z)$ can be analytically continuable on any non problematic point, since there can always be a path to get there. We call the set of such problematic points the *exceptional set* of p and the singularities of f definitely belong there. In order to compute exceptional set we can use the algebraic construction of the *resultant*.

> The resultant of two polynomials $p(x) = p_0x^d + p_2x^{d-1} + ... + p_d,\ q(x) = q_0x^{d'}$ $+q_2x^{d'-1} + ... + p_{d'}$ in $\mathbb{C}[x]$ is a polynomial on the coefficients p_i, q_i that vanishes if and only if either the two polynomials have a common solution or $p_0 = q_0 = 0$.[6]

Returning to our quest for singularities, if on the fixed z_0 there is a multiple root y_i, then it is a solution of $\partial_x p(z, x)$ as well. So, considering z as a parameter and computing the resultant $\mathbf{R}(p, \partial_x p, x)$, we have a polynomial on z that vanishes exactly at the critical points.[7]

[6] More details exist in [2] or [3] for the generalized construction.
[7] In fact, generally this is called the *discriminant* of a polynomial.

Now we have to highlight the fact that generating functions in our context will indeed have a finite singularity. An immediate explanation comes from the fact that our combinatorial structures will have exponential growth and thus the first principle of coefficient asymptotics can be applied.[8] Hence, what is left for us to do is to figure out which of the points of the exceptional set really are singularities of $y(z)$. This can be accomplished by ad hoc arguments in the simple cases or else there is a systematic numeric way in order to do it (see note VII.36 from [9]).

After the singularities are recognized, the following theorem can be applied:

Theorem 5. *(Newton-Puiseux expansions at a singularity). Let $f(z)$ be a branch of an algebraic function $P(z, f(z)) = 0$. In a circular neighbourhood of a singularity ζ slit along a ray emanating from ζ, $f(z)$ admits a fractional series expansion (Puiseux expansion) that is locally convergent and of the form*

$$f(z) = \sum_{k \geq k_0} c_k (z-\zeta)^{k/\kappa},$$

for a fixed determination of $(z-\zeta)^{1/\kappa}$, where $k_0 \in \mathbb{Z}$ and κ is an integer ≥ 1, called the branching type.

In the above representation it is noted that if the branching type is 2, then we say that there is a *square root type of singularity*, a term that will be recurrent in our context. Also, the first appearing non integer power is called the *critical exponent.*

In order to compute the Puiseux series, we first translate the algebraic curve in order to transfer the singularity on $(0, 0)$. Then, using the Newton polygon method we can find the first exponent of the expansion, then apply indeterminate coefficients, and then iterate the same procedure until we have as many terms of the expansion as wanted. In short, the Newton polygon method consists of considering the exponents of the monomials as points in the plane, and then taking all the inverse slopes of the leftmost convex envelope as possible candidates for the exponents of the expansion.

An important detail about algebraic series is that they are still considered algebraic if they are defined by a system of polynomial equations of the form

$$\begin{cases} P_1(z, x_1, ..., x_m) & = & 0 \\ \vdots & \vdots & \vdots \\ P_m(z, x_1, ..., x_m) & = & 0 \end{cases}$$

The reason comes from the theory of *algebraic elimination* where, generically, we can eliminate any $m-2$ of the above variables and end up with a defining equation of the form $P(z, x_i)$ (see [2] and [3]).

[8] The reason here is nevertheless deaper, since it is a known fact from complex analysis that the only algebraic and entire functions are polynomials.

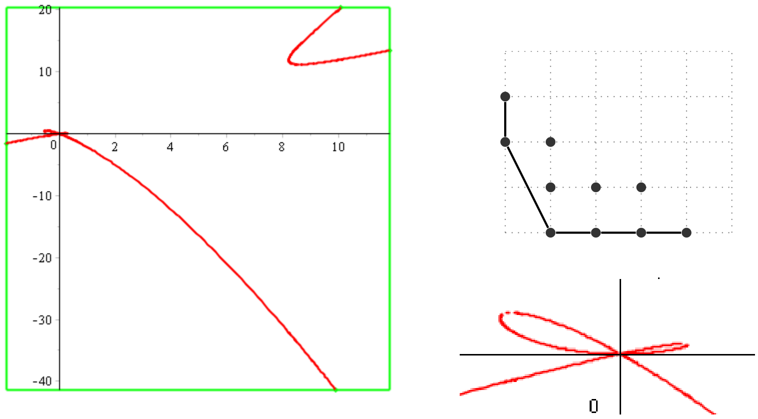

Figure 2.3: The plot of the real algebraic curve $x^3+x^2y+x(-z^2-z^3)+z^4$, with a zoom in the interval $[-1,1]$ next to it. The exceptional set in this case is comprised of $z\approx 0.2, -0.5, 8$ and $z=0$. Above is the Newton diagram for its translation $z\approx 0.2-Z$ and $x\approx 0.11+X$, that gives (with approximate constants) the equation $X^3+0.64X^2-X^2Z+0.60XZ-0.15Z-1.88XZ^2+XZ^3+0.29Z^2-1.05Z^3+Z^4$, leading to the critical exponent $1/2$.

Now, under some assumptions, polynomial systems may be amenable to the more general theorem 5 that will be stated shortly. In order to do so, we must first introduce the notion of the dependency graph for a system of equations. Let S be a system of the form

$$\begin{cases} y_1 & = & F_1(z, y_1, ..., y_m) \\ \vdots & \vdots & \vdots \\ y_m & = & F_m(z, y_1, ..., y_m) \end{cases}$$

The dependency graph of such a system is a graph with vertices $y_1, ..., y_N$ and a directed edge $y_i \to y_j$ if the function for y_i depends in a non trivial way on y_j. The graph is then strongly connected if for every pair of vertices there is a directed path between them (see figure 2.4).

Theorem 6.[9] *Let* $\mathbf{F}(z,\mathbf{y}) = (F_1(z,\mathbf{y}), ..., F_N(z,\mathbf{y}))$ *be a non-linear system of functions analytic around* $z=0$, $\mathbf{y}=(y_1,...,y_N)=\mathbf{0}$ *whose Taylor coefficients are all non-negative, such that* $\mathbf{F}(0,\mathbf{y})=\mathbf{0}$, $\mathbf{F}(z,\mathbf{0})\neq\mathbf{0}$, $\mathbf{F}_z(z,\mathbf{y})\neq\mathbf{0}$. *Furthermore assume that the dependency graph of* $\mathbf{F}$ *is strongly connected and that the system*

$$\mathbf{y} = \mathbf{F}(z,\mathbf{y})$$

$$0 = det(\mathbf{I}-\mathbf{F}_{\mathbf{y}}(z,\mathbf{y}))$$

has solutions z_0 *and* $\mathbf{y}_0$ *that are real and positive. Then there exists* $\epsilon>0$ *such that* $y_j(z)$ *admit a representation of the form*

$$y_j(z) = g_j(z) - h_j(z)\sqrt{1-z/z_0}$$

[9] First presented in [5].

for $|z-z_0| < \epsilon$ *and* $|arg(z-z_0) \neq 0|$, *where* $g_j(z) \neq 0$ *and* $h_j(z) \neq 0$ *are analytic funtions with* $g_j(z_0) = y_j(z_0) = (\mathbf{y}_0)_j$. *Furthermore, if* $[z^n]y_j(z) > 0$ *for* $1 \leq j \leq N$ *and for sufficiently large* $n \geq n_1$, *then there exists* $0 < \delta < \epsilon$ *such that* $y_j(z)$ *is analytic for* $|z-z_0| \geq \epsilon$ *but* $|z| \leq |z_0| + \delta$ *(this condition guarantees that* $\mathbf{y}(z)$ *has a unique smallest singularity with* $|z| = z_0$*).*

The above suggests that we can a priori suppose a square root type of singularity, without resorting to the Newton polygon method, for a wide variety of combinatorial classes including the so-called irreducible context-free structures.

$$y_1 = y_2 + y_3 + y_4$$
$$y_2 = z + \frac{y_1^5}{1-y_1}$$
$$y_3 = y_2^2(1+2y_2)$$
$$y_4 = (y_2+y_4)^3$$

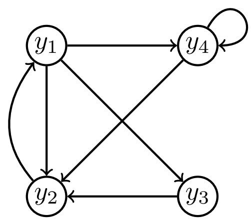

Figure 2.4: A system of equations and its corresponding dependency graph

Singularity Analysis

The previous two sections have established certain expansions for specific types of functions, in order to show here how they are analysed using singularity analysis. Just like with the symbolic method, the spirit in the methods of analytic combinatorics here is the same: building theorems that serve as general *black boxes* and apply automatically to a wide variety of combinatorial classes.

In particular, singularity analysis offers the tools and conditions in order to transfer error terms near singularity expansions to coefficients, which is the ultimate goal. The notion of a *dented* of $\Delta-$*domain* on $z = r$ is essential here (see figure 2.2). Formally, it is defined as

$$\Delta_r(\phi, R) = \{z \ : \ |z| < R, \ z \neq r, \ |arg(z-r)| > \phi\}$$

for some $R > r$ and $0 < \phi < \frac{\pi}{2}$.

Theorem 7. *(Singularity analysis, single singularity). Let* $f(z)$ *be a function analytic at* $z = 0$, *such that* $f(z)$ *can be continued to a dented domain* Δ_ζ *on* ζ. *Assume that there exist two functions* σ, τ, *where* σ *is a finite linear combination of functions in the standard scale, so that*

$$f(z) = \sigma(z/\zeta) + \mathcal{O}(\tau(z/\zeta)) \qquad \text{as } z \to \zeta \text{ in } \zeta \cdot \Delta_\zeta.$$

Then, the coefficients of $f(z)$ *satisfy the asymptotic estimate*

$$f_n = \zeta^{-n}\sigma_n + \mathcal{O}(\zeta^{-n}\tau_n).$$

Note that if $f(z) = \sigma(z/\zeta) + \mathcal{O}(\tau(z/\zeta))$, then $[z^n]f(z) = [z^n]\sigma(z/\zeta) + [z^n]\mathcal{O}(\tau(z/\zeta))$, so indeed an asymptotic transfer on the coefficients has been made, using scaled versions of σ, τ. These are used in their scaled version, since the functions of the *standard scale* typically contain functions with singularities at $z = 1$. Standard scale functions consist in this case the analytic dictionary, i.e. they are a set of functions for which we know exactly the asymptotic expansion of their coefficients. In our context, the case of $\sigma(z) = (1-z)^{-\alpha}$ for non-positive α will be very useful:

$$[z^n]\sigma(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}.$$

In order to connect this with the previous sections, it is enough to conclude from the above that if

$$A(z) = a_0 + a_1(1 - z/\rho)^{-\alpha} + \mathcal{O}\big((1 - z/\rho)^{-\alpha+1}\big),$$

for a non-positive α, then

$$[z^n]A(z) = \alpha_1 \frac{n^{\alpha-1}}{\Gamma(\alpha)} \rho^{-n} \Big(1 + \mathcal{O}\Big(\frac{1}{n}\Big)\Big).$$

The above simplified version will be regularly referred to as the *transfer theorem for singularity analysis.*[10]

Limit laws and the quasi-powers theorem

A random real variable Y is completely defined by its distribution function

$$F(x) = \mathbf{P}[Y \le x].$$

An important distribution in our context is the Gaussian, or normal $\mathcal{N}(0,1)$, whose distribution function is:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-w^2/2} dw.$$

If $Y$is $\mathcal{N}(0,1)$, then the variable $Y' = \mu + \sigma Y$ defines the normal distribution with mean μ and standard deviation σ, denoted $\mathcal{N}(\mu, \sigma)$.

Let Y be a continuous random variable with distribution function $F_Y(x)$. A sequence of random variables Y_n with distribution functions $F_{Y_n}(x)$ is said to converge in distribution to Y if pointwise for each x :

$$\lim_{n\to\infty} F_{Y_n}(x) = F_Y(x).$$

Then one writes $F_{Y_n} \Rightarrow F_Y$, and convergence is said to take place with speed ϵ_n if

$$\sup_{x\in\mathbb{R}} |F_{Y_n}(x) - F_Y(x)| \le \epsilon_n.$$

In particular, it is a known fact that convergence of distribution functions to a continuous limit is always uniform.

[10] $\Gamma(\alpha)$ is the Euler Gamma function, defined as $\Gamma(\alpha) := \int_0^\infty e^{-t} t^{\alpha-1} dt$ for $Re(\alpha) > 0$, which coincides with $(\alpha-1)!$ when α is an integer.

We will now turn again to bivariate generating functions and state the so-called *Quasi-Powers* theorem, which provides details for the limiting distribution of the random variable defined by the parameter χ when $n \to \infty$.

In our context, the important thing is that the analytic expansions obtained in the single variable case are preserved in form and critical exponent when we add the second variable u as a parameter in an infinitesimal neighbourhood of $u = 1$. In fact, while the critical exponent remains still, the singularity $\rho(u)$ moves smoothly with u. We can then apply the Quasi-Powers theorem which can be stated as follows, in its simplified version for algebraic singularities (Th.IX.12 from [9]):

Theorem 8. *Let $F(z,u)$ be a function that is bivariate analytic at $(z,u) = (0,0)$ and has non-negative coefficients. Assume the following conditions:*

(i) Analytic perturbation: there exist three functions A, B, C, analytic in a domain $\mathcal{D} = \{|z| \leq r\} \times \{|u-1| < \epsilon\}$, such that, for some r_0 with $0 < r_0 \leq r$, and $\epsilon > 0$, the following representation holds, with $a \notin \mathbb{Z}_{\leq 0}$,

$$F(z,u) = A(z,u) + B(z,u)C(z,u)^{-\alpha};$$

futhermore, assume that, in $|z| < r$, there exists a unique root ρ of the equation $C(z,1) = 0$, that this root is simple, and that $B(\rho,1) \neq 0$.

(ii) Non-degeneracy: one has $\partial_z C(\rho,1) \cdot \partial_u C(\rho,1) \neq 0$, ensuring the existence of a non-constant $\rho(u)$ analytic at $u = 1$, such that $C(\rho(u),u) = 0$ and $\rho(1) = \rho$.

(iii) Variability: one has

$$b\Big(\frac{\rho(1)}{\rho(u)}\Big) = -\frac{\rho''(1)}{\rho(1)} - \frac{\rho'(1)}{\rho(1)} + \Big(\frac{\rho'(1)}{\rho(1)}\Big)^2 \neq 0 \tag{2.3}$$

Then, the random variable with probability generating function

$$p_n(u) = \frac{[z^n]F(z,u)}{[z^n]F(z,1)}$$

converges to a Gaussian variable with a speed of convergence that is $\mathcal{O}(n^{-1/2})$. The mean μ_n and the variance σ_n^2 converge asymptotically to

$$m\Big(\frac{\rho(1)}{\rho(u)}\Big)n \quad \text{and} \quad b\Big(\frac{\rho(1)}{\rho(u)}\Big)n$$

where $m\Big(\frac{\rho(1)}{\rho(u)}\Big) = -\frac{\rho'(1)}{\rho(1)}$ and $b\Big(\frac{\rho(1)}{\rho(u)}\Big)$ as in (2.3)

2.3 Application: labelled outerplanar graphs

> An outerplanar graph is a planar graph that can be embedded on the plane in such a way that all the vertices lie on the outer face. A second definition: outerplanar graphs are graphs that do not contain K_4 or $K_{2,3}$ as a minor[11].

The two definitions are equivalent: If a graph contains $K_{2,3}$ as a minor and is outerplanar, then we can insert a vertex on the outer face, and attach it with the three suitable vertices of $K_{2,3}$ in order to obtain a planar embedding of $K_{3,3}$ - impossible by Kuratowski's theorem. The same reasoning applies to the case of K_4, building in this way a planar embedding of K_5 which is also impossible. Conversely: If a graph does not contain $K_{2,3}$ as a minor or K_4, then we insert an extra vertex and connect it with all the existing vertices. This graph cannot contain K_5 or $K_{3,3}$, since then the original graph would contain a $K_{2,3}$ or K_4, so we can conclude it is planar. Then there is an embedding of it with the extra vertex on the outer face. Since it is connected in a planar way with all the original vertices, removing it will create an embedding for the original graph with all the vertices on the outer face.

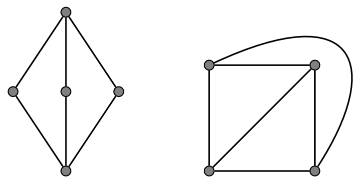

Figure 2.5: The graphs $K_{2,3}$ and K_4 in a planar embedding

This section demonstrates and explains the way general, labelled outerplanar graphs were specified and analysed asymptotically in [1].

The combinatorial structure

The process in order to reach a specification for our class is the following:

1. Find a specification for the biconnected outerplanar graphs, which corresponds to the generating function $D(z)$.
2. Place labels on them, and obtain $B(z)$.
3. Attach graphs of the above class in a tree-like way, in order to obtain connected labelled outeplanar graphs and the corresponding $C(z)$.
4. Take sets of the above, in order to obtain general labelled outerplanar graphs and the corresponding $G(z)$.

For number 1: It is known that biconnected graphs can be constructed from a cycle by successively adding $G-$paths to G graphs already constructed (see Pr.3.1.1 from [4]). Since outerplanar graphs do not contain $K_{2,3}$ or K_4 as minors, the paths can be either non-crossing

[11] A graph H is a minor of G if it can be obtained from G by applying edge contactions and deletions, and deletions of vertices.

chords, or paths attached on the ends of an existing edge. This way we can view 2-connected outerplanar graphs as composed of a unique hamilton cycle with non-crossing chords, implying that such $n-$graphs are isomorphic as a combinatorial structure to dissections of $n-$polygons. We suppose on the starting $n-$cycle the regular $\{1, 2, ..., n\}$-clockwise numbering on the vertices. Then, we can mark the edge $e = (1, 2)$ at all dissections and divide them to dissections where e lies alone, or on a triangle, on a quadrangle etc. If e lies on a $k-$gon (or, else said, on a $k-$gon base) then we can identify each dissection with a sequence of $k - 1$ dissections that are attached around it. Thus, letting z^2 represent an edge, we have:

$$D = z^2 + \frac{D^2}{z} + \frac{D^3}{z^2} + ... + \frac{D^r}{z^{r-1}} + ... \tag{2.4}$$

$$\Rightarrow \quad D = z^2 + \frac{D^2}{z - D} \quad \Rightarrow \quad 2D^2 - z(1 + z)D + z^3 = 0.$$

where the denominators erase the vertices that are double counted when attaching two neighbouring dissections on the base.

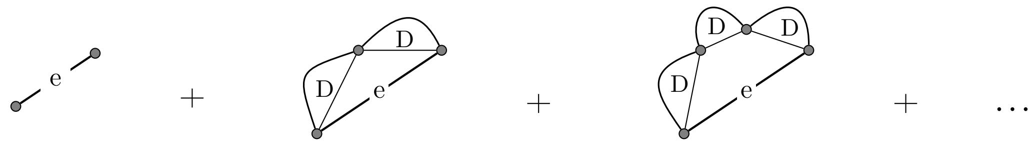

Figure 2.6: The construction of dissections D.

The last equation can be solved directly as a quadratic equation, and gives:

$$D(z) = z/4 + z^2/4 - z/4\sqrt{z^2 - 6\,z + 1}$$

since the other solution has negative coefficients.

For number 2: Now we must place labels on the dissections. All circular labellings without orientation give a different graph (as they give a different Hamilton cycle), so for each dissection there are $b_n = d_n\frac{(n-1)!}{2}$ biconnected labelled outerplanar graphs, except for the case of $n = 2$ where $b_2 = 1$. But $B(z)$ is an exponential generating function, so, having

$$b_n = d_n\frac{(n-1)!}{2} \quad \Rightarrow \quad n\frac{b_n}{n!} = \frac{d_n}{2} \quad \text{for} \quad n \geq 3$$

and $b_2 = 1$, we can say that

$$zB'(z) = \frac{D(z)}{2} + \frac{z^2}{2} \quad \Rightarrow \quad B'(z) = \frac{\frac{1}{z}D(z) + z}{2}.$$

For number 3: To pass to the connected outerplanar graphs, the following relation is used:

$$zC'(z) = z\exp(B'(zC'(z))).$$

This depicts the fact that we can decompose a rooted connected outerplanar graph into its rooted maximal biconnected components that lie on the main root, while each one of the components is carrying other rooted connected outerplanar graphs on its nodes.

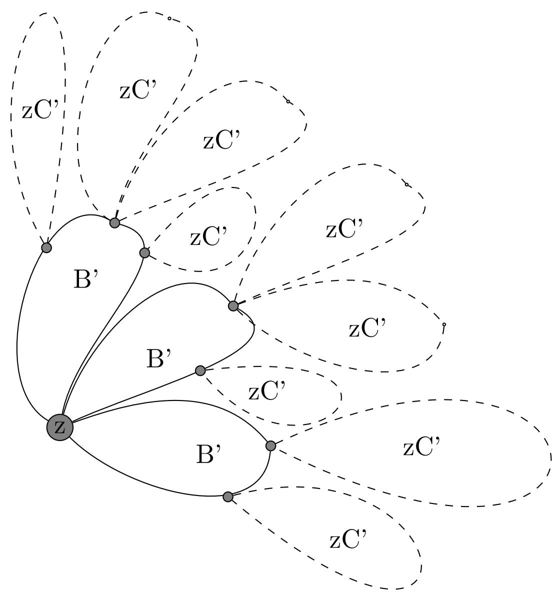

Figure 2.7: The decomposition of a rooted connected outerplanar graph

In more detail, the left side suggests that a vertex is being pointed to, as a root, while the right side suggests taking an atom z as a root, and then taking sets of rooted biconnected outerplanar graphs attached to it. Note that the roots of the biconnected components have no size, since they are attached to the atom z. Now, the substistution inside B shows that each vertex of the attached biconnected components carries a rooted connected outerplanar graph.

For number 4: The last equation is straightforward, since, in order to obtain general labelled outerplanar graphs, we only have to take sets of labelled connected ones:

$$G(z) = \exp(C(z)).$$

What is more, the equations for 3 and 4 can also be applied in the enumeration of general planar graphs (see [10]), since any planar graph has a similar unique tree-like decomposition into its maximal biconnected components, and any set of planar graphs consists a planar graph as well. Notice that this decomposition is particular to planar graphs. In fact is does not apply for graphs of higher genus.

A specification for labelled outerplanar graphs	
2-connected components	$D(z) = z/4 + z^2/4 - z/4\sqrt{z^2 - 6z + 1}$
Labelled 2-connected	$B'(z) = \frac{1}{2z}D(z) + \frac{z}{2}$
Labelled connected	$zC'(z) = z\exp(B'(zC'(z)))$
General labelled	$G(z) = \exp(C(z))$

Table 2.3: The complete specification for labelled outerplanar graphs that is derived in [1].

Outline of the asymptotic analysis

The first equation of table 2.3 is algebraic, with a unique real and positive singularity $r = 3 - 2\sqrt{2}$. It then admits a Puiseux expansion on r, with critical exponent $1/2$:

$$a_0 + a_1(1 - z/r)^{1/2} + a_2(1 - z/r) + ... \quad \text{with} \quad a_1 \neq 0.$$

On the other hand, $B'(z)$ inherts exactly the same singularities as $A(z)$ and the same type of expansion on r, since it suffices to consider the taylor expansion of $\widetilde{B}'(z) = \frac{u}{2z} + \frac{z}{2}$ on r and then substitute u with the singular expansion of $A(z)$. Thus, B' admits a singular expansion of the form:

$$b_0 + b_1(1 - z/r)^{1/2} + b_2(1 - z/r) + ... \quad \text{with} \quad b_1 \neq 0.$$

Let us now consider $F(z) = zC'(z)$. The inverse of F is $\psi(u) = u\exp(-B'(u))$, and differentiating we have $\psi'(u) = \exp(-B'(u))(1 - uB''(u))$. In the case where ψ' becomes zero for some $z = \tau$, we know that $\psi(\tau)$ is a singularity for F, and for C as well. It can be argued that there is such a τ and in fact it is smaller than r, which makes $\rho = \psi(\tau)$ its dominant singularity: differentiating B', we come up with a leading $-\frac{1}{2}$ term in the expansion, so B'' tends to infinity near r, while $\frac{1}{u}$ is decreasing, so there must be a point before r where the two functions meet, let us call it τ. Then, by singular inversion, F admits a singular expansion on ρ of the form:

$$F_0 + F_1\sqrt{1 - z/\rho} + \mathcal{O}(1 - z/\rho) \quad \text{for} \quad F_0 = \tau, \quad \text{and} \quad F_1 := -\sqrt{\frac{2\phi(\tau)}{\phi''(\tau)}}. \tag{2.5}$$

Now, F and C have the same singularities, and thus ρ is the dominant singularity of C. This gives C a singular expansion of critical exponent $3/2$ on ρ, since it is the integral of F divided by z. The singular expansion of F can be obtained from that of C by differentiating and multiplying by z, and setting $Z = \sqrt{1 - z/\rho} \Rightarrow z = \rho(1 - Z^2)$, we have

$$\rho(1 - Z^2)C'(z) = \rho(1 - Z^2)\Big(-\frac{C_2}{\rho} - \frac{3C_3}{2\rho}Z + \mathcal{O}(Z^2)\Big) = -C_2 + \Big(\frac{1}{2}C_1 - \frac{3}{2}C_3\Big)Z + \mathcal{O}(Z^2),$$

and equating with (2.5) gives $C_2 = -F_0$ and $C_3 = -\frac{2}{3}F_1$.
In order to find C_0 we integrate $\frac{F(s)}{s}$, having:

$$C(z) = \int_0^z \frac{F(s)}{s}ds = F(z)\log z - \int_0^z F'(s)\log s ds.$$

Changing variables to $t = F(s)$ so that $s = \psi(t) = t\exp(-B'(t))$, the last integral becomes:

$$\int_0^{F(z)} \log\psi(t)dt = \int_0^{F(z)} (\log t - B'(t))dt = F(z)\log(F(z)) - F(z) - B(F(z)) + B(0).$$

Since $B(0) = 0$, we have

$$C(z) = F(z)(\log z + 1 - \log(F(z)) + B(F(z)),$$

and taking in mind $F(\rho) = \tau$ leads to

$$C_0 = \tau(\log(\rho) + 1 - \log(\tau)) + B(\tau). \tag{2.6}$$

In the end, $G(z)$ has the same singularities as $C(z)$, since the exponential is an entire function, and the singular expansion of G follows immediately:

$$G(z) = \exp(C_0 + C_2 Z^2 + C_3 Z^3 + \mathcal{O}(Z^4)) = \exp(C_0)\exp(C_2 Z^2 + C_3 Z^3) + \mathcal{O}(Z^4)$$

$$\Rightarrow G(z) = \exp(C_0)(1 + C_2 Z^2 + C_3 Z^3) + \mathcal{O}(Z^4). \tag{2.7}$$

The analysis can be now completed by an application of the transfer theorem for singularity analysis, and the final asymptotic result is of the form

$$g_n \sim g \cdot n^{-5/2} \cdot \rho^{-n} \cdot n!$$

where $g = \exp(C_0) \cdot C_2$.

In fact, the actual constants found in [1] are $\rho \approx 0.13659$ and $g \approx 0.01821$.

Chapter 3

Outerplanar graphs with forbidden cycles

The results from this point onwards are original results of this thesis. In the present chapter we will build specifications for outerplanar graphs that avoid certain $k-$cycles, in the spirit of section 2.3. The first step will be to build specifications for the biconnected components, which is equivalent to building specifications for dissections d that avoid the same cycles, considering at first $\hat{d} = d/z$ in order to avoid the denominators and have clearer equations (mainly in the more involved cases of 5 and 6 cycles). In combinatorial terms, this can be seen as not counting one of the two atoms of the base edge e. Having accomplished the specification for d, the equations to pass to labelled biconnected, to connected, and then to general outerplanar graphs are the same as the ones in Table 2.3.

The appearing generating functions	
Symbols	Combinatorial structures
d	dissections of a restricted kind
$\hat{d}$	dissections in d/z
$\hat{d}_2$	dissections in d/z that exclude as a base all j-cycles for $3 \leq j \leq n-1$
$\hat{d}_j$ for $3 \leq j \leq n-1$	dissections in d/z with a j-cycle base
B	labelled biconnected outerplanar graphs of a restricted kind
C	labelled connected outerplanar graphs of a restricted kind
G	labelled general outerplanar graphs of a restricted kind

Table 3.1: The appearing combinatorial structures and their respective generating function symbols

3.1 The case of 3 and 4-cycles

3.1.1 When 3-cycles are excluded

At first, we will study the biconnected components. Excluding the case of a triangle base in the original equation for general dissections (2.4) and dealing with $\hat{d} = d/z$, the dissections with no triangles in $\hat{d}$ are specified by the equation below:

$$\hat{d} = z + \hat{d}^3 + \hat{d}^4 + ... = z + \frac{\hat{d}^3}{1-\hat{d}} \quad \Rightarrow \quad \hat{d}^3 + \hat{d}^2 + \hat{d}(-z-1) + z = 0$$

Multiplying by z^3, the equation becomes

$$d^3 + d^2 z + d(-z^2 - z^3) + z^4 = 0 \tag{3.1}$$

and the above polynomial $P(d,z)$ vanishes identically when substituting our generating function $d(z)$ for d. The discriminant of $P(d,z)$ with z as a parameter and d as the main variable is equal to the resultant:

$$\mathbf{R}\Big(P, \frac{\partial P}{\partial d}, d\Big) = -z^6(4z^3 - 32z^2 - 8z + 5),$$

and the exceptional set Ξ consists of its solutions. Rounding the values up to 5 digits, we have

$$\Xi = \{0.29336, 8.22469, -0.51805, 0\}.$$

There is a unique real root of minimum positive modulus $r \approx 0.29336$, so r is $d's$ dominant singularity.[1] Our function $d(z)$ is a branch of P so it admits a fractional Puiseux expansion of the form

$$d(z) = \sum_{k \geq k_0} a_k \Big(1 - \frac{z}{r}\Big)^{\frac{k}{\kappa}} \text{ for some positive } \kappa \in \mathbb{N}$$

in a dented domain on r. In order to compute the expansion we use iteratively the Newton polygon method along with undetermined coefficients, to obtain in the end the following expansion (see also figure 2.3):

$$a_0 + a_1\sqrt{1 - \frac{z}{r}} + \mathcal{O}\Big(1 - \frac{z}{r}\Big) \text{ for } a_0 \approx 0.11823, \text{ and } a_1 \approx -0.08260.$$

Using the transfer theorem for singularity analysis, one has:

$$[z^n]d(z) \sim a_1 \frac{r^{-n} n^{-3/2}}{\Gamma(\frac{-1}{2})} = a \cdot r^{-n} n^{-3/2} \text{ for } a \approx 0.02330.$$

In order to pass to labelled dissections and then to labelled outerplanar graphs we will use

[1] The analysis here could also be done with the explicit form of d, but not in the next cases where the degree of the defining polynomial equation becomes immediately much larger.

the equations described in Section 2.3, namely

$$B'(z) = \frac{\frac{1}{z}d(z) + z}{2} \qquad zC'(z) = z\exp(B'(zC'(z))) \qquad \text{and} \qquad G(z) = \exp(C(z)). \quad (3.2)$$

Now consider $F(z) := z\exp(B'(zC'(z)))$ and its inverse $\psi(u) = u\exp(-B'(u))$. Following the same reasoning as in 2.3, we expect to find some $\tau < r$ as a root of ψ and thus a smaller singularity for F and by extension for C.

The only remaining thing is to compute τ.
By differentiating (3.1) we find an expression of d' in terms of d :

$$d'(z) = -\frac{-3dz^2 + 4z^3 + d^2 - 2dz}{-z^3 + 3d^2 + 2dz - z^2} \quad (3.3)$$

and then

$$\psi'(u) = \frac{-18du^3 + 9u^4 + 3du^2 - 6u^3 + 3u^5 - 9d^2u^2 + 9d^3 + 27d^2u}{6z(-u^3 + 3d^2 + 2du - u^2)} \exp\Big(-\frac{\frac{d}{u} + u}{2}\Big), \quad (3.4)$$

hence the zeros of ψ' are the zeros of the above numerator. We solve the polynomial system of (3.1) and the above using the computational program `maple`, which gives us only one suitable pair of solutions: $(d', \tau) \approx (0.11010, 0.29118)$. It follows that the dominant singularity is on $\rho = \psi(\tau) \approx 0.20836$. Then, by singular inversion, F admits a singular expansion on ρ of the form:

$$F_0 + F_1\sqrt{1 - \frac{z}{\rho}} + \mathcal{O}\Big(1 - \frac{z}{\rho}\Big) \quad \text{for} \quad F_0 = \tau, \quad \text{and} \quad F_1 := -\sqrt{\frac{2\phi(\tau)}{\phi''(\tau)}} \approx 0.05546. \quad (3.5)$$

We know that C has the same dominant singularities, and a singular expansion of the form

$$C(z) = C_0 + C_2\Big(1 - \frac{z}{\rho}\Big) + C_3\Big(1 - \frac{z}{\rho}\Big)^{3/2} + \mathcal{O}\Big(\Big(1 - \frac{z}{\rho}\Big)^2\Big)$$

with $C_2 = -F_0$, $C_3 = -\frac{2}{3}F_1$, and $C_0 = \tau(\log(\rho) - \log(\tau) + 1) + B(\tau) \approx 0.01982$. It must be noted that $B(\tau)$ was computed using the Taylor expansion of b from that of d, and is equal to approximately 0.10232.

The singular expansion of G follows from $G(z) = \exp(C(z))$:

$$G(z) = \exp(C_0)(1 + C_2Z^2 + C_3Z^3) + \mathcal{O}(Z^4).$$

Using the transfer theorem for singularity analysis it can be deduced that:

$$g_n \sim \frac{g \cdot n^{-\frac{5}{2}} \cdot \rho^{-n} \cdot n!}{\Gamma(-\frac{3}{2})} \approx g' \cdot n^{-\frac{5}{2}} \cdot \rho^{-n} \cdot n!$$

for $\rho = \psi(\tau) \approx 0.20836$ and $g' = C_3 \cdot \exp(C_0) \cdot \Gamma(-\frac{3}{2})^{-1} \approx 0.01578$.

3.1.2 When 4-cycles are excluded

In this case it is not enough to exclude the 4−bases or both the 3 and 4−bases, since the 3−bases are allowed if a triangle is not attached to them. So we seperate cases. There is the case where the base is bigger than a 5−cycle, so anything from the class d can be attached around it and the case where we have a triangle base, and then we can attach anything except for $\hat{d}$−elements bearing a triangle base. This justifies the specification below:

$$\begin{aligned} \hat{d} &= \hat{d}_2 + \hat{d}_3 \\ \hat{d}_2 &= z + \hat{d}^4 + \hat{d}^5 + ... = z + \frac{\hat{d}^4}{1-\hat{d}} \\ \hat{d}_3 &= (\hat{d} - \hat{d}_3)^2 = \hat{d}_2^2 \end{aligned}$$

Multiplying by z to the right power each time, the system becomes:

$$\begin{aligned} d &= d_2 + d_3 \\ d_2 z^3 &= z^5 + \frac{zd^4}{z-d} \\ d_3 z &= d_2^2 \end{aligned}$$

Eliminating the other variables, it is found that d satisfies identically the equation:

$$d^2z^8 - 2dz^9 + z^{10} - 2d^5z^4 + 2d^4z^5 + d^2z^7 - 2dz^8 + z^9 + d^8 - d^5z^3 + d^4z^4 - d^3z^5 + 2d^2z^6 - dz^7 = 0. \tag{3.6}$$

The discriminant of $P(d, z)$ with d as the main variable and z as a parameter is equal to

$$\mathbf{R}\left(P, \frac{\partial P}{\partial d}, d\right) = z^{56}(233280z^9 - 3373191z^8 + 4822712z^7 + 46343900z^6 +$$

$$72539810z^5 + 43394958z^4 + 8355236z^3 + 793033z^2 + 1407918z - 908503).$$

We compute the exceptional set with `maple` and find only one real root with modulus smaller than one, $r \approx 0.26488$. By Pringsheim's theorem we can deduce that this is one of the dominant sigularities and, since all the other roots have larger modulus, it is the only one. We can then compute the Puiseux expansion of P on a dented domain around r as in the previous case, and find an expansion of the form:

$$a_0 + a_1\sqrt{1-\frac{z}{r}} + \mathcal{O}\Big(1-\frac{z}{r}\Big) \text{ for } a_0 \approx 0.11156,\ a_1 \approx -0.07717.$$

Using the transfer theorem for singularity analysis, one has:

$$[z^n]d(z) \sim a_1 \frac{r^{-n}n^{-3/2}}{\Gamma(\frac{-1}{2})} = a \cdot r^{-n} \cdot n^{-3/2} \quad \text{for } a \approx 0.02177.$$

The analysis required in order to pass from the 2-connected components to labelled outerplanar graphs is done again using the equations in Section 2.3, and the same arguments for the singular expansions of F, B' and C as before. We can still express the derivative d' in terms of d and then compute $\psi'(u)$ in terms of d. In this way we find the value of τ by solving the system of $\psi'(u)$ and the defining equation for d, (3.6). We have $\tau \approx 0.26280$ and $\rho = \psi(\tau) \approx 0.18919$. We compute $B(\tau)$ using the Taylor expansion of B from that of d and it is found equal to approximately 0.03835. Through it we can compute $C_0 \approx 0.00676$, and in the end we have:

$$g_n \sim \frac{g \cdot n^{-\frac{5}{2}} \cdot \rho^{-n} \cdot n!}{\Gamma(-\frac{3}{2})} = g' \cdot n^{-\frac{5}{2}} \cdot \rho^{-n} \cdot n!$$

for $\rho = \psi(\tau) \approx 0.18919$ and $g' = C_3 \cdot \exp(C_0) \cdot \Gamma(-\frac{3}{2})^{-1} \approx 0.01462$.

3.2 The case of 5 and 6-cycles

In this section, instead of eliminating the extra variables d_i in order to get a single equation with d, z, we will use immediately Theorem 5. The reason is that the defining polynomials now become quickly too big to handle computationally with relative ease.

3.2.1 When 5-cycles are excluded

Here we divide the dissections $\hat{d}$ with no $5-$cycles into 3 categories according to their base:

- $\hat{d}_2$ denotes the dissections that have as a base a $j-$polygon with $j \geq 5$. The base is then big enough to attach anything around it.
- $\hat{d}_3$ denotes the dissections that have as a base a triangle. Then we can either attach only members of d_2, or attach one triangle on it (2 ways to do that) and then attach members of $\hat{d}_2$.
- $\hat{d}_4$ denotes the dissections that have as a base a quadrangle, and the only thing to avoid there is attaching a triangle.

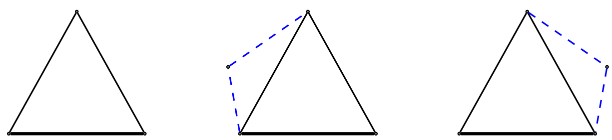

Figure 3.1: The possible bases for d_3.

The above are reflected in the following specification:

$$\begin{aligned} \hat{d} &= \hat{d}_2 + \hat{d}_3 + \hat{d}_4 \\ \hat{d}_2 &= z + \hat{d}^5 + \hat{d}^6 + ... = z + \frac{\hat{d}^5}{1-\hat{d}} \\ \hat{d}_3 &= \hat{d}_2^2 + 2(\hat{d} - \hat{d}_3 - \hat{d}_4)^3 = \hat{d}_2^2 + 2\hat{d}_2^3 = \hat{d}_2^2\Big(1 + 2\hat{d}_2\Big) \\ \hat{d}_4 &= (\hat{d} - \hat{d}_3)^3 = (\hat{d}_2 + \hat{d}_4)^3 \end{aligned} \tag{3.7}$$

Multiplying by z to the suitable power and simplifying, the system becomes:

$$\begin{aligned} d - d_2 - d_3 - d_4 &= 0 \\ d^5 - z^5 d + z^6 - z^4 d_2 + z^3 d d_2 &= 0 \\ 2d_2^3 + d_2^2 z - z^2 d_3 &= 0 \\ z^2 d_4 - (d_2 + d_4)^3 &= 0 \end{aligned} \tag{3.8}$$

Consider the right-hand side of (3.7) as a system of equations $\mathbf{F}$ on the variables $z, \hat{\mathbf{d}}$. The equations of $\mathbf{F}$ are non-linear, analytic around zero with positive Taylor coefficients, and with a strongly connected dependency graph (see Figure 2.4). We also have $\mathbf{F}(z, \mathbf{0}) \neq 0$ and $\mathbf{F}_z(z, \hat{\mathbf{d}}) \neq 0$, so if we also had $\mathbf{F}(0, \hat{\mathbf{d}}) = 0$, then we could apply theorem 6. But in our combinatorial context we know that all $\hat{d}_i$ are of the form $z\hat{d}_i'$ for some d_i', which leads to an equivalent system that also fulfills $\mathbf{F}(0, \hat{\mathbf{d}}') = 0$. [2] Hence we can apply the theorem, and the next step is find the solution $(r, \hat{d}_1, ..., \hat{d}_5)$ of the characteristic system

$$\begin{aligned} \hat{\mathbf{d}} &= \mathbf{F}(z, \hat{\mathbf{d}}) \\ 0 &= \det(\mathbf{I} - \mathbf{F}_{\hat{\mathbf{d}}}(z, \mathbf{d})). \end{aligned}$$

The result is $r \approx 0.25383$ and then knowing that d has the same singularities and of the same type as $\hat{d}$, we can establish a singular expansion for $d(z)$ of the form

$$d(z) = a_0 + a_1\sqrt{1 - z/r} + \mathcal{O}(1 - z/r) \quad \text{for } a_0 \approx 0.11255 \quad \text{and } a_1 \approx -0.07861$$

in a dented domain around r. Via the transfer theorem for singularity analysis we obtain

$$[z^n]d(z) \sim a_1 \frac{r^{-n} n^{-3/2}}{\Gamma(\frac{-1}{2})} = a \cdot r^{-n} \cdot n^{-3/2} \quad \text{for } a \approx 0.02217$$

The computation for a_1 was done by substituting the singular expansions up to two terms for all d_i in (3.8), and then solving the system for the unknown coefficients.

The rest of the analysis is done in the same way as before, using the Taylor expansion of d (this time for the computation of τ as well), giving in the end that

$$g_n \sim \frac{g \cdot n^{-\frac{5}{2}} \cdot \rho^{-n} \cdot n!}{\Gamma(-\frac{3}{2})} = g' \cdot n^{-\frac{5}{2}} \cdot \rho^{-n} \cdot n!$$

for $\rho = \psi(\tau) \approx 0.18054$ and $g' = C_3 \cdot \exp(C_0) \cdot \Gamma(-\frac{3}{2})^{-1} \approx 0.01804$.

[2] The systems are indeed equivalent, since $\hat{d}_i$ and $\hat{d}_i'$ have the same singularities and of the same type.

3.2.2 When 6-cycles are excluded

The specification now gets a little more complex. Starting from the easier parts, the specifications for $\hat{d}, \hat{d}_2$ follow the same reasoning as in the previous cases and thus

$$\hat{d} = \hat{d}_2 + \hat{d}_3 + \hat{d}_4 + \hat{d}_5, \qquad \hat{d}_2 = z + \hat{d}^6 + \hat{d}^7 + ... = z + \frac{\hat{d}^6}{1 - \hat{d}}$$

For pentagon based dissections $\hat{d}_5$, the only thing to worry about is attaching a triangle-based dissection, so we have

$$\hat{d}_5 = (\hat{d} - \hat{d}_3)^4.$$

For the quadrangle base case $\hat{d}_4$ we have the option either to attach only elements of d_2 and d_5 which are too big to cause a problem, or attach some smaller polygons and deal with all the new restrictions. The first option yields the $(\hat{d}_2 + \hat{d}_5)^3$ summand, while the second one leads to three other cases (see also Figure 3.2):

1. attach exactly one triangle around the base (3 ways for that). Then no 3,4-gons can be placed on the quadrangle, and no 3,4,5-gons can be placed on the triangle. This gives the $(\hat{d} - \hat{d}_3 - \hat{d}_4)^2(\hat{d} - \hat{d}_3 - \hat{d}_4 - \hat{d}_5)^2$ summand.

2. attach one triangle and one quadrangle upon it (6 ways for that). Then no 3,4,5-gons can be placed on the triangle, and no 3,4-gons can be placed on the quadrangle. This leads to the $(\hat{d} - \hat{d}_3 - \hat{d}_4 - \hat{d}_5)(\hat{d} - \hat{d}_3 - \hat{d}_4)^5$ summand.

3. attach one triangle and then exactly two quadrangles (3 ways for that), which gives $(\hat{d} - \hat{d}_3 - \hat{d}_4)^8$.

Written in full, we have the following for $\hat{d}_4$:

$$\hat{d}_4 = (\hat{d}_2+\hat{d}_5)^3+3(\hat{d}-\hat{d}_3-\hat{d}_4)^2(\hat{d}-\hat{d}_3-\hat{d}_4-\hat{d}_5)^2+6(\hat{d}-\hat{d}_3-\hat{d}_4-\hat{d}_5)(\hat{d}-\hat{d}_3-\hat{d}_4)^5+3(\hat{d}-\hat{d}_3-\hat{d}_4)^8.$$

For the triangle base case we have the option either to attach only elements of $\hat{d}_2$, or at-

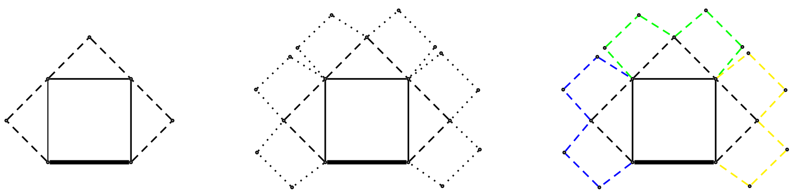

Figure 3.2: The possible bases for d_4 (we can use only one dashed and one dotted shape, while the coloured shapes are taken as whole)

tach smaller polygons and deal with all the new restricitons. The first option yields the $\hat{d}_2^2$ summand, while the second one leads to four other cases:

1. attach exactly one triangle on the triangle base (2 ways). Then no 3,4,5-gons are allowed around them. This corrresponds to the $(\hat{d} - \hat{d}_3 - \hat{d}_4 - \hat{d}_5)^3$ summand.

2. attach exactly two triangles (5 ways for that). Then no 3,4,5-gons are allowed and that gives $(\hat{d} - \hat{d}_3 - \hat{d}_4 - \hat{d}_5)^4$.

3. attach exactly one quadrangle (2 ways). Then no 3,4,5-gons are allowed on the triangle and no 4,5-gons are allowed on the quadrangle, leading to $(\hat{d}-\hat{d}_3-\hat{d}_4)^3(\hat{d}-\hat{d}_3-\hat{d}_4-\hat{d}_5)$.

4. attach two quadrangles (1 way). Then no 3,4-gons are allowed, and we have a $(\hat{d}-\hat{d}_3-\hat{d}_4)^6$ summand.

The result for $\hat{d}_3$ is

$$\hat{d}_3 = \hat{d}_2^2+2(\hat{d}-\hat{d}_3-\hat{d}_4-\hat{d}_5)^3+5(\hat{d}-\hat{d}_3-\hat{d}_4-\hat{d}_5)^4+2(\hat{d}-\hat{d}_3-\hat{d}_4)^3(\hat{d}-\hat{d}_3-\hat{d}_4-\hat{d}_5)+(\hat{d}-\hat{d}_3-\hat{d}_4)^6.$$

The complete specification is then:

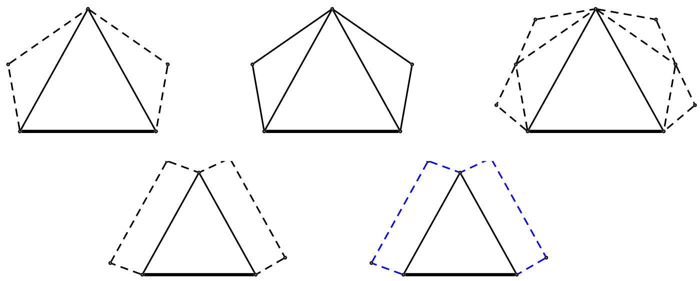

Figure 3.3: The possible bases for d_3 (the same rules apply as in Figure 2.1)

$$\begin{aligned}
\hat{d} &= \hat{d}_2+\hat{d}_3+\hat{d}_4+\hat{d}_5 \\
\hat{d}_2 &= z+\hat{d}^6+\hat{d}^7+\ldots = z+\frac{\hat{d}^6}{1-\hat{d}} \\
\hat{d}_3 &= \hat{d}_2^2+2(\hat{d}-\hat{d}_3-\hat{d}_4-\hat{d}_5)^3+5(\hat{d}-\hat{d}_3-\hat{d}_4-\hat{d}_5)^4+2(\hat{d}-\hat{d}_3-\hat{d}_4)^3(\hat{d}-\hat{d}_3-\hat{d}_4-\hat{d}_5) \\
&\quad +(\hat{d}-\hat{d}_3-\hat{d}_4)^6 \\
\hat{d}_4 &= (\hat{d}_2+\hat{d}_5)^3+3(\hat{d}-\hat{d}_3-\hat{d}_4)^2(\hat{d}-\hat{d}_3-\hat{d}_4-\hat{d}_5)^2+6(\hat{d}-\hat{d}_3-\hat{d}_4-\hat{d}_5)(\hat{d}-\hat{d}_3-\hat{d}_4)^5 \\
&\quad +3(\hat{d}-\hat{d}_3-\hat{d}_4)^8 \\
\hat{d}_5 &= (\hat{d}-\hat{d}_3)^4
\end{aligned}$$

equivalent to

$$\begin{aligned}
\hat{d} &= \hat{d}_2+\hat{d}_3+\hat{d}_4+\hat{d}_5 \\
\hat{d}_2 &= z+\frac{\hat{d}^6}{1-d} \\
\hat{d}_3 &= \hat{d}_2^2+2\hat{d}_2^3+5\hat{d}_2^4+2(\hat{d}_2+\hat{d}_5)^3\hat{d}_2+(\hat{d}_2+\hat{d}_5)^6 \\
\hat{d}_4 &= (\hat{d}_2+\hat{d}_5)^3+3(\hat{d}_2+\hat{d}_5)^2\hat{d}_2^2+6\hat{d}_2(\hat{d}_2+\hat{d}_5)^5+3(\hat{d}_2+\hat{d}_5)^8 \\
\hat{d}_5 &= (\hat{d}_2+\hat{d}_4+\hat{d}_5)^4
\end{aligned} \tag{3.9}$$

and, by multiplying with the right power of z each time and simplifying:

$$\begin{aligned}
d - d_2 - d_3 - d_4 - d_5 &= 0\\
d^6 + z^7 - z^6 d - d_2 z^5 + d d_2 z^4 &= 0\\
z^4 d_2^2 + 2d_2^3 z^3 + 5d_2^4 z^2 + 2(d_2+d_5)^3 d_2 z^2 + (d_2+d_5)^6 - d_3 z^5 &= 0\\
(d_2+d_5)^3 z^5 + 3z^4 (d_2+d_5)^2 d_2^2 + 6z^2 d_2 (d_2+d_5)^5 + 3(d_2+d_5)^8 - d_4 z^7 &= 0\\
(d_2+d_4+d_5)^4 - z^3 d_5 &= 0
\end{aligned} \tag{3.10}$$

Consider the system (3.9) as $\mathbf{F}$. This satisfies the premises of theorem 5, and so the dominant singularity of $\hat{d}$ is the r from the solution $(r, d_1, ...d_5)$ of the system

$$\begin{aligned}
\hat{\mathbf{d}} &= \mathbf{F}(z, \hat{\mathbf{d}})\\
0 &= \det(\mathbf{I} - \mathbf{F}_{\hat{\mathbf{d}}}(z, \mathbf{d}),
\end{aligned}$$

that is $r \approx 0.24835$. But d has the same singularities and of the same type as d, and thus admits a singular expansion of the form

$$d(z) = a_0 + a_1\sqrt{1 - z/r} + \mathcal{O}(1 - z/r) \quad \text{for} \quad a_0 \approx 0.11620 \quad \text{and } a_1 \approx -0.08227.$$

in a dented domain around r, which via the transfer theorem for singularity analyisis yields

$$[z^n]d(z) \sim a_1 \frac{r^{-n} n^{-3/2}}{\Gamma(\frac{-1}{2})} = a \cdot r^{-n} \cdot n^{-3/2} \quad \text{for } a \approx 0.02321.$$

The computation for a_1 is done as in the previous case, by substituting the cut singular expansions for all d_i in (3.10) and then solving the system for the unknown coefficients.

The rest of the analysis is done the same way as before, using the Taylor expansion for d, giving in the end:

$$g_n \sim \frac{g \cdot n^{-\frac{5}{2}} \cdot \rho^{-n} \cdot n!}{\Gamma(-\frac{3}{2})} = g' \cdot n^{-\frac{5}{2}} \cdot \rho^{-n} \cdot n!$$

for $\rho = \psi(\tau) \approx 0.17516$ and $g' = C_3 \cdot \exp(C_0) \cdot \Gamma(-\frac{3}{2})^{-1} \approx 0.01870$.

3.2.3 Synopsis

The type of asymptotic growth for all the cases mentioned in this chapter is of the form

$$g \cdot n^{-\frac{5}{2}} \cdot \rho^{-n} \cdot n!,$$

while for their 2-connected counterparts it is of the form $g_{2con} \cdot n^{-\frac{3}{2}} \cdot \rho_{2con}^{-n} \cdot n!$. We note that the reason for this *universal* law of $n^{-\frac{5}{2}}$ lies in the fact that B'' diverges on $d's$ singularity r, which in turn comes from the fact that B' admits a square-root type of singularity on r.

In short, if we could argue sufficiently for such a specification for any excluded $k-$cycle, then by theorem 6 we would have a square-root type of singularity, and thus this kind of law. In fact, for any *block-stable*[3] graph class we could argue for the same subexponential growth, if B'' diverges on B's singularity (see [7]).

The constants computed in this chapter are summarized in the table below:

Class	ρ	g	ρ_{2con}	g_{2con}
3-cycles	0.20836	0.01578	0.29336	0.02330
4-cycles	0.18919	0.01462	0.26488	0.02177
5-cycles	0.18054	0.01804	0.25383	0.02217
6-cycles	0.17516	0.01870	0.24835	0.02321

Table 3.2: A summary of the constants for both general and 2-connected restricted outerplanar graphs.

We note that in the Appendix one can find the first thirty terms of the countinig sequences for all the combinatorial classes studied in this chapter.

[3]A block is a maximal 2-connected component and a block-stable class $\mathcal{G}$ is a class that contains the edge-graph and $G \in \mathcal{G}$ if and only if each of its blocks belongs in $\mathcal{G}$.

Chapter 4

Statistics and limit laws

In this chapter we study parameters of general outerplanar graphs, namely the number of triangles and quadrangles. We will be able to extract gaussian limit laws for them with linear mean and variance

$$\mu \sim \alpha n, \quad \sigma^2 \sim \beta n,$$

as well as compute explicitely[1] the constants α and β. It is noted that recently, in [6], normality is established in a general scenario for a wide class of graphs including outerplanar graphs and their subgraph parameters, but unfortunately it does not give a way to compute the explicit constants.

4.1 Triangles and quadrangles in outerplanar graphs

In this chapter we will use directly dissections of D instead of $\hat{D}$, since the equations here have a less complex look.

4.1.1 The case of triangles

The analysis begins once more with the 2-connected components.
A bivariate specification for general dissections with u denoting the number of triangles is the following:

$$D = z^2 + u\frac{D^2}{z} + \frac{D^3}{z^2} + ... + \frac{D^r}{z^{r-1}} + ... = z^2 + u\frac{D^2}{z} + \frac{D^3}{z(z-D)}, \tag{4.1}$$

since it is enough to add the triangles of a dissection's components, and add one more when the base is a triangle. This is simplified to

$$-uD^3 + uD^2z - Dz^3 + z^4 + D^3 + D^2z - Dz^2 = 0$$

and the discriminant of the above polynomial $P(D, z, u)$ equals

$$\mathbf{R}\left(P, \frac{\partial P}{\partial D}, d\right) = -z^6(u-1)(4u^3z+8u^2z^2+4uz^3-8u^2z-44uz^2-4z^3-u^2+20uz+32z^2+2u+8z-5).$$

[1] up to five digits, but could be done in any degree of accuracy.

We will focus on the last term of the resultant, let us call it $res(u, z)$, in order to see how to singularities move when we change the parameter u. Setting $u = 1$, the possible roots for z are $\{3 - 2\sqrt{2}, 3 + 2\sqrt{2}\} \approx \{0.17157, 5.82842\}$ so the first one is the dominant singularity of $D(z, 1)$. We see that the derivative of res with respect to z does not vanish for $u = 1, z = 3 - 2\sqrt{2}$, hence the implicit function theorem applies. There is then a function $r(u)$ such that $z = r(u)$, analytic in a domain around $(1, 3 - 2\sqrt{2})$. In our case this function gives the dominant singularity for every fixed u. Via the Newton polygon method, the singularity is of a square root type for every fixed u, so D admits a singular expansion of the form

$$a_0(u) + \sum_{k\geq 1} a_k(u)\sqrt{1 - \frac{z}{r(u)}}^{k}$$

where $a_k(u)$ is a rational function in u for every k, analytic in a neighbourhood of $u = 1$. The above can be grouped as

$$\sum_{k\geq 0} a_{2k}(u)\Big(1 - \frac{z}{r(u)}\Big)^k + \sqrt{1 - \frac{z}{r(u)}}\sum_{k\geq 0} a_{2k+1}(u)\Big(1 - \frac{z}{r(u)}\Big)^k,$$

so in the end we can say that D admits a singular expansion of the form

$$A(z, u) + B(z, u)\sqrt{1 - \frac{z}{r(u)}} \tag{4.2}$$

where $A(z, u), B(z, u)$ are analytic in a region around $|z| < r$ and $|u - 1| < r'$ for positive r, r'. But now the first two conditions of the Quasi-Power theorem are fullfilled (Theorem 7), and we only have to compute the quantities:

$$m\Big(\frac{\rho(1)}{\rho(u)}\Big) = -\frac{r'(1)}{r(1)} \quad \text{and} \quad b\Big(\frac{\rho(1)}{\rho(u)}\Big) = -\frac{r''(1)}{r(1)} - \frac{r'(1)}{r(1)} + \Big(\frac{r'(1)}{r(1)}\Big)^2$$

so as to implement it. In order to compute $r'(1)$ we will set $z \equiv r(u)$ in res, differentiate, and then solve for $r'(1)$. The result is equal to $\sqrt{2} - 3/2 \approx -0.08578$. Differentiating again we can solve for $r''(u)$ and then $r''(1) \approx 0.06066$. The condition of variability is fullfiled, and thus a Gaussian limit law holds for our random variable, and the mean and variance are asymptotically linear in n. In particular,

$$\mu_n \sim m\Big(\frac{\rho(1)}{\rho(u)}\Big)n \text{ and } \sigma_n^2 \sim b\Big(\frac{\rho(1)}{\rho(u)}\Big)n.$$

with $m\Big(\frac{\rho(1)}{\rho(u)}\Big) \approx 0.39644$ and $b\Big(\frac{\rho(1)}{\rho(u)}\Big) \approx 0.50000$.

To pass to the connected and then general outerplanar graphs, the same three equations apply as in part 2, only now D has a parameter u:

$$B'(z, u) = \frac{\frac{1}{z}D(z, u) + z}{2}, \quad zC'(z, u) = z\exp(B'(zC'(z), u)) \quad \text{and} \quad G(z, u) = \exp(C(z, u)).$$

Now, B' has the same singularities as D, while C has the same as $F(z,u) \equiv zC'(z,u)$. But F is the inverse of $\Psi(z,u) = z\exp(-B'(z,u))$ so we have to check also for the zeroes of Ψ'. We have $\Psi'(z,u) = \exp(-B'(z,u))(1 - zB''(z,u))$ so it is enough to find the zeros of $(1 - zB''(z,u))$. But $B''(z,u)$ depends on D', which can be expressed in terms of D by differentiating its defining relation (4.1) and solving for D'. Setting $u = 1$ and solving the system we have a root $\tau(1) \approx 0.17076$, while $D(\tau(1),1) \approx 0.04709$. Differentiating $(1 - zB''(z,u))$ for z again and substituting the latter values, we see that it does not become zero, so by the implicit function theorem we can deduce that there is an analytic function $\tau(u)$ around $u = 1$ that corresponds to the zeros of $\Psi'_u(z)$, considering u as a parameter. Then, for each fixed u, the dominant singularity of F is $\rho(u) = \Psi(\tau(u))$. The same applies to C and then to G by the same reasoning as in section 2, while G inherits from F and C a singular expansion with critical exponent $a = -\frac{3}{2}$. To apply again the Quasi-Power theorem we need the values $\rho(1), \rho'(1), \rho''(1)$ and $\tau'(1)$. The first is already found. Since $\rho(u) = \Psi(\tau(u), u)$, for the other two we can say that

$$\rho'(u) = \frac{\partial \Psi}{\partial z}(\tau(u), u)\tau'(u) + \frac{\partial \Psi}{\partial u}(\tau(u), u) = \frac{\partial \Psi}{\partial u}(\tau(u), u),$$

and

$$\rho''(u) = \frac{\partial^2 \Psi}{\partial z \partial u}(\tau(u), u)\tau'(u) + \frac{\partial^2 \Psi}{\partial u^2}(\tau(u), u).$$

The computations can be done using

$$\frac{\partial}{\partial u} D(z,u) = -\frac{D^2(D-z)}{3\,uD^2 - 2\,Duz + z^3 - 3\,D^2 - 2\,Dz + z^2} \quad \text{and}$$

$$\frac{\partial}{\partial z} D(z,u) = \frac{uD^2 - 3\,Dz^2 + 4\,z^3 + D^2 - 2\,Dz}{3\,uD^2 - 2\,Duz + z^3 - 3\,D^2 - 2\,Dz + z^2}$$

With the above relations and $\tau(u)B''(u) = 1$ we can also find $\tau'(1)$. The final computations give $\rho'(1) \approx -0.05564$, $\rho''(1) \approx -37.94307$ and $\tau'(1) \approx -0.08493$. Then

$$\mu_n \sim m\Big(\frac{\rho(1)}{\rho(u)}\Big)n \text{ and } \sigma_n^2 \sim b\Big(\frac{\rho(1)}{\rho(u)}\Big)n$$

with $m\Big(\frac{\rho(1)}{\rho(u)}\Big) \approx 0.34793$ and $b\Big(\frac{\rho(1)}{\rho(u)}\Big) \approx 0.40737$.

4.1.2 The case of quadrangles

The specification here needs to augment the exponent of u each time a new quadrangle is formed. We have to take care of two cases: when an *empty* quadrangle is added and when a quadrangle is formed by two adjacent triangles. With this in mind, we now define D_2 to represent all the dissections without triangle base and D_3 to represent all dissections with triangle base. Of course then

$$D = D_2 + D_3.$$

Now, D_2 is formed by taking all kinds of bases except for triangular ones and attaching on them any object of D. In the meantime, when we take a quadrangle base, we have to raise the exponent of u by one. This corresponds to

$$D_2 = z^2 + u\frac{D^3}{z} + \frac{D^4}{z^3} + \frac{D^5}{z^4}... = z^2 + u\frac{D^3}{z} + \frac{D^4}{z^2(z-D)}.$$

On the other hand, D_3 is formed by attaching to a triangle base any object of D_2 or D_3 and raise the exponent of u when an object of D_3 is used. This leads to

$$D_3 = \frac{(D_2 + uD_3)^2}{z}.$$

As in the previous section, we use algebraic elimination and end up with an irreducible polynomial that defines $D(z,u)$ uniquely (look at the end of this section). Then, working like in the previous case, we can say there is an analytic function $r(u)$ near $u = 1$ which, for each fixed u, gives the unique dominant singularity of $D(z,u)$. Then, as before, a singular expansion of D is granted, of the form

$$A(z,u) + B(z,u)\sqrt{1 - \frac{z}{r(u)}}, \tag{4.3}$$

where $A(z,u), B(z,u)$ are analytic in a domain around $|z| < r$ and $|u - 1| < r'$ for positive r, r'. The Quasi-Powers theorem can then be applied and we can argue that the number of quadrangles in biconnected outerplanar graphs is asymptotically normal, with linear mean and variance. The computations give

$$\mu_n \sim m\Big(\frac{\rho(1)}{\rho(u)}\Big)n \text{ and } \sigma_n^2 \sim b\Big(\frac{\rho(1)}{\rho(u)}\Big)n$$

with $m\Big(\frac{\rho(1)}{\rho(u)}\Big) \approx 0.43933$ and $b\Big(\frac{\rho(1)}{\rho(u)}\Big) \approx 0.44710$.

For the general labelled outerplanar case, we work again as before and establish the existence of an analytic function $\rho(u)$ around $u = 1$, which gives the unique dominant sigularity for fixed u and helps us deduce a singular expansion for F (note that for $u = 1$ the τ is the same as in the previous case, since $D(z,1)$ is the same function). Then C and G have the same singularities and inherit from F through integration a singular expansion with critical exponent $\frac{3}{2}$. This allows us to implement again the Quasi-Power theorem, after computing the quantities $m\Big(\frac{\rho(1)}{\rho(u)}\Big)$ and $b\Big(\frac{\rho(1)}{\rho(u)}\Big)$. Thus, we can argue that the number of quadrangles X_n is an asymptotically normal random variable, with linear mean and variance:

$$\mu_n \sim m\Big(\frac{\rho(1)}{\rho(u)}\Big)n \text{ and } \sigma_n^2 \sim b\Big(\frac{\rho(1)}{\rho(u)}\Big)n$$

for $m\Big(\frac{\rho(1)}{\rho(u)}\Big) \approx 0.33705$ and $b\Big(\frac{\rho(1)}{\rho(u)}\Big) \approx 0.36145$.

In order to give a feeling of the form and size such defining equations have, we state below the defining equation for $D(z,u)$ and highlight that it constitutes an irreducible polynomial in u, z, x_1 that, when setting $u = 1$, gives a multiple by some power of z of the irreducible polynomial derived in (2.3):

$$u^4z^2{x_1}^6 - 2\,u^4z{x_1}^7 + u^4{x_1}^8 + 2\,u^3z^6{x_1}^3 - 4\,u^3z^5{x_1}^4 + 2\,u^3z^4{x_1}^5+$$
$$u^2z^{10} - 2\,u^2z^9x_1 + u^2z^8{x_1}^2 - 2\,u^3z^4{x_1}^4 + 4\,u^3z^3{x_1}^5 - 4\,u^3z^2{x_1}^6 + 6\,u^3z{x_1}^7 - 4\,u^3{x_1}^8-$$
$$2\,u^2z^8x_1 + 4\,u^2z^7{x_1}^2 - 6\,u^2z^6{x_1}^3 + 10\,u^2z^5{x_1}^4 - 6\,u^2z^4{x_1}^5 - 2\,uz^{10}$$
$$+4\,uz^9x_1 - 2\,uz^8{x_1}^2 + u^2z^6{x_1}^2 - 2\,u^2z^5{x_1}^3 + 3\,u^2z^4{x_1}^4 - 6\,u^2z^3{x_1}^5+$$
$$5\,u^2z^2{x_1}^6 - 6\,u^2z{x_1}^7 + 6\,u^2{x_1}^8 + 2\,uz^8x_1 - 4\,uz^7{x_1}^2 + 4\,uz^6{x_1}^3 - 8\,uz^5{x_1}^4 + 6\,uz^4{x_1}^5$$
$$+z^{10} - 2\,z^9x_1 + z^8{x_1}^2 + uz^5{x_1}^3 - 2\,uz^4{x_1}^4 + 3\,uz^3{x_1}^5 - 2\,uz^2{x_1}^6+$$
$$2\,uz{x_1}^7 - 4\,u{x_1}^8 + z^9 - 2\,z^8x_1 + z^7{x_1}^2 + 2\,z^5{x_1}^4 - 2\,z^4{x_1}^5 - z^7x_1+$$
$$2\,z^6{x_1}^2 - z^5{x_1}^3 + z^4{x_1}^4 - z^3{x_1}^5 + {x_1}^8$$

4.1.3 Synopsis

The number of triangles and quadrangles in general outerplanar graphs is, as expected from [6], asymptotically normal with linear mean and variance

$$\mu \sim \alpha n, \quad \sigma^2 \sim \beta n.$$

The constants α and β are computed explicitely with a $5-$digit accuracy, and given in the table 4.1 below. We have to highlight that the computations performed can be done in any degree of accuracy, since all the constants are introduced by explicit analytic expressions.

Parameter	α	β
triangles	0.34793	0.40737
quadrangles	0.33705	0.36145

Table 4.1: The constants for the mean and variance.

One must also note that although the normality of the limiting distribution is expected from [6], there is not a general way to compute the explicit constants for the mean and variance. In fact, it is not the case that the Quasi-Power theorem always applies: an interesting case is when the parameter is not additive in terms of the objects components (you can see an example for the case of P_2[2] in [6]).

[2] P_2 denotes a path of length two.

Chapter 5

Conclusions

In this thesis we explored the nature of the counting formulas for outerplanar graphs under cycle constraints and obtained exact asymptotic expansions for the cases of excluded triangles, quadrangles, pentagons, and hexagons (see Table 3.2). The asymptotics for all these cases were of the form

$$g \cdot n^{-\frac{5}{2}} \cdot \rho^{-n} \cdot n!,$$

which lies crucially to the algebraicity of the 2-connected components, as well as their concrete Puiseux expansions. The algebraicity offers a square-root type of singularity to the derivative of the 2-connected components' generating function B that, in turn, leads B'' to diverge there, giving the particular subexponential growth $n^{-\frac{5}{2}}$. This has been explored extensively in [7] for general block-stable graph classes. In our context, this means that if a similar representation was well established for general $k-$cycle exclusions, then we would provably be able to deduce the universality of such a law in our constrained classes.

Another thing that would be an advance is finding systematic ways to compute the necessary constants τ and $B(\tau)$ when there is no access at all to an explicit representation of d. Moreover, maybe a systematic way to create all these specifications could be found and then an efficient mechanical procedure to produce their asymptotics. This would certainly come with a big advance in understanding the problem exposed here, and would encompass both combinatorial and computational leaps.

In this thesis we also found limiting distributions for the parameters of 3 and 4-cycles in general outerplanar graphs and computed explicitely the relevant constants (see Table 4.1). It was mentioned that these limit laws are expected to be normal from the general result of [6], which nevertheless provides no way to obtain the constants for the mean and variance. In fact, what we explored here is the computational aspect of this result in the case of our parameters. However, it's certainly not the case that the Quasi-Power theorem, which we used, is the rule when it comes to studying parameters in this general so-called *subcritical* class of graphs. We note that a crucial condition for this to work is the 2-connectivity of the subgraph that we have chosen as a parameter. Else, if the chosen subgraph has cut-vertices, then we must have in mind not only what happens inside the 2-connected components, but also what happens when they are attached on a cut vertex. The parameter thus ceases to be additive and the classical translation schemas do not apply. This was illlustrated as a case study in [6] for the number of 2-paths in series-parallel graphs, where an infinite system of generating functions was used, with an infinite number of variables in order to encompass all the different cases. This could be endeavoured for outerplanar graphs as well.

Appendix

The counting sequences of the restricted outerplanar graphs				
n	3-cycles excluded	4-cycles excluded	5-cycles excluded	6-cycles excluded
2	1	1	1	1
3	0	1	1	1
4	1	0	3	3
5	1	1	0	11
6	4	7	4	0
7	8	22	8	15
8	25	49	65	37
9	64	130	229	85
10	191	468	946	651
11	540	1651	2850	2498
12	1616	5240	9367	10556
13	4785	16485	28068	46112
14	14512	55184	97408	167100
15	44084	190724	339694	621677
16	135545	652359	1276467	2215039
17	418609	2213044	4659990	7524303
18	1302340	7584939	17107629	26414280
19	4070124	26346522	61200635	92579458
20	12785859	91951596	220323189	332018450
21	40325828	321079035	792549890	1236600966
22	127689288	1124304217	2894544436	4661052146
23	405689020	3956244997	10636412377	17856973980
24	1293060464	13976729976	39402675095	68811536633
25	4133173256	49496496226	146035611491	264020825996
26	13246527139	175658247703	542050702586	1010795150433
27	42557271268	624958280698	2011105221340	3849919742470
28	137032656700	2229032360888	7476068631638	14606666124827
29	442158893833	7967018628527	27845071637132	55392530805786
30	1429468244788	28527676814989	104020601265688	210180466079635
31	4629713966452	102329475730993	389476031130949	799736665155904
32	15019870618329	367690587142491	1461344460329491	3054909064626511

Table 6.1: The first thirty terms of the counting sequences appearing in Chapter 3.

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