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End Extensions of Models of Weak Arithmetics

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ΔΙΔΑΚΤΟΡΙΚΗ ΔΙΑΤΡΙΒΗ

Τελικές Επεκτάσεις Μοντέλων Υποσυστημάτων της Αριθμητικής

Βασίλειος Σ. Πασχάλης

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Abstract

The subject of the Ph.D Thesis is the study of problems concerning end extensions of models of subsystems of first-order Peano arithmetic (PA) in the first order language of arithmetic $\mathcal{L}_A$. More specifically the problem first posed by J. Paris, *Is every model of* Σ_1*-Collection a proper initial segment of a model of bounded induction?* remains unanswered.

This problem was stated in an effort to miniaturize the famous McDowell-Specker Theorem that every model of PA has a proper elementary end extension. The main problem was studied by J. Paris and A. Wilkie who showed that a sufficient condition for a positive answer is that the model is $I\Delta_0$-full (where $I\Delta_0$ denotes the theory of Δ_0-induction).

We show that the notion of $I\Delta_0$-fullness can be by-passed by alternative proofs to these results which employ the classical argument of the Completeness theorem in its arithmetised form (Hilbert-Bernays) together with consistency statements referring to semantic tableaux methods.

Furthermore, using the same methodology suitably modified we prove the generalisation of the result, namely that every countable model of Σ_n-Collection, $n > 1$, has a proper Σ_n-elementary end extension to a model of bounded induction.

SUBJECT AREA: 03F30 First-order arithmetic and fragments
03H15 Nonstandard models of arithmetic
03C62 Models of arithmetic and set theory

KEY WORDS: Arithmetized completeness theorem, Fragments of Peano Arithmetic, End extensions, Elimination lemma, Bounded Induction

Περίληψη

Η διδακτορική διατριβή ασχολείται με τη μελέτη προβλημάτων που αφορούν τελικές επεκτάσεις μοντέλων υποσυστημάτων της πρωτοβάθμιας αριθμητικής Peano. Πιο συγκεκριμένα, το πρόβλημα του J. Paris: «*Υπάρχει, για κάθε αριθμήσιμο μοντέλο της Σ_1 συλλογής γνήσια τελική επέκτασή του που ικανοποιεί την Δ_0 επαγωγή;*» παραμένει ανοικτό.

Το πρόβλημα μελέτησαν οι J. Paris και A. Wilkie (1989), οι οποίοι απέδειξαν ότι ικανή συνθήκη για θετική απάντηση είναι το μοντέλο να είναι $I\Delta_0$-πλήρες (όπου με $I\Delta_0$ συμβολίζεται η θεωρία της Δ_0-επαγωγής).

Αποδεικνύουμε ότι η χρήση της έννοιας της $I\Delta_0$-πληρότητας μπορεί να παρακαμφθεί και στη θέση της να χρησιμοποιηθεί η τυποποίηση του κλασικού επιχειρήματος του θεωρήματος πληρότητας (θεώρημα Hilbert-Bernays), με χρήση σημασιολογικών πινάκων (semantic tableaux).

Επιπλέον, με την ίδια μεθοδολογία κατάλληλα τροποποιημένη αποδεικνύουμε τη γενίκευση του αποτελέσματος, δηλαδή ότι για κάθε αριθμήσιμο μοντέλο της Σ_n-συλλογής, $n > 1$, υπάρχει γνήσια Σ_n-στοιχειώδης τελική επέκτασή του που ικανοποιεί την Δ_0-επαγωγή.

ΘΕΜΑΤΙΚΗ ΠΕΡΙΟΧΗ: 03F30 Πρωτοβάθμια αριθμητική και υποσυστήματα
03H15 Μη συμβατικά (Nonstandard) μοντέλα της αριθμητικής
03C62 Μοντέλα της αριθμητικής και της θεωρίας συνόλων

ΛΕΞΕΙΣ ΚΛΕΙΔΙΑ: Αριθμητικοποιημένο θεώρημα πληρότητας, Υποσυστήματα αριθμητικής Peano, Τελικές επεκτάσεις, Λήμμα απαλοιφής, Φραγμένη επαγωγή

To my Parents

To my Brother

To Marika

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Contents

Introduction

The famous McDowell-Specker Theorem states that every model of Peano arithmetic (PA) has a proper elementary end extension. Questions related to complexity theory, existence of solutions to diophantine equations and the MRDP theorem led to efforts to miniaturise the problems for subsystems of first-order Peano arithmetic. The motivation for this thesis and one of the main questions in the area is a problem first posed by J. Paris, namely

> *Is every model of Σ_1-Collection a proper initial segment of a model of bounded induction?*

The main problem remains unanswered. However, some partial results have been obtained by J. Paris and A. Wilkie who studied the problem in a classical paper (1989). We give alternative proofs to these results which employ the classical argument of the Completeness theorem in its arithmetised form (Hilbert-Bernays) together with consistency statements referring to semantic tableaux methods.

The first chapter is dedicated to the basic notions needed for the rest of this work. So we start with the necessary notions that enable us to define the arithmetical hierarchy and the induction and collection schemes, which makes it possible to draw the picture of the connections between subsystems of PA. Next, we describe briefly the arithmetization of syntax in order to formalize the notions of semantic tableaux proofs and tableaux consistency arguments. We then define satisfaction, another notion fundamental for the sequel. The chapter is concluded with a brief overview of the Arithmetized Completeness Theorem (ACT). As a rule in this chapter, we will try to avoid details and give suitable references instead.

In the second chapter, we study a significant tool, tableaux proofs, that will assist us to prove the main results. Since we work at a very low level of the arithmetical hierarchy, a lot of work needs to be done in order to show that the formalization of tableaux proofs is available. Therefore, one goal of this chapter is to obtain the necessary formal statements and bounds. A second goal is to calculate the complexity of the so-called Elimination Lemma. Armed with the Elimination Lemma and formal consistency statements we can set off for modifications of the ACT suitable for weak theories.

Chapter three is the heart of this thesis. The main results and methods are presented in this Chapter. We begin with an alternative proof of a result by J. Paris and A. Wilkie, which states that every countable model of Σ_1-collection with exponentiation can be properly end extended to a model of bounded induction. This is an instance of the main problem and the "closest" we can get to it thus far. The idea behind the proof is that given a model of Σ_1-collection, one can start with a tableau consistent extension of the theory of bounded induction as a base theory. This theory includes the diagram of the initial model and a set of sentences that guarantee that its models are different from the initial one. We can then proceed to construct a maximal tableau consistent extension in a way that the initial segment property is preserved. This is possible by a modification of the ACT. The same method is used to show that there is always a proper end extension of a countable model which satisfies Σ_1-collection together with conditions other than exponentiation. Generally, these conditions include a notion of (weak) recursive saturation or the existence of specific powers in the presence of a very strong condition, namely that the Arithmetic hierarchy provably collapses in $I\Delta_0$.

In the fourth and final, chapter we examine the generalisation of the problem, namely for $n > 1$ is every countable model of Σ_n-collection, properly and Σ_n-elementarily end extendable to a model of bounded induction? The problem was first studied by J. Paris and L. Kirby who, used a restricted ultrapower construction in order to obtain a proper Σ_n-elementary end extension. By adjusting the methods of chapter 3 in the new context where superexponantiation is available, we obtain an alternative proof for for $n = 2$ and go on to generalise the result for all $n \geq 2$.

The main problem that motivated this work remains unanswered but we feel that our results provide a uniform and in a sense simpler method to study instances of the main problem and, hence, contribute to the better understanding of the main problem itself.

1 Basics

1.1 Preliminaries

Throughout this thesis, $\mathcal{LA}$ denotes the *first-order language of arithmetic* whose nonlogical symbols consist of the following: the constant symbols, 0 and 1; the binary relation symbol, $<$; and the two binary function symbols, $+$ and $\cdot$. The standard model for the language $\mathcal{LA}$ will be denoted by $\mathbb{N}$.

1.1.1 Axioms of Peano Arithmetic

The base theory satisfied in models of Arithmetic is denoted by PA^- and it consists of some simple axioms that are obviously true in every model of Arithmetic. If we add the axiom schema of induction to the base theory, we obtain Peano Arithmetic, which is denoted by PA. We work with weaker theories than Peano Arithmetic. These theories are obtained by adding to the base theory weak induction axioms. This area of model theory is better known as *Subsystems of Peano Arithmetic* or *Weak Arithmetics*.

The first five axioms of PA^- state that the binary functions $+$ and $\cdot$ of $\mathcal{LA}$ are associative and commutative, and satisfy the distributive law:

Ax_1 $\forall x, y, z((x + y) + z = (x + (y + z))$,

Ax_2 $\forall x, y(x + y = y + x)$,

Ax_3 $\forall x, y, z((x \cdot y) \cdot z = (x \cdot (y \cdot z))$,

Ax_4 $\forall x, y(x \cdot y = y \cdot x)$ and

Ax_5 $\forall x, y, z((x + y) \cdot z = x \cdot z + y \cdot z)$.

The next two axioms state that the constant symbols 0 and 1 of $\mathcal{LA}$ are the identity of $+$ and $\cdot$ respectively:

Ax_6 $\forall x[(x + 0 = x) \wedge (x \cdot 0 = 0)]$ and

Ax_7 $\forall x(x \cdot 1 = 1)$.

$\mathbb{N}$ is linearly ordered and the following axioms state that the 2-placed relation symbol $<$ of $\mathcal{LA}$ is a linear order:

Ax_8 $\forall x, y, z(((x < y) \wedge (y < z)) \to (x < z))$,

Ax_9 $\forall x(\neg(x < x))$ and

Ax_{10} $\forall x, y((x < y) \vee (x = y) \vee (y < x))$.

The following three axioms state the relation between the function symbols and the relation symbol of $\mathcal{LA}$:

Ax_{11} $\forall x, y, z((x < y) \to (x + z < y + z))$,

Ax_{12} $\forall x, y((0 < z) \wedge ((x < y) \to (x \cdot z < y \cdot z)))$ and

Ax_{13} $\forall x, y((x < y) \to (\exists z(x + z = y)))$.

Hence, $+$ and $\cdot$ respect $<$, and we can subtract x from y, if $x < y$.

The last two axioms of the base theory state the order is discrete and that 0 is the least natural number:

Ax_{14} $0 < 1 \wedge \forall x(x > 0 \to x \geq 1)$ and

Ax_{15} $\forall x(x \geq 0)$.

1.1.2 Induction, collection and exponentiation

Our aim in this section is to define the *Induction schema*, some of its alternatives, like the *Least number principle* and the *Collection axioms*, the axiom of the totality of the exponential function denoted by exp and some weaker forms of it denoted by Ω_r.

PA is the first-order theory we get when we add to the our base theory, PA^-, the induction axiom for all formulas of $\mathcal{LA}$. More precisely, let Γ be a class of formulas of $\mathcal{LA}$ we denote by $I\Gamma$ the class of first-order formulas of the form:

$$\forall \bar{y}(\phi(0, \bar{y}) \wedge \forall x(\phi(x, \bar{y}) \to \phi(x + 1, \bar{y})) \to \forall x \phi(x, \bar{y}))$$

for all $\phi \in \Gamma$.

Throughout the history of mathematics, induction has taken many forms which were later proven to be equivalent. These equivalent forms are usually called *alternative induction schemes*. The first alternative induction scheme is usually called *induction up to z* and it is the scheme:

$$\forall \bar{y}, z(\phi(0, \bar{y}) \wedge \forall x < z(\phi(x, \bar{y}) \to \phi(x + 1, \bar{y})) \to \forall x \leq z(\phi(x, \bar{y})))$$

for all $\phi \in \Gamma$.

The *Least number principle* is, perhaps, the most commonly used alternative to the induction scheme. It is the class of all sentences of the form:

$$\forall \bar{y}(\exists x \phi(x, \bar{y}) \to \exists z(\phi(z, \bar{y}) \wedge \forall w < z \neg \phi(w, \bar{y})))$$

for all $\phi \in \Gamma$ and it is usually denoted by $L\Gamma$ where Γ is a class of $\mathcal{LA}$-formulas.

Another alternative induction schema is the *principle of complete induction.* Denoted by $T\Gamma$, for a class of $\mathcal{LA}$-formulas Γ, the principle of complete induction is the class of sentences:

$$\forall \bar{y}(\forall x(\forall z < x \phi(z, \bar{y}) \to \phi(x, \bar{y})) \to \forall x \phi(x, \bar{y}))$$

for all $\phi \in \Gamma$.

All the induction schemes defined so far are proven to be equal in the presence of PA^-, for a proof see Chapter 4 in [9].

The arithmetic hierarchy and induction

Gödel's *First Incompleteness theorem* made evident that induction for all $\mathcal{LA}$ formulas was too much to ask for. So questions were raised about the strength of induction on classes of $\mathcal{LA}$-formulas whose complexity is restricted. This leads us to define the *arithmetic hierarchy* which is a hierarchy of formula classes. The complexity measure used in the arithmetic hierarchy is the number of alternations of existential and universal quantifiers.

The definition of the bounded quantifiers is a prerequisite in order to define the base of the hierarchy.

Definition 1. If t is an $\mathcal{LA}$-term then $\forall x < t\phi$ is an abbreviation for $\forall x(x < t \rightarrow \phi)$ and $\exists x < t\phi$ is an abbreviation for $\exists x(x < t \wedge \phi)$. Similarly, $\forall x \leq t\phi$ and $\exists x \leq t\phi$ are shorthand for $\forall x(x \leq t \rightarrow \phi)$ and $\exists x(x \leq t \wedge \phi)$ respectively. These quantifiers are said to be *bounded.*

The base class of the hierarchy consists of all $\mathcal{LA}$-formulas defined in the next definition.

Definition 2. An $\mathcal{LA}$ formula ϕ is Δ_0 iff all quantifiers in ϕ are bounded.

The class Δ_0 of $\mathcal{LA}$ formulas is also denoted by Σ_0 and Π_0 in order to recursively define the *arithmetic hierarchy* of classes of $\mathcal{LA}$ formulas. So the classes Σ_n and Π_n are defined by recursion on $n \in \mathbb{N}$ in the next definition.

Definition 3. Let ϕ be an $\mathcal{LA}$ formula. The formula ϕ is Σ_{n+1} iff it is of the form $\exists x\phi$ with $\phi \in \Pi_n$. The formula ϕ is Π_{n+1} iff it is of the form $\forall x\phi$ with $\phi \in \Sigma_n$.

We say that the formula $\theta(\bar{x})$ is equivalent to a Σ_n formula $\phi(\bar{x})$ in the theory T or the model M if

$$T \vdash \forall\bar{x}(\theta(\bar{x}) \leftrightarrow \phi(\bar{x})) \quad \text{or} \quad M \models \forall\bar{x}(\theta(\bar{x}) \leftrightarrow \phi(\bar{x}))$$

and, if necessary, we write that $\theta(\bar{x}) \in \Sigma_n(T)$ or $\theta(\bar{x}) \in \Sigma_n(M)$. The class of $\Pi_n(T)$ and $\Pi_n(M)$ formulas is defined similarly. Finally, the formula ϕ is $\Delta_n(T)$ (respectively $\Delta_n(M)$) iff it is equivalent to both a $\Sigma_n(T)$ (respectively $\Sigma_n(M)$) formula and a $\Pi_n(T)$ (respectively $\Pi_n(M)$) formula. In the previous notation the theory T and the model M will be omitted when they are clear from the context.

We are now able to define weaker classes of induction axioms. Let T be a theory T and M a model of a theory. According to the definition of the induction schema we denote by $I\Sigma_n$, $I\Pi_n$ and $I\Delta_n$ the class of all sentences of the form:

$$\forall\bar{y}(\phi(0,\bar{y}) \wedge \forall x(\phi(x,\bar{y}) \rightarrow \phi(x+1,\bar{y})) \rightarrow \forall x\phi(x,\bar{y}))$$

where $\phi \in \Sigma_n$ (or $\phi \in \Sigma_n(T)$ or $\phi \in \Sigma_n(M)$), $\phi \in \Pi_n$ ($\phi \in \Pi_n(T)$, $\phi \in \Pi_n(M)$) and $\phi \in \Delta_n(T)$ (or $\phi \in \Delta_n(M)$) respectively. The alternative induction schemata for the restricted formula classes can be defined similarly.

Collection

The *collection scheme* is the class of sentences:

$$\forall \bar{z} \forall a (\forall x < a \exists y \phi(x, y, \bar{z}) \to \exists t \forall x < a \exists y < t \phi(x, y, \bar{z}))$$

for all formulas ϕ in Γ, and it is denoted by $B\Gamma$ where Γ is a class of $\mathcal{LA}$-formulas. The restricted collection scheme can be defined as before, e.g. $B\Sigma_n$ is the class of sentences of the form:

$$\forall \bar{z} \forall a (\forall x < a \exists y \phi(x, y, \bar{z}) \to \exists t \forall x < a \exists y < t \phi(x, y, \bar{z}))$$

where $\phi \in \Sigma_n$. We also denote by $B\Sigma_n$ the theory with axioms PA^-; induction for Δ_0 formulas; and collection for Σ_n formulas.

The unrestricted collection scheme is equivalent to the induction schemata over the weak theory $PA^- + I\Delta_0$, for a proof see Chapter 7 in [9]. At this point, a natural question to ask is how are the collection scheme and the classical induction schemata related when we restrict the classes of formulas to which they are applied to. The relation between collection subsystems and the traditional induction subsystems of Peano Arithmetic was proved in [15] and it is as follows.

Theorem 1. *Let $n \geq 0$. The following implications hold in the presence of $PA^- + I\Delta_0$:*

$$\begin{array}{ccccccc} I\Sigma_{n+1} & & & & & & \\ \Downarrow & & & & & & \\ B\Sigma_{n+1} & \Longleftrightarrow & B\Pi_n & & & & \\ \Downarrow & & & & & & \\ I\Sigma_n & \Longleftrightarrow & I\Pi_n & \Longleftrightarrow & L\Sigma_n & \Longleftrightarrow & L\Pi_n \end{array}$$

However, the converses to the two vertical arrows are false.

Exponentiation

We conclude this section with the definition of some exponentiation axioms that will be used later on. We denote by exp the axiom expressing that exponentiation is total, i.e.

$$\forall x, y \exists z (z = x^y).$$

Recall that there is a Δ_0 formula representing the graph of the function 2^x for details see chapter 2 of [6] or the exercises of chapter 5 in [9]. Hence, $\exp \in \Pi_2$.

As with induction, there are also restricted versions of the exponentiation axiom. An example of a weaker exponential is Ω_1 expressing that the function $x^{|x|}$ is total, where

,for now, $|x|$ denotes the logarithm of x. To define the axioms denoted by Ω_n we first need the following definition.

Definition 4. For $n \in \omega$, the $n+1$-place function e_n is defined as follows:

$$e_0(x_1) = x_1$$
$$e_{n+1}(x_1, \ldots, x_{n+2}) = x_1^{e_n(x_2,\ldots,x_{n+2})}.$$

Thus

$$e_n(x_1, \ldots, x_{n+1}) = x_1^{x_2^{\cdot^{\cdot^{\cdot^{x_{n+1}}}}}}.$$

Also, set

$$\omega_n(x) = e_n(x, |x|, ||x||, \ldots, |x|^{(n)})$$

where $|x|^{(n)}$ denotes the result of applying the length function n times to x.

The graph of the function ω_n can be represented by a Δ_0 formula, but the axiom Ω_n, expressing that ω_n is total, is Π_2.

Finally, the superexponential function is defined as follows.

Definition 5. For all $x, y \in \mathbb{N}$ the *superexponential* function, denoted by supexp, is defined by the following recursion:

$$\text{supexp}(x, 0) = x$$
$$\text{supexp}(x, y+1) = x^{\text{supexp}(x,y)}.$$

It easy to show that the graph of supexp can be expressed by a Δ_0 formula and whenever the function is defined we can prove, in the presence of $I\Delta_0$ that the recursive equations hold. However, the formula expressing the totality of the supexp function is Π_2 and so;

$$I\Sigma_1 \vdash \forall x \forall y \exists z (z = \text{supexp}(x, y)). \tag{1.1}$$

1.1.3 Model theory concepts

If M, N are models for the same first-order language $\mathcal{L}$, Then M is a *submodel* of N (or a substructure of N), $M \subseteq N$, iff the domain of M is a subset of the domain of N containing the constants of N and closed under the functions of N, and each relation symbol in $\mathcal{L}$ is interpreted in M according to the restriction of its interpretation in N.

Definition 6. M is an *elementary submodel* of N, $M \prec N$, iff $M \subseteq N$, and for each formula $\phi(\bar{x})$ and each $\bar{a} \in M$

$$M \models \phi(\bar{a}) \iff N \models \phi(\bar{a}).$$

Definition 7. Let Γ be a class of $\mathcal{LA}$-formulas. M is an Γ-*elementary submodel* of N, $M \prec_\Gamma N$, iff $M \subseteq N$, and for each Γ formula $\phi(\bar{x})$ and each $\bar{a} \in M$

$$M \models \phi(\bar{a}) \iff N \models \phi(\bar{a}).$$

Definition 8. if M and N are $\mathcal{LA}$-structures with N a substructure of M, then N *is an initial segment of* M, or M *is an end-extension of* N, or (in symbols) $N \subseteq_e M$ iff for all $x \in N$ and for all $y \in M$,

$$M \models y < x \Rightarrow y \in N.$$

N is a proper initial segment if, in addition, $N \neq M$.

Theorem 2. *Let* $M \subseteq_e N$ *both be* $\mathcal{LA}$*-structures, with* N *an end-extension of* M*. Then* $M \prec_{\Delta_0} N$*.*

Notice also that for all n

$$M \prec_{\Sigma_n} N \iff M \prec_{\Pi_n} N$$

therefore, we will write $M \prec_n N$ when $\Gamma = \Sigma_n$ or $\Gamma = \Pi_n$.

1.2 Arithmetization of syntax

PA is such strong a system that it can code many of its aspects. In this section, we will define formulas that express syntactical notions of the language. We will omit the most common or obvious definitions for the sake of clarity. Since our resources will be restricted, we will also note, when important, the complexity of the formula defined.

Coding function

There are many functions that have been proposed for coding. We will briefly introduce the coding function used in [17].

Let M be a model of bounded induction i.e. $M \models I\Delta_0$. We can Gödel number the basic logical symbols of the language using the alphabet $\{3, 4, \ldots, B\}$ as follows:

(	)	$\neg$	$\rightarrow$	$\forall$	v	$_1$	$'$	$+$	$\cdot$	$\leq$	0	$=$
3	4	5	6	7	8	9	$\overline{10}$	$\overline{11}$	$\overline{12}$	$\overline{13}$	$\overline{14}$	$\overline{15}$

Then the Gödel number of a formula or a term will be its natural B-adic code. To see an example let w be any non-zero element of M. Then

$$M \models w = m_0 B^0 + m_1 B^1 + \cdots + m_t B^t$$

where for $i = 0, 1, \ldots, t$ $t \in M$ and $m_i \in M$ are unique and such that $M \models 1 \leq m_i \leq B$.

We will write $(w)_i = x$ if x is the i-th element of the sequence with code w. The *length* of the empty word is 0 and the length of every non-zero $w \in M$, denoted by $|w|$, is equal to $t \in M$ if and only if

$$w = m_0 B^0 + m_1 B^1 + \cdots + m_t B^t$$

where $m_i \in M$ and $M \models 1 \leq m_i \leq B$ for $i = 0, 1, \ldots, t$. Finally, we write $x * y = z$ when z is the code of the sequence deriving from the *concatenation* of the word with code x with the word with code y. Note that there are Δ_0 formulas that express syntactically the three functions that were presented above.

From now on we will not make a distinction between an expression of $\mathcal{LA}$ and its code when it is clear from the context to which of both we are referring to. For instance, suppose t is an $\mathcal{LA}$-term, we will often write "the term t" instead of the correct "the term with Gödel number t".

Formalization of basic functions and relations

Let $\mathcal{L}$ be a language extending $\mathcal{LA}$. The following functions and relations are recursive in every model of $I\Delta_0 + \omega_1$ for $\mathcal{L}$:

Var(x) x is the Gödel number of a variable,

EX(x) x is the Gödel number of an expression,

MP(x, y, z) z is the Gödel number of an expression derived from the expressions with Gödel numbers x and y by the use of the Modus Ponens rule,

Term(x) x is the Gödel number of a term of $\mathcal{L}$,

ATF(x) x is the Gödel number of an atomic formula of $\mathcal{L}$,

Form(x) x is the Gödel number of a formula of $\mathcal{L}$,

SUB(x, y, u, v) x is the Gödel number of the expression that results from the substitution of the term with Gödel number v for all free occurrences of the variable with Gödel number u in the expression with Gödel number y,

LA(x) x is the Gödel number of a logical axiom,

LEAxiom(x) x is the Gödel number of a logical equality axiom,

PA$^-(x)$ x is the Gödel number of an axiom of PA$^-$,

Proof$_T(x)$ x is the Gödel number of a formal proof from a recursive set of sentences T and

Proof$_T(x, y)$ x is the Gödel number of a formal proof, from a recursive set of sentences T, ending with a formula with Gödel number y.

Definition 9. Let $\mathcal{L}$ be a language extending $\mathcal{LA}$ and $k \in \mathbb{N}$. A formula θ of a language $\mathcal{L}$ is said to be a k-*formula* iff there is an $\mathcal{L}$ formula ϕ with Gödel number less than k and θ is obtained from ϕ by substituting some or all of its free variables with terms of $\mathcal{L}$.

For this restricted kind of formulas, the following functions and relations are also recursive:

$\mathrm{ReForm}(k, x)$ x is the Gödel number of a k-formula of $\mathcal{L}$,

$\mathrm{ReProof}_T(k, x)$ x is the Gödel number of a formal proof from a recursive set of k-sentences T and

$\mathrm{ReProof}_T(k, x, y)$ x is the Gödel number of a formal proof, from a recursive set of k-sentences T, ending with a formula with Gödel number y.

Definition 10. We say that the variable x *occurs* in the $\mathcal{L}$ term t if and only if:

- $t \equiv x$ or
- $t \equiv s'$, where s is an $\mathcal{L}$ term and x occurs in s or
- $t \equiv s_1 + s_2$, where s_1 and s_2 are $\mathcal{L}$ terms and x occurs in either s_1 or s_2 or
- $t \equiv s_1 \cdot s_2$, where s_1 and s_2 are $\mathcal{L}$ terms and x occurs in either s_1 or s_2.

$$\begin{aligned}
\mathrm{TOccur}(x, t) \iff\ & (\mathrm{Var}(x) \land \mathrm{Var}(t) \land x = t) \lor \\
& \exists s < t(\mathrm{Term}(s) \land \mathrm{TOccur}(x, s) \land t = s * \ulcorner ' \urcorner) \lor \\
& \exists t_1, t_2 < t(\mathrm{Term}(t_1) \land \mathrm{Term}(t_2) \land \\
& (\mathrm{TOccur}(x, t_1) \lor \mathrm{TOccur}(x, t_2)) \land t = t_1 * \ulcorner + \urcorner * t_2) \lor \\
& \exists t_1, t_2 < t(\mathrm{Term}(t_1) \land \mathrm{Term}(t_2) \land \\
& (\mathrm{TOccur}(x, t_1) \lor \mathrm{TOccur}(x, t_2)) \land t = t_1 * \ulcorner \cdot \urcorner * t_2).
\end{aligned}$$

Definition 11. We say that the variable x *occurs* in the $\mathcal{L}$ formula ϕ if and only if:

- $\phi \equiv t_1 = t_2$ and x occurs in either t_1 or t_2 or
- $\phi \equiv \neg\phi'$ and x occurs in ϕ' or
- $\phi \equiv \phi_1 \to \phi_2$ and x occurs either in ϕ_1 or in ϕ_2 or
- $\phi \equiv \forall y \phi'$ and x occurs in ϕ' or $x \equiv y$.

$$\begin{aligned}
\mathrm{FOccur}(x,\phi) \iff\ & \mathrm{Var}(x) \wedge \mathrm{Form}(\phi) \wedge [\\
& \exists t_1, t_2 < \phi(\mathrm{Term}(t_1) \wedge \mathrm{Term}(t_2) \wedge\\
& (\mathrm{TOccur}(x,t_1) \vee \mathrm{TOccur}(x,t_2)) \wedge \phi = t_1 * \ulcorner = \urcorner * t_2) \vee\\
& \exists \psi < \phi(\mathrm{Form}(\psi) \wedge \mathrm{FOccur}(x,\psi) \wedge \phi = \ulcorner \neg \urcorner * \psi) \vee\\
& \exists \psi, \theta < \phi(\mathrm{Form}(\psi) \wedge \mathrm{Form}(\theta) \wedge (\mathrm{FOccur}(x,\psi) \vee \mathrm{FOccur}(x,\theta)) \wedge\\
& \phi = \psi * \ulcorner \rightarrow \urcorner * \theta) \vee\\
& \exists y, \psi < \phi(\mathrm{Form}(\psi) \wedge \mathrm{Var}(y) \wedge ((\mathrm{FOccur}(x,\psi) \vee (x = y)) \wedge\\
& \phi = \ulcorner \forall \urcorner * y * \psi)].
\end{aligned}$$

The previously defined formulas can be combined in one by the definition:

Definition 12.

$$\mathrm{Occur}(x,y) \iff \mathrm{TOccur}(x,y) \vee \mathrm{FOccur}(x,y).$$

Definition 13. We say that the variable x is *free* in the $\mathcal{L}$ formula ϕ if and only if:

- ϕ is atomic and x occurs in ϕ or
- $\phi \equiv \neg\phi'$ and x is free in ϕ' or
- $\phi \equiv \phi_1 \rightarrow \phi_2$ and x is free either in ϕ_1 or in ϕ_2 or
- $\phi \equiv \forall y \phi'$ and x is free in ϕ' and y is a variable different from x.

$$\begin{aligned}
\mathrm{Free}(x,\phi) \iff\ & \mathrm{Var}(x) \wedge \mathrm{Form}(\phi) \wedge [\\
& \exists t_1, t_2 < \phi(\mathrm{Term}(t_1) \wedge \mathrm{Term}(t_2) \wedge (\mathrm{Occur}(x,t_1) \vee \mathrm{Occur}(x,t_2)) \wedge\\
& \phi = t_1 * \ulcorner = \urcorner * t_2) \vee\\
& \exists \psi < \phi(\mathrm{Form}(\psi) \wedge \mathrm{Free}(x,\psi) \wedge \phi = \ulcorner \neg \urcorner * \psi) \vee\\
& \exists \psi, \theta < \phi(\mathrm{Form}(\psi) \wedge \mathrm{Form}(\theta) \wedge (\mathrm{Free}(x,\psi) \vee \mathrm{Free}(x,\theta)) \wedge\\
& \phi = \psi * \ulcorner \rightarrow \urcorner * \theta) \vee\\
& \exists y, \psi < \phi(\mathrm{Form}(\psi) \wedge \mathrm{Var}(y) \wedge (\mathrm{Free}(x,\psi) \wedge \neg(x = y) \wedge\\
& \phi = \ulcorner \forall \urcorner * y * \psi)].
\end{aligned}$$

We define the formula $\mathrm{FreeFor}(x, y, z)$ which expresses that the term with code x is free for the variable with code y in the formula with code z.

Definition 14. We say that the $\mathcal{L}$ term t is *free for* the variable x in the $\mathcal{L}$ formula ϕ if and only if:

- ϕ is atomic or
- $\phi \equiv \neg\phi'$ and t is free for x in ϕ' or
- $\phi \equiv \phi_1 \to \phi_2$ and t is free for x in both ϕ_1 and ϕ_2 or
- $\phi \equiv \forall y \phi'$ and one of the following conditions holds:
  - x is not free in ϕ,
  - x is free in ϕ, t is free for x in ϕ' and y does not occur in t.

$$\begin{aligned}
\mathrm{FreeFor}(t, x, \phi) \iff\ & \mathrm{Term}(t) \wedge \mathrm{Var}(x) \wedge \mathrm{Form}(\phi) \wedge [\\
& \exists t_1, t_2 < \phi(\mathrm{Term}(t_1) \wedge \mathrm{Term}(t_2) \wedge \phi = t_1 * \ulcorner = \urcorner * t_2) \vee \\
& \mathrm{AF}(\phi) \vee \\
& \exists \psi < \phi(\mathrm{Form}(\psi) \wedge \mathrm{FreeFor}(t, x, \psi) \wedge \phi = \ulcorner \neg \urcorner * \psi) \vee \\
& \exists \psi, \theta < \phi(\mathrm{Form}(\psi) \wedge \mathrm{Form}(\theta) \wedge \mathrm{FreeFor}(t, x, \psi) \wedge \\
& \mathrm{FreeFor}(t, x, \theta) \wedge \phi = \psi * \ulcorner \to \urcorner * \theta) \vee \\
& (\neg\mathrm{Free}(x, \phi) \vee \exists y, \psi < \phi(\mathrm{Free}(x, \phi) \wedge \mathrm{FreeFor}(t, x, \psi) \wedge \\
& \neg\mathrm{Occur}(y, t)) \wedge \phi = \ulcorner \forall \urcorner * y * \psi)]
\end{aligned}$$

1.3 Tableaux proofs

The method of *Tableaux proofs* is an alternative to the classic Hilbert system which avoids the use of the Modus Ponens rule.

Let $\mathcal{L}$ be a language extending $\mathcal{LA}$. To give a precise definition first we need to introduce the equality axioms:

Reflexivity for each variable x: $x = x$,

Substitution for functions for all variables x and y, and any function symbol f:

$$x = y \to f(\ldots, x, \ldots) = f(\ldots, y, \ldots) \quad \text{and}$$

Substitution for formulas (Leibniz's law) for any variables x and y, and any formula $\phi(x)$, if ϕ' is obtained by replacing any number of free occurrences of x in ϕ with y, such that these remain free occurrences of y, it holds that:

$$x = y \rightarrow (\phi \rightarrow \phi').$$

We are now ready to give a precise definition of a tableau proof from a set of formulas Σ of a contradiction. The definition following was first given by A.J Wilkie and J.B. Paris in [17].

Definition 15. Let Σ be a set of sentences (or formulas). We say that a sequence of sets of sets of formulas $\Gamma_0, \Gamma_1, \ldots, \Gamma_s$ is a *tableau proof from* Σ of a contradiction if the following hold:

1. For all $X \in \Gamma_s$ there is an atomic formula θ such that $\theta \in X$ and $\neg\theta \in X$.

2. $X \in \Gamma_0$ implies $X \subseteq \Sigma \cup \{$the logical equality axioms$\}$.

3. For all $X \in \Gamma_i$ with $i < s$ one of the following holds:
   a) $X \in \Gamma_{i+1}$,
   b) $X \cup \{\theta(x)\} \in \Gamma_{i+1}$ for some $\neg\neg\theta(x) \in X$,
   c) $X \cup \{\neg\theta_1\}$, $X \cup \{\theta_2\} \in \Gamma_{i+1}$ for some $(\theta_1 \rightarrow \theta_2) \in X$,
   d) $X \cup \{\theta_1, \neg\theta_2\} \in \Gamma_{i+1}$ for some $\neg(\theta_1 \rightarrow \theta_2) \in X$,
   e) $X \cup \{\theta(t)\} \in \Gamma_{i+1}$ for some $\forall x\theta(x) \in X$ and some term t free for x in $\theta(x)$
   f) $X \cup \{\neg\theta(y)\}$ for some $\neg\forall x\theta(x) \in X$ and some variable y which does not occur in any formula in X.

4. For all $Y \in \Gamma_{i+1}$ with $i < s$ there is an $X \in \Gamma_i$ such that Y is obtained from X by one of the rules 3. a)-f).

A sequence Π of sets of sets of formulas $\Gamma_0, \Gamma_1, \ldots, \Gamma_s$ is a *tableau from* Σ, if 2., 3. and 4. of definition 15 hold. Furthermore, if T is a tableau proof from Σ of a contradiction we will say that the tableau T is *closed* and that T is a *first-order confutation* of Σ. The *depth* of a tableau proof T, denoted by $\mathrm{dp}(T)$ is the height of the tree representation of the tableau proof. So if T is the tableau in the definition above, then $\mathrm{dp}(T) = s + 1$.

Remark 1. If we change 3. in definition 15 with "For all $X \in \Gamma_i$ with $i < s$ exactly one of the following holds:", then the deriving tableau could be represented by a binary tree.

Let Π be a tableau from Σ. We shall say that Π is *pure* if the following two conditions hold:

1. The terms t used in the applications of the $\forall$-rule in Π do not contain any variable that occurs bound in Σ.

2. The critical variables of the applications of the $\neg\forall$-rule in Π do not occur bound in Σ.

The rest of this paragraph is dedicated to the formalization of the notion of tableaux proofs.

Definition 16. Let $\mathcal{T}$ be a theory for $\mathcal{L}$ and let $A(x)$ be a Δ_0 formula of $\mathcal{L}$. We say that $\mathcal{T}$ *is coded by* $A(x)$, if $\mathcal{T} \vdash \mathrm{I}\Delta_0$ and

$$\mathrm{I}\Delta_0 \vdash \forall x(A(x) \to \mathrm{Sent}(x))$$

and

$$\{\ulcorner \delta \urcorner : \delta \in \mathcal{T}\} = \{m \in \omega : \mathbb{N} \models A(m)\}.$$

Similarly, we will say that Σ *is coded by* $\Sigma(x)$ if Σ is a recursive set of $\mathcal{L}$-formulas and there is a Δ_0 formula $\Sigma(x)$ satisfying

$$\mathrm{I}\Delta_0 \vdash \forall x(\Sigma(x) \to \mathrm{Form}(x))$$

and

$$\{\ulcorner \phi \urcorner : \phi \in \Sigma\} = \{m \in \omega : \mathbb{N} \models \Sigma(m)\}.$$

Each node of the tableaux proof tree is a set of formulas. So we need a formula to identify the codes that are sets of formulas.

$$\mathrm{SForm}(x) \iff \forall i \leq \mathrm{lh}(x)\mathrm{Form}((x)_i).$$

Furthermore, since each Γ_i, in the definition of tableaux proofs, is a set of sets of formulas, we define $\mathrm{SSForm}(x)$ to stand for the codes of sets of sets of formulas:

$$\mathrm{SSForm}(x) \iff \forall i \leq \mathrm{lh}(x)\mathrm{SForm}((x)_i).$$

The $\mathcal{L}$ formula $\mathrm{FUnion}(Z, X, y)$ holds if and only if Z is the code of a sequence (set) derived from the set with code X by adding the formula with code y to it, i.e.

$$\begin{aligned}\mathrm{FUnion}(Z, X, y) \iff\ & \mathrm{SForm}(Z) \wedge \mathrm{SForm}(X) \wedge \mathrm{Form}(y)\wedge \\ & (\forall i < \mathrm{lh}(Z))[(\exists j < \mathrm{lh}(X))[(Z)_i = (X)_j]\vee \\ & (Z)_i = y]]\end{aligned}$$

We now have all the necessary relations to formalize tableaux and tableau proofs for a set of formulas Σ. The subscript of each Tp relation below denotes the formalization of the corresponding number in definition 15. Notice that in the definitions below we can replace the set of sentences A with the set of formulas Σ.

$$\begin{aligned}\mathrm{Tp}_1(\gamma) \iff\ & \forall y \leq \mathrm{lh}(\gamma)\exists i_1, i_2, z_1, z_2 < \gamma[z_1 = \ulcorner \neg \urcorner * z_2\wedge \\ & z_1 = ((\gamma)_y)_{i_1} \wedge z_2 = ((\gamma)_y)_{i_2} \wedge \mathsf{ATF}(z_1) \wedge \mathsf{ATF}(z_2)]\end{aligned}$$

$$\begin{aligned}\mathrm{Tp}_2(A,\gamma) \iff & (\forall i \leq \mathrm{lh}(\gamma))(\exists X < \gamma)[X = (\gamma)_i \wedge \\ & (\forall j \leq \mathrm{lh}(X))[A((X)_j) \vee \mathrm{LEAxiom}((X)_j)]]\end{aligned}$$

$$\mathrm{Tp}_{3_a}(\gamma_i,\gamma_{i+1},k) \iff \exists l \leq \mathrm{lh}(\gamma_{i+1})[(\gamma_i)_k = (\gamma_{i+1})_l]$$

$$\begin{aligned}\mathrm{Tp}_{3_b}(\gamma_i,\gamma_{i+1},k) \iff & (\exists x < \mathrm{lh}(\gamma_i))(\exists X < \gamma_i)(\exists y < \mathrm{lh}(\gamma_{i+1}))(\exists Y < \gamma_{i+1}) \\ & (\exists z < (\gamma_i)_x)\{\mathrm{Form}(z) \wedge [X = (\gamma_i)_x] \wedge [Y = (\gamma_{i+1})_y] \wedge \\ & [(X)_k = \ulcorner\neg\urcorner * \ulcorner\neg\urcorner * z] \wedge \mathrm{FUnion}(Y,X,z)\}.\end{aligned}$$

$$\begin{aligned}\mathrm{Tp}_{3_c}(\gamma_i,\gamma_{i+1},k) \iff & (\exists x < \mathrm{lh}(\gamma_i))(\exists X < \gamma_i)(\exists y_1,y_2 < \mathrm{lh}(\gamma_{i+1})) \\ & (\exists Y_1,Y_2 < \gamma_{i+1})(\exists z_1,z_2 < (\gamma_i)_x)\{\mathrm{Form}(z_1) \wedge \mathrm{Form}(z_2) \wedge \\ & [X = (\gamma_i)_x] \wedge [Y_1 = (\gamma_{i+1})_{y_1}] \wedge [Y_2 = (\gamma_{i+1})_{y_2}] \wedge \\ & [(X)_k = z_1 * \ulcorner\rightarrow\urcorner * z_2] \wedge \mathrm{FUnion}(Y_1,X,\ulcorner\neg\urcorner * z_1) \wedge \\ & \mathrm{FUnion}(Y_2,X,z_2)\}.\end{aligned}$$

$$\begin{aligned}\mathrm{Tp}_{3_d}(\gamma_i,\gamma_{i+1},k) \iff & (\exists x < \mathrm{lh}(\gamma_i))(\exists X < \gamma_i)(\exists y_1,y_2 < \mathrm{lh}(\gamma_{i+1})) \\ & (\exists Y_1,Y_2 < \gamma_{i+1})(\exists z_1,z_2 < (\gamma_i)_x)\{\mathrm{Form}(z_1) \wedge \mathrm{Form}(z_2) \wedge \\ & [X = (\gamma_i)_x] \wedge [Y_1 = (\gamma_{i+1})_{y_1}] \wedge [Y_2 = (\gamma_{i+1})_{y_2}] \wedge \\ & [(X)_k = \ulcorner\neg\urcorner * \ulcorner(\urcorner * z_1 * \ulcorner\rightarrow\urcorner * z_2 * \ulcorner)\urcorner] \wedge \\ & \mathrm{FUnion}(Y_1,X,z_1) \wedge \mathrm{FUnion}(Y_2,Y_1,\ulcorner\neg\urcorner * z_2)\}.\end{aligned}$$

$$\begin{aligned}\mathrm{Tp}_{3_e}(\gamma_i,\gamma_{i+1},k) \iff & (\exists z_1,z_2 < \gamma_i)(\exists t < \gamma_{i+1})(\exists v < \gamma_i)(\exists x < \mathrm{lh}(\gamma_i)) \\ & (\exists X < \gamma_i)(\exists y < \mathrm{lh}(\gamma_{i+1}))(\exists Y < \gamma_{i+1}) \\ & \{\mathrm{Form}(z_1) \wedge \mathrm{Form}(z_2) \wedge \mathrm{Term}(t) \wedge \mathrm{Var}(v) \wedge \\ & [X = (\gamma_i)_x] \wedge [Y = (\gamma_{i+1})_y] \wedge \mathrm{Free}(v,z_1) \wedge \\ & \mathrm{FreeFor}(t,v,z_1) \wedge [(X)_k = \ulcorner\forall\urcorner * v * \ulcorner(\urcorner * z_1 * \ulcorner)\urcorner] \wedge \\ & \mathrm{SUB}(z_2,z_1,v,t) \wedge \mathrm{FUnion}(Y,X,z_2)\}.\end{aligned}$$

$$\begin{aligned}\mathrm{Tp3}_f(\gamma_i,\gamma_{i+1},k) \iff & (\exists z_1, z_2 < \gamma_i)(\exists w < \gamma_{i+1})(\exists v < \gamma_i)(\exists x < \mathrm{lh}(\gamma_i)) \\ & (\exists X < \gamma_i)(\exists y < \mathrm{lh}(\gamma_{i+1}))(\exists Y < \gamma_{i+1}) \\ & \{\mathrm{Form}(z_1) \wedge \mathrm{Form}(z_2) \wedge \mathrm{Var}(w) \wedge \mathrm{Var}(v) \wedge \\ & [X = (\gamma_i)_x] \wedge [Y = (\gamma_{i+1})_y] \wedge \mathrm{Free}(v, z_1) \wedge \\ & (\forall l < \mathrm{lh}(X))[\neg \mathrm{Occur}(w, (X)_l)] \wedge \\ & [(X)_k = \ulcorner \neg\forall \urcorner * v * z_1] \wedge \mathrm{SUB}(z_2, z_1, v, w) \wedge \\ & \mathrm{FUnion}(Y, X, \ulcorner \neg \urcorner * z_2)\}.\end{aligned}$$

$$\begin{aligned}\mathrm{Tp3}(\gamma_i,\gamma_{i+1}) \iff & \forall k \leq \mathrm{lh}(\gamma_i)[\mathrm{Tp3}_a(\gamma_i,\gamma_{i+1},k) \vee \\ & \mathrm{Tp3}_b(\gamma_i,\gamma_{i+1},k) \vee \ldots \vee \mathrm{Tp3}_f(\gamma_i,\gamma_{i+1},k)]\end{aligned}$$

The formalization of a tableau x for the set of formulas Σ is the relation:

$$\begin{aligned}\mathrm{Tableau}(\Sigma, x) \iff & \exists s < x\, \mathrm{lh}(x) = s \wedge (\forall i \leq s)[\mathrm{SSForm}((x)_i) \wedge \\ & \mathrm{Tp}_2(\Sigma, (x)_0) \wedge (\forall j < s)\mathrm{Tp}_3((x)_j, (x)_{j+1})].\end{aligned}$$

Furthermore, the formalization of a closed tableau x from A is the relation:

$$\begin{aligned}\mathrm{Tabinconseq}(A, x) \iff & \exists s < x\, \mathrm{lh}(x) = s \wedge (\forall i \leq s)[\mathrm{SSForm}((x)_i) \wedge \mathrm{Tp}_1((x)_s) \wedge \\ & \mathrm{Tp}_2(A, (x)_0) \wedge (\forall j < s)\mathrm{Tp}_3((x)_j, (x)_{j+1})].\end{aligned}$$

We are now able to define the Π_1 formula $\mathrm{Tabcon}(T)$ which says that there is no tableau proof of a contradiction from the theory T.

Definition 17. For any theory T as in 16, set:

$$\mathrm{Tabcon}(T) \quad \iff_{df} \quad \forall x \neg \mathrm{Tabinconseq}(T, x).$$

Restricted consistency statements

If we replace in $\mathrm{Tabcon}(T)$ the occurrences of the formula $\mathrm{Form}(x)$ by the formula $\mathrm{Reform}(k, x)$, where $k \in \mathbb{N}$, the formula derived, denoted by $k{-}\mathrm{Tabcon}(T)$, is the formalization of the statement "there is no tableau proof of a contradiction from T, using only substitution instances of formulas with Gödel number $\leq k$"; note that this is strongly reminiscent of the formula $\mathsf{Con}(X, k)$, which was introduced and used extensively in [17].

The difference between Tabcon and $k{-}$Tabcon resembles the difference between unrestricted consistency statements and consistency statements that involve only substitution instances of formulas with Gödel number less than a fixed bound. The unrestricted consistency of a theory implies the restricted version for all $k \in \mathbb{N}$. However, the converse implication does not necessarily hold; indeed, as shown in [17], for all $k \in \omega$

$$I\Delta_0 + \exp \vdash \mathrm{Con}(I\Delta_0, k)$$

but

$$I\Delta_0 + \exp \nvdash \mathrm{Con}(I\Delta_0).$$

1.4 Satisfaction

Another fact that is necessary for the sequel is a result of H. Lessan ([10], see also Theorem 2 of [14]), concerning the satisfaction of Δ_0 formulas (in models of $I\Delta_0$).

Theorem 3. *There exists a Δ_0 formula $\mathrm{Sat}_0(x, y, z)$ such that, for any $M \models I\Delta_0$, $\varphi(\vec{x}) \in \Delta_0$ and $\vec{a}, b \in M$,*

$$M \models b \geq 2^{(\max(\vec{a})+2)^{\ulcorner \varphi \urcorner}} \rightarrow [\varphi(\vec{a}) \leftrightarrow \mathrm{Sat}_0(b, \langle \vec{a} \rangle, \ulcorner \varphi(\vec{x}) \urcorner)].$$

Remark 2. Sat_0 acts like a satisfaction relation, for formulas in the sense of M. For example, for any $d, e \in M$, if, in the sense of M, d is the Gödel number of a Δ_0 formula of the form $\exists y \leq x_1 \psi(y, \vec{x})$ and e is the Gödel number of the formula $\psi(y, \vec{x})$, then

$$M \models \forall \vec{z} \forall t {\geq} 2^{(\max(\vec{z})+2)^d} [\mathrm{Sat}_0(t, \langle \vec{z} \rangle, d) {\leftrightarrow} \exists y {\leq} z_1 \mathrm{Sat}_0(t, \langle y, \vec{z} \rangle, e)].$$

Remark 3. The particular value of b is insignificant, as long as it exceeds $2^{(\max(\vec{a})+2)^{\ulcorner \varphi \urcorner}}$.

Recall that if $\theta_0(\vec{y}) \in \Delta_0$, then there is an open formula $\psi(\vec{x}, \vec{y})$ such that:

$$\theta_0(\vec{y}) \equiv Q_1 x_1 < t_1(\vec{y}) \dots Q_n x_n < t_n(\vec{y}) \psi(\vec{x}, \vec{y}) \tag{1.2}$$

where Q_i is either $\forall$ or $\exists$ for all $i = 1, 2, \dots, n$.

We will show by induction on the complexity of ψ that:

Lemma 1. *For all open formulas ψ if θ is as 1.2, then there are polynomials $f, g \in \mathbb{N}[\vec{x}, \vec{y}, \vec{u}]$ such that θ is equivalent in the presence of $I\Delta_0$ to:*

$$Q_1 x_1 < t_1(\vec{y}) \dots Q_n x_n < t_n Q_{n+1} u_1 < t_{n+1}(\vec{x}, \vec{y}) \dots \\ Q_{n+m} u_m < t_{n+m}(\vec{x}, \vec{y})[f(\vec{x}, \vec{y}, \vec{u}) = g(\vec{x}, \vec{y}, \vec{u})].$$

Proof. **Base**

- It holds, if $\psi(\vec{x}, \vec{y}) \equiv t(\vec{x}, \vec{y}) = s(\vec{x}, \vec{y})$.
- If $\psi(\vec{x}, \vec{y}) \equiv t(\vec{x}, \vec{y}) < s(\vec{x}, \vec{y})$, we can take the equivalent formula

$$\exists r \leq t(\vec{x}, \vec{y}) + s(\vec{x}, \vec{y})[t(\vec{x}, \vec{y}) + r = s(\vec{x}, \vec{y})]$$

IS

- If $\psi(\vec{x}, \vec{y}) \equiv \psi_1(\vec{x}, \vec{y}) \vee \psi_2(\vec{x}, \vec{y})$, then by the induction hypothesis there are polynomials $f_1, g_1, f_2, g_2 \in \mathbb{N}[\vec{x}, \vec{y}, \vec{u}]$ such that

$$\psi_1(\vec{x}, \vec{y}) \equiv Q_{11} u_{11} < t_{11}(\vec{x}, \vec{y}) \dots Q_{1n} u_{1n} < t_{1n}(\vec{x}, \vec{y})[f_1(\vec{x}, \vec{y}, \vec{u}) = g_1(\vec{x}, \vec{y}, \vec{u})]$$

and

$$\psi_2(\vec{x},\vec{y}) \equiv Q_{21}u_{21} < t_{21}(\vec{x},\vec{y}) \ldots Q_{2k}u_{2k} < t_{2k}(\vec{x},\vec{y})[f_2(\vec{x},\vec{y},\vec{u}) = g_2(\vec{x},\vec{y},\vec{u})]$$

If necessary, we can rename the bounded variables so that $u_{1i} \not\equiv u_{2j}$ for all $i = 1, \ldots, n$ and for all $j = 1, \ldots, k$. Then for

$$u_1 = u_{11}, \quad u_2 = u_{12}, \ldots, \quad u_n = u_{1n}, \quad u_{n+1} = u_{21}, \ldots, \quad u_m = u_{2k}$$

we have that

$$\psi(\vec{x},\vec{y}) \equiv Q_1 u_1 < t_1(\vec{x},\vec{y}) \ldots Q_m u_m < t_m(\vec{x},\vec{y}) \\ [f_1(\vec{x},\vec{y},\vec{u}) = g_1(\vec{x},\vec{y},\vec{u}) \vee f_2(\vec{x},\vec{y},\vec{u}) = g_2(\vec{x},\vec{y},\vec{u})]$$

which is equivalent to

$$\psi(\vec{x},\vec{y}) \equiv Q_1 u_1 < t_1(\vec{x},\vec{y}) \ldots Q_m u_m < t_m(\vec{x},\vec{y}) \\ [(f_1(\vec{x},\vec{y},\vec{u}) - g_1(\vec{x},\vec{y},\vec{u}))(f_2(\vec{x},\vec{y},\vec{u}) - g_2(\vec{x},\vec{y},\vec{u})) = 0]$$

Then $f(\vec{x},\vec{y},\vec{u})$ is the positive part of the above equation and $g(\vec{x},\vec{y},\vec{u})$ the negative part.

- If $\psi(\vec{x},\vec{y}) \equiv \neg\psi_1(\vec{x},\vec{y})$, then there are polynomials $f_1, g_1 \in \mathbb{N}[\vec{x},\vec{y},\vec{u}]$ such that

$$\psi_1(\vec{x},\vec{y}) \equiv Q_1 u_1 < t_1(\vec{x},\vec{y}) \ldots Q_m u_m < t_m(\vec{x},\vec{y})[f_1(\vec{x},\vec{y},\vec{u}) = g_1(\vec{x},\vec{y},\vec{u})]$$

hence

$$\neg\psi_1(\vec{x},\vec{y}) \equiv Q_1' u_1 < t_1(\vec{x},\vec{y}) \ldots Q_m' u_m < t_m(\vec{x},\vec{y})\neg[f_1(\vec{x},\vec{y},\vec{u}) = g_1(\vec{x},\vec{y},\vec{u})]$$

which is equivalent to

$$\neg\psi_1(\vec{x},\vec{y}) \equiv Q_1' u_1 < t_1(\vec{x},\vec{y}) \ldots Q_m' u_m < t_m(\vec{x},\vec{y}) \\ [(f_1(\vec{x},\vec{y},\vec{u}) < g_1(\vec{x},\vec{y},\vec{u})) \vee (f_1(\vec{x},\vec{y},\vec{u}) > g_1(\vec{x},\vec{y},\vec{u}))]$$

where Q_i' is $\exists$ if Q_i is $\forall$ and vice versa for $i = 1, 2, \ldots, m$. Continuing as in the cases for disjunction and inequality we can get the required polynomials f and g. □

We denote by $\Delta_{0,k}$ the class of Δ_0 formulas with k alternations of bounded quantifiers i.e.

$$Q_1 x_1 < t_1(\vec{y}) \ldots Q_k x_k < t_k \phi(\vec{x},\vec{t})$$

where ϕ is open and Q_i is either $\exists$ or $\forall$ for all $i = 1, 2, \ldots, k$. By Lemma 1 and the first part of the proof of proposition 4 in [14] we get the following theorem.

Theorem 4. *There is a* Δ_0 *formula* $\mathsf{Sat}_0(x, y, z)$ *such that, for any* $M \models I\Delta_0$*,* $\varphi(\vec{x}) \in \Delta_{0,k}$ *and* $\vec{a}, b \in M$*,*

$$M \models b \geq (max(\vec{a}) + 2)^{\ulcorner \varphi \urcorner} \to [\varphi(\vec{a}) \leftrightarrow \mathsf{Sat}_0(b, \langle \vec{a} \rangle, \ulcorner \varphi(\vec{x}) \urcorner)].$$

This means that if we care for the satisfaction of Δ_0 formulas with only k alternations of bounded quantifiers we only need weak exponentiation and for standard formulas we need no exponentiation at all.

1.5 Arithmetized Completeness Theorem

Having proved his first incompleteness theorem, Gödel realized that the proof could be formalized and thus, he obtained his second incompleteness theorem. The same fundamental insight works for other results, including Gödel's completeness theorem for the predicate calculus. This idea led to the so-called Arithmetized Completeness Theorem (ACT), first formulated by D. Hilbert and P. Bernays ([8]).

The ACT is undoubtedly an important result, as it can be applied to construct arithmetical models and give alternative proofs of the incompleteness theorems (see, e.g., [9]). Its statement has two forms, a syntactic and a semantic one. Since in the sequel we will be considering models of theories in $\mathcal{LA}$, the semantic form seems more appropriate (see, e.g. section 13.2 in [9]). In what follows, T will denote a theory in $\mathcal{LA}$.

Theorem 5. *(ACT-Semantic Form) Let* M *be a model of PA and* T *be a theory definable in* M*. If* $M \models \mathsf{Con}(T)$*, then there exists a model* K *of* T *such that* K *is "strongly definable" in* M*.*

Here, *strong definability* means, roughly speaking, that

(a) the universe of K may be taken to be the same as that of M and

(b) the satisfaction relation for K is parametrically definable in M, i.e. there is a formula $\mathsf{Sat}(x, y, z)$ and some $b \in M$ such that for all formulas $\phi(\vec{x})$ of $\vec{\alpha} \in K$

$$N \models \phi[\vec{\alpha}] \iff M \models \mathsf{Sat}(b, \langle \vec{\alpha} \rangle, \ulcorner \phi(\vec{x}) \urcorner).$$

If the theory T contains PA, the relationship between M and K is much nicer; indeed, one can prove (6.12 in [16]) the following.

Lemma 2. *If* M, K *are models of PA and* K *is strongly definable in* M*, then* M *is isomorphic to an initial segment of* K*.*

By condition (b) of strong definability and the (well-known) fixed-point lemma, it follows that M cannot be isomorphic to an elementary substructure of K. However, the ACT can be applied in such a way that M is isomorphic to a Σ_n elementary substructure of K. Indeed, the following result, first stated explicitly by K. McAloon ([12]), refers to this fact.

Theorem 6. *Let* M *be a model of PA and* T *be a theory definable in* M *such that* $\mathsf{M} \models \mathsf{Con}(\mathsf{T}+\mathsf{Tr}(\Pi_n))$, *where* $\mathsf{Tr}(\Pi_n)$ *denotes the set of (Gödel numbers of)* Π_n *sentences true in* M*. Then there exists a model* K *of* T *such that*

1. K *is strongly definable in* M *(and, therefore,)*

2. M *is isomorphic to a proper* Σ_n *elementary initial segment of* K.

We will continue this (historical) review in chapter 3, where we will also see the ACT play the main role in the effort to answer questions concerning the end extendability of a model.

2 Tableau proof elimination

2.1 Elimination Lemma

This chapter is dedicated to the formalization of the *Elimination Lemma*. The aim is to formalize the proofs presented in chapter 2 sections 5 and 6 of [1] in an arbitrary model of $I\Delta_0 + \Omega_1$. If we restrict ourselves to standard formulas the *Elimination Lemma* states that for any set of formulas T and any formula ϕ if we can get a confutation for $T+\phi$ and $T + \neg\phi$, then we can confute T alone. It is an immediate corollary of the Completeness theorem that the *Elimination Lemma* holds. This is because any valuation satisfies either ϕ or $\neg\phi$. So if both $T + \phi$ and $T + \neg\phi$ are unsatisfiable then T is unsatisfiable. Hence, by the Completeness theorem, there exists a confutation for T. In this chapter $\mathcal{L}$ will denote a language extending $\mathcal{LA}$. Since we will be working with variants of the Arithmetized Completeness Theorem we will also require an arithmetized version of the Elimination Lemma.

2.1.1 Some “book-keeping” lemmas

When we use the $\forall$-rule in a tableau proof the term has to be free for the variable that is replaced in the formula. A similar situation should be considered in the use of the $\neg\forall$-rule. A way to avoid this complication is to use terms in the applications of the $\forall$-rule in a tableau proof that do not contain bounded variables of Σ and also when the $\neg\forall$-rule is applied the critical variables used are not among the bounded variables of Σ.

The above discussion leads us to the definition of the notion of *being a variant* of a formula, which is done by recursion on the construction of the formula. This notion can be formalized by a Δ_0 formula $\mathrm{Variant}(x, y)$ which holds when x and y are codes of formulas of $\mathcal{L}$ and $\ulcorner x \urcorner$ is a variant of $\ulcorner y \urcorner$, where $\ulcorner x \urcorner$ denotes the formula with Gödel number x.

Definition 18.

- If ϕ is atomic, ψ is a variant of ϕ if and only if $\psi \equiv \phi$
- If $\phi \equiv \neg\phi'$, ψ is a variant of ϕ if and only if $\psi \equiv \neg\psi'$ and ψ' is a variant of ϕ'.
- If $\phi \equiv \phi_1 \to \phi_2$, ψ is a variant of ϕ if and only if $\psi \equiv \psi_1 \to \psi_2$, ψ_1 is a variant of ϕ_1 and ψ_2 is a variant of ϕ_2.
- If $\phi \equiv \forall x\phi'$, ψ is a variant of ϕ if and only if ψ' is a variant of ϕ' and $\psi \equiv \forall x\psi'$ or $\psi \equiv \forall z\psi'(x/z)$ where z is a variable which is not free in ψ' but is free for x in ψ'.

$$
\begin{aligned}
\mathrm{Variant}(x,y) \iff\ & \mathrm{Form}(x) \wedge \mathrm{Form}(y) \wedge [\\
& (\mathrm{AF}(x) \wedge \mathrm{AF}(y) \wedge x = y) \vee \\
& \exists x_1 < x \exists y_1 < y (\mathrm{Form}(x_1) \wedge \mathrm{Form}(y_1) \wedge \mathrm{Variant}(x_1, y_1) \wedge \\
& x = \ulcorner \neg \urcorner * x_1 \wedge y = \ulcorner \neg \urcorner * y_1) \vee \\
& \exists x_1, x_2 < x \exists y_1, y_2 < y (\mathrm{Form}(x_1) \wedge \mathrm{Form}(x_2) \wedge \mathrm{Form}(y_1) \\
& \wedge \mathrm{Form}(y_2) \wedge \mathrm{Variant}(x_1, y_1) \wedge \mathrm{Variant}(x_2, y_2) \wedge \\
& x = x_1 * \ulcorner \rightarrow \urcorner * x_2 \wedge y = y_1 * \ulcorner \rightarrow \urcorner * y_2) \vee \\
& \exists x_1, v < x \exists y_1, y_2, z < y (\mathrm{Form}(x_1) \wedge \mathrm{Form}(y_1) \wedge \mathrm{Form}(y_2) \wedge \\
& \mathrm{Var}(v) \wedge \mathrm{Var}(z) \wedge \mathrm{Variant}(x_1, y_1) \wedge x = \ulcorner \forall \urcorner * v * x_1 \wedge \\
& (y = \ulcorner \forall \urcorner * v * y_1 \vee (\neg \mathrm{Free}(z, y_1) \wedge \mathrm{FreeFor}(z, v, y_1) \wedge \\
& \mathrm{SUB}(y_2, y_1, v, z) \wedge y = \ulcorner \forall \urcorner * z * y_2)))]
\end{aligned}
$$

We will write $\phi \sim \psi$ whenever ϕ is a variant of ψ. Notice that if ϕ is a variant of ψ then the formulas have the same complexity.

Definition 19. The *complexity* of a formula is the height of the tree representation of the formula; that is to say:

- $\mathrm{cpl}(\phi) = 0$, if ϕ is atomic,
- $\mathrm{cpl}(\phi \vee \psi) = \mathrm{cpl}(\phi \wedge \psi) = \mathrm{cpl}(\phi \rightarrow \psi) = \max\{\mathrm{cpl}(\phi), \mathrm{cpl}(\psi)\} + 1$,
- $\mathrm{cpl}(\neg\phi) = \mathrm{cpl}(\forall x \phi) = \mathrm{cpl}(\exists x \phi) = \mathrm{cpl}(\phi) + 1$.

Hence, if $\phi \sim \psi$, then $\mathrm{cpl}(\phi) = \mathrm{cpl}(\psi)$. However, the notion of being a variant says more about the relation between the variant formulas. If $\phi \sim \psi$, then ϕ and ψ have the same tree representation.

The following lemma shows that in every model of $I\Delta_0 + \Omega_1$ we can prove that if ϕ is a variant of ψ, then $\phi(x/t)$ is a variant of $\psi(x/t)$, where x is a variable and t a term of $\mathcal{LA}$ and $\phi(x/t)$ is the formula that results from the substitution of the term t for all the free occurrences of the variable x in ϕ.

Lemma 3. *For every model* M *of* $I\Delta_0 + \Omega_1$,

$$
\begin{aligned}
M \models & \forall c_1, c_2, z, t, c_3, c_4 [(\mathrm{Variant}(c_1, c_2) \wedge \mathrm{Var}(x) \wedge \mathrm{Term}(t) \wedge \\
& \mathrm{SUB}(c_3, c_1, x, t) \wedge \mathrm{SUB}(c_4, c_2, x, t) \rightarrow \mathrm{Variant}(c_3, c_4)].
\end{aligned}
$$

Proof. We will show by complete induction on k that if $M \models I\Delta_0 + \Omega_1$,

$$M \models \forall c_1, c_2, z, t, c_3, c_4 < k[(\mathrm{Variant}(c_1, c_2) \wedge \mathrm{Var}(x) \wedge \mathrm{Term}(t) \wedge \mathrm{SUB}(c_3, c_1, x, t) \wedge \mathrm{SUB}(c_4, c_2, x, t) \rightarrow \mathrm{Variant}(c_3, c_4)].$$

Base It holds trivially for $k = 0$.

IH Suppose it holds for all $k \leq n$, i.e.

$$M \models \forall c_1, c_2, x, t, c_3, c_4 < n[(\mathrm{Variant}(c_1, c_2) \wedge \mathrm{Var}(x) \wedge \mathrm{Term}(t) \wedge \mathrm{SUB}(c_3, c_1, x, t) \wedge \mathrm{SUB}(c_4, c_2, x, t) \rightarrow \mathrm{Variant}(c_3, c_4)].$$

IS We show that it holds for $k = n + 1$, i.e.

$$M \models \forall c_1, c_2, x, t, c_3, c_4 < n + 1[(\mathrm{Variant}(c_1, c_2) \wedge \mathrm{Var}(x) \wedge \mathrm{Term}(t) \wedge \mathrm{SUB}(c_3, c_1, x, t) \wedge \mathrm{SUB}(c_4, c_2, x, t) \rightarrow \mathrm{Variant}(c_3, c_4)].$$

It suffices to prove the statement of $c_1 = n$ or $c_2 = n$ and $M \models \mathrm{Form}(c_1) \wedge \mathrm{Form}(c_2)$, for otherwise either the hypothesis of the implication is false and so the implication is true or both c_1 and c_2 are less than n and so the implication holds by the Induction Hypothesis.

We will consider the most interesting case where the formula with code c_1 is universal. Suppose $c_1 = \ulcorner \forall w \beta \urcorner$, then by definition 18 $c_2 = \ulcorner \forall w \beta' \urcorner$ or $c_2 = \ulcorner \forall z(\beta'(w/z)) \urcorner$; where z is not free in β' but is free for w in β'.

Let v be the variable of $\mathcal{LA}$ such that $\ulcorner v \urcorner = x$. We may assume that v is free in the formula with Gödel number c_1 because otherwise we have that

$$M \models \mathrm{SUB}(c_1, c_1, \ulcorner v \urcorner, t) \wedge \mathrm{SUB}(c_2, c_2, \ulcorner v \urcorner, t)$$

and the assertion of the lemma is trivial.

For the time being, we will assume that s, where $t = \ulcorner s \urcorner$, is free for w in both formulas with codes c_1 and c_2. So w cannot occur in s, hence

$$M \models \neg \mathrm{Occur}(\ulcorner w \urcorner, t)$$

and it is easy to show by induction on $k = n + 1$ that:

$$M \models \exists c < n + 1[\mathrm{SUB}(c_3, c_1, x, t) \wedge \mathrm{SUB}(c, \ulcorner \beta \urcorner, x, t) \wedge c_3 = \ulcorner \forall w \urcorner * c]$$

which means that we can show in M that

$$(\forall w \beta)(v/s) \equiv \forall w(\beta(v/s)).$$

Similarly,

$$\begin{aligned} M \models &\exists d < n+1[\text{SUB}(c_4, c_2, x, t) \wedge \\ &\text{SUB}(d, \ulcorner \beta' \urcorner, x, t) \wedge c_4 = \ulcorner \forall w \urcorner * d]. \end{aligned}$$

By definition 18 we have that

$$M \models \text{Variant}(c_1, c_2) \rightarrow \text{Variant}(\ulcorner \beta \urcorner, \ulcorner \beta' \urcorner)$$

and $\ulcorner \beta \urcorner < c_1 < n+1$, $\ulcorner \beta' \urcorner < c_2 < n+1$ hence by the induction hypothesis

$$M \models \text{Variant}(c, d).$$

Now suppose $c_2 = \ulcorner \forall z(\beta'(w/z)) \urcorner$, then z cannot occur in s, s is free for v in $\beta'(w/z)$ it is easy to show by induction that:

$$\begin{aligned} M \models &\exists d, c_2, c_4, x, t < n+1[\text{SUB}(c_4, c_2, x, t) \wedge \\ &\text{SUB}(d, \ulcorner \beta'(w/z) \urcorner, x, t) \wedge c_4 = \ulcorner \forall z(\urcorner * d * \ulcorner) \urcorner]. \end{aligned}$$

Since z is free for y in β' and s is free for v in $\beta'(w/z)$ it follows that in going from β' to $\beta'(w/z)(v/s)$ no alphabetic changes are made. Thus, β' and $\beta'(w/z)(v/s)$ have exactly the same bound occurrences of variables. Next, we observe that by induction on $k = n+1$ we can show:

$$\begin{aligned} M \models &\exists b, c, d, x, t < n+1[\text{SUB}(c, \ulcorner \beta' \urcorner, x, t) \wedge \text{SUB}(d, c, \ulcorner w \urcorner, \ulcorner z \urcorner) \wedge \\ &\text{SUB}(b, \ulcorner \beta' \urcorner, \ulcorner w \urcorner, \ulcorner z \urcorner) \wedge \text{SUB}(d, b, x, t)]. \end{aligned}$$

For, x is different from both w and z and y does not occur in s; so it makes no difference whether we first substitute z for w and then t for x, or vice versa. Thus,

$$\begin{aligned} M \models &\exists e, d, c_2, c_4, x, t < n+1[\text{SUB}(c_4, c_2, x, t) \wedge \text{SUB}(d, \ulcorner \beta' \urcorner, x, t) \wedge \\ &\text{SUB}(e, d, \ulcorner w \urcorner, \ulcorner z \urcorner) \wedge c_4 = \ulcorner \forall z(\urcorner * e * \ulcorner) \urcorner]. \end{aligned}$$

By the definition of variants and the fact that z is not free in β'

$$\begin{aligned} M \models &\text{Variant}(c_1, c_2) \rightarrow \text{Variant}(\ulcorner \beta \urcorner, \ulcorner \beta'(w/z) \urcorner) \wedge \\ &\neg\text{Free}(z, \ulcorner \beta' \urcorner) \wedge \text{FreeFor}(z, w, \ulcorner \beta' \urcorner) \end{aligned}$$

and $\ulcorner \beta \urcorner < c_1 < n+1$, $\ulcorner \beta'(w/z) \urcorner < c_2 < n+1$. By the induction hypothesis

$$\begin{aligned} M \models &\text{Variant}(\ulcorner \beta(v/s) \urcorner, \ulcorner \beta'(w/z)(v/s) \urcorner) \wedge \\ &\neg\text{Free}(z, \ulcorner \beta' \urcorner) \wedge \text{FreeFor}(z, w, \ulcorner \beta' \urcorner) \end{aligned}$$

which by the discussion above implies

$$\begin{aligned} M \models &\mathrm{Variant}(\ulcorner \beta(\nu/s) \urcorner, \ulcorner \beta'(\nu/s)(w/z) \urcorner) \wedge \\ &\neg\mathrm{Free}(\ulcorner z \urcorner, \ulcorner \beta' \urcorner) \wedge \mathrm{FreeFor}(\ulcorner z \urcorner, \ulcorner w \urcorner, \ulcorner \beta' \urcorner). \end{aligned}$$

Now z is not free in $\beta'(\nu/s)$, because z was not free in β' and z does not occur in s and it is easy to show that in M, i.e. for any formula β' and any variable z

$$M \models \neg\mathrm{Free}(\ulcorner z \urcorner, \ulcorner \beta' \urcorner) \wedge \neg\mathrm{Occur}(\ulcorner z \urcorner, \ulcorner s \urcorner) \to \neg\mathrm{Free}(\ulcorner z \urcorner, \ulcorner \beta'(\nu/s) \urcorner))$$

Also, z is free for w in $\beta'(\nu/s)$, because z was free for w in β' and the substitution of s for ν in β' cannot change matters in this respect since s does not contain w and no alphabetic changes are made by the substitution. Hence, we can show that for any formula β', any term s and all variables z and w

$$\begin{aligned} M \models &\mathrm{FreeFor}(\ulcorner z \urcorner, \ulcorner w \urcorner, \ulcorner \beta' \urcorner) \wedge \\ &\neg\mathrm{Occur}(\ulcorner w \urcorner, \ulcorner s \urcorner) \to \mathrm{FreeFor}(\ulcorner z \urcorner, \ulcorner w \urcorner, \ulcorner \beta'(\nu/s) \urcorner)). \end{aligned}$$

Therefore,

$$\begin{aligned} M \models &\mathrm{Variant}(\ulcorner \beta(\nu/s) \urcorner, \ulcorner \beta'(\nu/s)(w/z) \urcorner) \wedge \\ &\neg\mathrm{Free}(z, \ulcorner \beta'(\nu/s) \urcorner) \wedge \mathrm{FreeFor}(z, w, \ulcorner \beta'(\nu/s) \urcorner). \end{aligned}$$

Hence, by definition 18

$$M \models \mathrm{Variant}(\ulcorner \forall w(\beta(\nu/s)) \urcorner, \ulcorner \forall w(\beta'(\nu/s)(w/z)) \urcorner)$$

and it is trivial to show

$$M \models \mathrm{Variant}(\ulcorner (\forall w \beta)(\nu/s)) \urcorner, \ulcorner (\forall w \beta'(w/z))(\nu/s) \urcorner).$$

□

The notion of being a variant can be extended to tableaux and sets of formulas.

Definition 20. Let Π and Π' be tableaux. We say that Π' is a *variant* of Π (briefly, $\Pi \sim \Pi'$) if and only if Π can be transformed into Π' by replacing each formula ϕ in Π by a variant ϕ', in such a way that each application of rule 3.(e) in Π is transformed into an application of the same rule with the same term t, and each application of the rule 3.(f) in Π is transformed into an application of the same rule with the same critical variable.

Similarly, we say that the set of sentences Σ is a *variant* of the set of sentences Σ' (briefly, $\Sigma \sim \Sigma'$) if and only if Σ can be transformed into Σ' by replacing each formula ϕ in Σ by a variant ϕ'.

Both notions defined above can be formalized in $I\Delta_0 + \Omega_1$. In the next definition we will demonstrate how this formalization can be achieved for tableau proof variants.

Definition 21. For a Δ_0 formula A such that

$$I\Delta_0 + \Omega_1 \vdash \forall x(A(x) \to \text{Sent}(x))$$

we define:

$$\begin{aligned}
\text{TVariant}(A, x, y) \iff\ & \text{Tabinconseq}(A, x) \wedge \text{Tabinconseq}(A, y) \wedge \\
& [\text{lh}(x) = \text{lh}(y)] \wedge (\forall i < \text{lh}(x))(\forall j < \text{lh}((x)_i)) \\
& (\exists l < \text{lh}((y)_i))\{ \\
& \\
& [[\text{Tp}_{3_a}((x)_i, (x)_{i+1}, j) \wedge \text{Tp}_{3_a}((y)_i, (y)_{i+1}, l)] \vee \\
& [\text{Tp}_{3_b}((x)_i, (x)_{i+1}, j) \wedge \text{Tp}_{3_b}((y)_i, (y)_{i+1}, l)] \vee \\
& [\text{Tp}_{3_c}((x)_i, (x)_{i+1}, j) \wedge \text{Tp}_{3_c}((y)_i, (y)_{i+1}, l)] \vee \\
& [\text{Tp}_{3_d}((x)_i, (x)_{i+1}, j) \wedge \text{Tp}_{3_d}((y)_i, (y)_{i+1}, l)]] \wedge \\
& [(\forall k < \text{lh}(((x)_i)_j))(\exists n < \text{lh}(((y)_i)_l)) \\
& \text{Variant}((((x)_i)_j)_k, (((y)_i)_l)_n)] \vee \\
& \\
& \text{VTp}_{3_e}(x, y, i) \vee \text{VTp}_{3_f}(x, y, i)\}
\end{aligned}$$

where

$$\begin{aligned}
\text{VTp}_{3_e}(x, y, i) \iff\ & \{[(\exists z_1, z_2 < (x)_i)(\exists t < (x)_{i+1})(\exists v < (x)_i) \\
& (\exists j_1 < \text{lh}((x)_i))(\exists X < (x)_i)(\exists j_2 < \text{lh}((x)_{i+1})) \\
& (\exists Y < (x)_{i+1})(\exists k < \text{lh}(((x)_i)_{j_1}))[\\
& \text{Form}(z_1) \wedge \text{Form}(z_2) \wedge \text{Term}(t) \wedge \text{Var}(v) \wedge \\
& [X = ((x)_i)_{j_1}] \wedge [Y = ((x)_{i+1})_{j_2}] \wedge \text{Free}(v, z_1) \wedge \\
& \text{FreeFor}(t, v, z_1) \wedge [(X)_k = \ulcorner\forall\urcorner * v * \ulcorner(\urcorner * z_1 * \ulcorner)\urcorner] \wedge \\
& \text{SUB}(z_2, z_1, v, t) \wedge \text{FUnion}(Y, X, z_2)]] \wedge \\
& \\
& [(\exists z_3, z_4 < (y)_i)(\exists t_1 < (y)_{i+1})(\exists w < (y)_i) \\
& (\exists l_1 < \text{lh}((y)_i))(\exists X_1 < (y)_i)(\exists l_2 < \text{lh}((y)_{i+1})) \\
& (\exists Y_1 < (y)_{i+1})(\exists n < \text{lh}(((x)_i)_{j_1}))[\\
& \text{Form}(z_3) \wedge \text{Form}(z_4) \wedge \text{Term}(t_1) \wedge \text{Var}(w) \wedge \\
& [X_1 = ((y)_i)_{l_1}] \wedge [Y_1 = ((y)_{i+1})_{l_2}] \wedge \text{Free}(w, z_3) \wedge \\
& \text{FreeFor}(t_1, w, z_3) \wedge [(X_1)_n = \ulcorner\forall\urcorner * w * \ulcorner(\urcorner * z_3 * \ulcorner)\urcorner] \wedge \\
& \text{SUB}(z_4, z_3, w, t_1) \wedge \text{FUnion}(Y_1, X_1, z_4)]] \wedge \\
& \\
& \text{Variant}(z_1, z_3) \wedge \text{Variant}(z_2, z_4) \wedge t = t_1\}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{VTp}_{3_f}(x,y,i) \iff & \{[(\exists z_1, z_2 < (x)_i)(\exists w < (x)_{i+1})(\exists v < (x)_i) \\
& (\exists j_1 < \mathrm{lh}((x)_i))(\exists X_1 < (x)_i)(\exists j_2 < \mathrm{lh}((x)_{i+1})) \\
& (\exists Y_1 < (x)_{i+1})(\exists k < \mathrm{lh}(((x)_i)_{j_1}))[\\
& \mathrm{Form}(z_1) \wedge \mathrm{Form}(z_2) \wedge \mathrm{Var}(w) \wedge \mathrm{Var}(v) \wedge \\
& [X = ((x)_i)_{j_1}] \wedge [Y = ((x)_{i+1})_{j_2}] \wedge \mathrm{Free}(v, z_1) \wedge \\
& (\forall l < \mathrm{lh}(X_1))[\neg \mathrm{Occur}(w, (X_1)_l)] \wedge \\
& [(X_1)_k = \ulcorner \neg\forall \urcorner * v * z_1] \wedge \mathrm{SUB}(z_2, z_1, v, w) \wedge \\
& \mathrm{FUnion}(Y_1, X_1, \ulcorner \neg \urcorner z_2)]] \wedge \\
& \\
& \{[(\exists z_3, z_4 < (x)_i)(\exists w_1 < (x)_{i+1})(\exists v_1 < (x)_i) \\
& (\exists l_1 < \mathrm{lh}((x)_i))(\exists X_2 < (x)_i)(\exists l_2 < \mathrm{lh}((x)_{i+1})) \\
& (\exists Y_2 < (x)_{i+1})(\exists k < \mathrm{lh}(((x)_i)_{l_1}))[\\
& \mathrm{Form}(z_3) \wedge \mathrm{Form}(z_4) \wedge \mathrm{Var}(w_1) \wedge \mathrm{Var}(v_1) \wedge \\
& [X = ((x)_i)_{l_1}] \wedge [Y = ((x)_{i+1})_{l_2}] \wedge \mathrm{Free}(v_1, z_3) \wedge \\
& (\forall l < \mathrm{lh}(X_2))[\neg \mathrm{Occur}(w_1, (X_2)_l)] \wedge \\
& [(X_2)_k = \ulcorner \neg\forall \urcorner * v_1 * z_3] \wedge \mathrm{SUB}(z_4, z_3, v_1, w_1) \wedge \\
& \mathrm{FUnion}(Y_2, X_2, \ulcorner \neg \urcorner z_4)]] \wedge \\
& \\
& \mathrm{Variant}(z_1, z_3) \wedge \mathrm{Variant}(z_2, z_4) \wedge w = w_1\}
\end{aligned}$$

We can now formalize and prove the lemmas needed for the proof of the Elimination Lemma. Most of the lemmas are obvious and so we will omit the details giving only the basic idea of each proof.

For the rest of this chapter, unless otherwise mentioned, the results are proven in a model M of Δ_0 induction that satisfies Ω_1. Let Σ be a recursive, in M, set of formulas.

Lemma 4. *If $\Sigma \sim \Sigma'$ then given a Tableau Π for Σ, we can construct a tableau Π' for Σ' such that:*

$$\Pi \sim \Pi', \quad \mathrm{dp}(\Pi') = \mathrm{dp}(\Pi) \quad \textit{and} \quad \Pi' \simeq \Pi.$$

Where $\Pi' \simeq \Pi$ means that the Gödel numbers of Π and Π' have approximately the same size.

Proof. The lemma is proven by induction on the number of nodes in Π. The idea is to make sure that when a bounded variable in Π is renamed in Π' the substitution by a term is not affected. So by lemma 3 $\Pi' \sim \Pi$. Furthermore, $\mathrm{dp}(\Pi') = \mathrm{dp}(\Pi)$. Finally, Π' is obtained by Π by renaming variables, hence $\Pi' \simeq \Pi$. □

Lemma 5. *Let $y_1, \dots, y_n$ be variables. Given a confutation T of Σ, we can construct a confutation T' for Σ in which none of the variables $y_1, \dots, y_n$ is used as a critical variable and it is such that:*

$$\mathrm{dp}(T') = \mathrm{dp}(T) \quad \textit{and} \quad T' \simeq T.$$

Proof. First, notice that we may assume that $y_1, \dots, y_n$ are not free in Σ. Then from a tableau proof T for Σ and lemma 4 we can get a pure tableau proof T_1 for a variant Σ' of Σ. Now we can replace $y_1, \dots, y_n$ in T_1 with any set of new variables that do not occur in T_1 and get a tableau proof T_2 for Σ'. Finally, since $\Sigma \sim \Sigma'$ by lemma 4 again we can get a variant T' of T which is a tableau proof for Σ and has the required properties. Furthermore, the process described above, involves only renaming of variables that doesn't affect the height and the size of the tableau proof. Hence, $\mathrm{dp}(T') = \mathrm{dp}(T)$ and $T' \simeq T$. □

Lemma 6. *Given a confutation T of Σ, a variable z and a term s, there exists a confutation T' for $\Sigma(z/s)$ such that:*

$$\mathrm{dp}(T') = \mathrm{dp}(T) \quad \textit{and} \quad T' \leq T \cdot s.$$

Proof. By the previous Lemmas we may assume that no variable occurring in the term s serves in T as a critical variable, T is pure and no variable occurring in s occurs bound in Σ. Hence, s is free for z in every formula of T.

Let T' be the tableau proof obtained by T by replacing each formula ϕ in T by the formula $\phi(z/s)$. T' is a tableau proof for $\Sigma(z/s)$ and by examining the application of each rule used in T we see that it can be transformed into an application of the same rule in T'. Hence, T' is a confutation for $\Sigma(z/s)$.

In the transformation of T to T' no new nodes are added and so $\mathrm{dp}(T') = \mathrm{dp}(T)$. However, the substitution of z by s may increase the the size of T', compared to the size of T. For the worst case we would have to substitute z by s for all formulas that appear in T. Hence $T' \leq T \cdot s$. □

2.1.2 Proof of the Elimination Lemma

Definition 22. For all formulas θ, for all sets of formulas Σ such that $\theta \notin \Sigma$ and for all tableaux T for $\Sigma + \theta$, set $n_T(\theta)$ to be the number of subformulas of θ in T.

Lemma 7. *Let Σ' be any finite set of formulas. Given a confutation T of Σ, there exists a confutation T' for $\Sigma \cup \Sigma'$ such that:*

$$\mathrm{dp}(T') = \mathrm{dp}(T) \quad \textit{and} \quad T' \simeq T.$$

Proof. In order to adjoin Σ' to the initial node of the given tableau, we have first to rename all the tableau variables that are free in Σ' and might have been used as critical variables. This is easily done because by Lemma 5 we can construct a confutation T' for Σ in which no free variable of Σ' is used as a critical variable. In this tableau we can adjoin Σ' to the initial node, getting a confutation of $\Sigma \cup \Sigma'$. Furthermore, since only alphabetic changes take place we have that $\mathrm{dp}(\mathsf{T}') = \mathrm{dp}(\mathsf{T})$ and $\mathsf{T}' \simeq \mathsf{T}$. $\square$

Lemma 8. *Given a confutation* T *of* $\Sigma, \neg\neg\theta$*, there is a confutation* T' *of* Σ, θ *such that:*

$$\mathrm{dp}(\mathsf{T}') \leq \mathrm{dp}(\mathsf{T}) \quad \textit{and} \quad \mathsf{T}' \leq \mathsf{T}.$$

Syntactically:

$$\mathsf{M} \models \forall \mathsf{T}\, \exists \mathsf{T}' \left((\mathrm{Tabinconseq}(\mathsf{T}, \Sigma, \neg\neg\theta) \to \mathrm{Tabinconseq}(\mathsf{T}', \Sigma, \theta)) \wedge \mathrm{dp}(\mathsf{T}') \leq \mathrm{dp}(\mathsf{T}) \right).$$

Proof. Let T be a confutation of $\Sigma, \neg\neg\theta$. We can derive a confutation T' of Σ, θ by replacing $\neg\neg\theta$ with θ in every node of T as in figure 2.1. The derived tableau T' has less nodes than T if $\neg\neg\theta$ appears in T. Hence, $\mathrm{dp}(\mathsf{T}') \leq \mathrm{dp}(\mathsf{T})$ and $\mathsf{T}' \leq \mathsf{T}$. $\square$

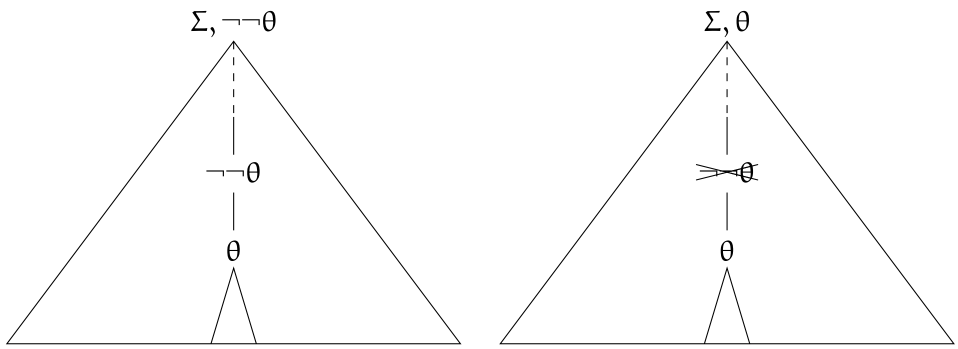

Figure 2.1: Transforming T to T' in Lemma 8

Lemma 9. *Given a confutation* T *of* $\Sigma, \theta \to \phi$*, there exist confutations* T' *and* T'' *for* $\Sigma, \neg\theta$ *and* Σ, ϕ *respectively such that:*

$$\mathrm{dp}(\mathsf{T}') \leq \mathrm{dp}(\mathsf{T}) \quad \textit{and} \quad \mathrm{dp}(\mathsf{T}'') \leq \mathrm{dp}(\mathsf{T}).$$

Furthermore,

$$\mathsf{T}' \leq \mathsf{T} \quad \textit{and} \quad \mathsf{T}'' \leq \mathsf{T}.$$

Syntactically:

$$\begin{aligned} \mathsf{M} \models & \forall \mathsf{T}\, \exists \mathsf{T}', \mathsf{T}'' \big(\\ & (\mathrm{Tabinconseq}(\mathsf{T}, \Sigma, \theta \to \psi) \to \mathrm{Tabinconseq}(\mathsf{T}', \Sigma, \neg\theta) \wedge \mathrm{Tabinconseq}(\mathsf{T}'', \Sigma, \neg\theta)) \wedge \\ & \mathrm{dp}(\mathsf{T}') \leq \mathrm{dp}(\mathsf{T}) \wedge \mathrm{dp}(\mathsf{T}'') \leq \mathrm{dp}(\mathsf{T}) \wedge \mathsf{T}' \leq \mathsf{T} \wedge \mathsf{T}'' \leq \mathsf{T} \big). \end{aligned}$$

Proof. Let T be a confutation of $\Sigma, \theta \to \phi$. In order to get a confutation T' for $\Sigma, \neg\theta$ we can erase $\theta \to \phi$ and the branch following ϕ wherever the $\to$-rule is applied in T as in Figure 2.2. The remaining tree T' is a confutation of $\Sigma, \neg\theta$ with fewer nodes than T. Hence, $\mathrm{dp}(\mathsf{T}') \leq \mathrm{dp}(\mathsf{T})$ and $\mathsf{T}' \leq \mathsf{T}$.

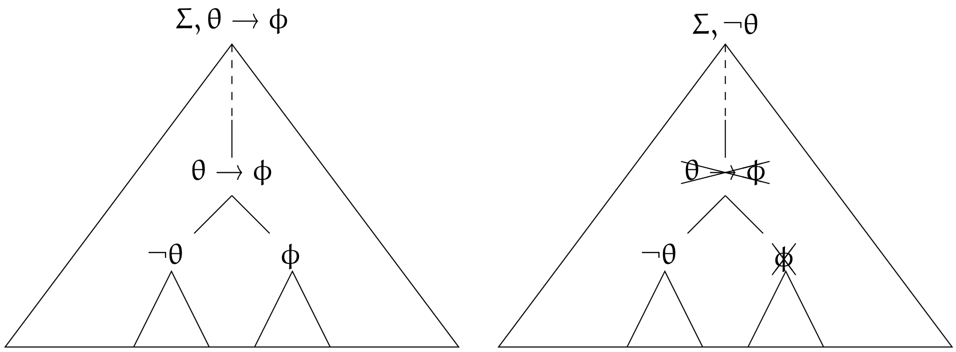

Figure 2.2: Transforming T to T′ in Lemma 9

Similarly, starting with T we can get a confutation T'' of Σ, ϕ which has fewer nodes than T. Hence, $\mathrm{dp}(T'') \leq \mathrm{dp}(T)$ and $T'' \leq T$. □

Lemma 10. *Given a confutation* T *of* $\Sigma, \neg(\theta \to \phi)$*, there exists a confutation* T' *for* $\Sigma, \theta, \neg\phi$ *such that:*

$$\mathrm{dp}(T') \leq \mathrm{dp}(T) \quad \textit{and} \quad T' \leq T.$$

Proof. Let T be a confutation of $\Sigma, \neg(\theta \to \phi)$. We can derive a confutation T' of $\Sigma, \theta, \neg\phi$ by erasing $\neg(\theta \to \phi)$ in every node of T as in figure 2.3. The derived tableau T' has fewer nodes than T if $\neg(\theta \to \phi)$ appears in T. Hence, $\mathrm{dp}(T') \leq \mathrm{dp}(T)$ and $T' \leq T$. □

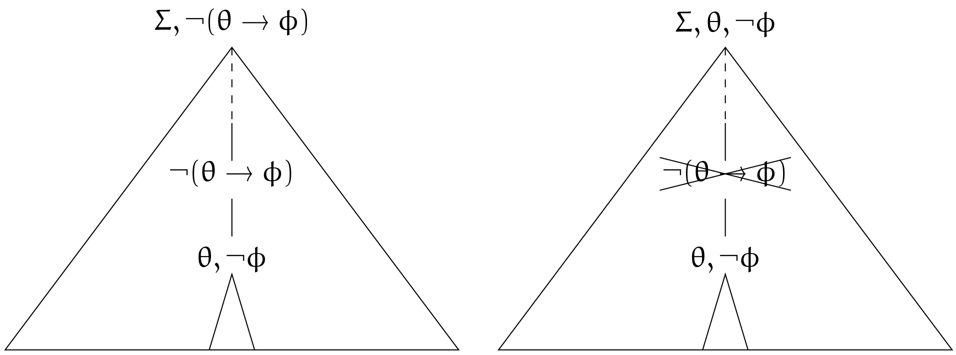

Figure 2.3: Transforming T to T′ in Lemma 10

Lemma 11. *If* T *is a confutation of* $\Sigma, \neg\forall x\theta$*, there exists a confutation* T' *for* $\Sigma, \neg\theta(x/t)$*, where* t *is any term, such that:*

$$\mathrm{dp}(T') \leq \mathrm{dp}(T) \quad \textit{and} \quad T' \leq T \cdot t$$

Proof. By complete induction on the depth of the given confutation, say T, we will show that there is a confutation T' of $\Sigma, \neg\theta(x/t)$, where t is any term and $\mathrm{dp}(T') \leq \mathrm{dp}(T)$. By

Lemma 5 we may assume that no variable occurring in the term t is used as a critical variable in the tableau T.

Base For $dp(T) = 0$ the hypothesis of the Lemma holds trivially since Σ is inconsistent.

I.H. Suppose that the hypothesis holds for every confutation T with $dp(T) < n$.

IS For the induction step we have to examine how the nodes of the first level in T could have been obtained.

First, suppose that the first level of T is obtained by an equality rule or by applying rules 3.b) or e) to a formula $\chi \in \Sigma$ and that ϕ is the derived formula. By erasing ϕ from the first level of T and adding it to the hypothesis we get a confutation S of $\Sigma, \phi, \neg\forall x\theta$. Since $dp(S) < dp(T)$ by the induction hypothesis there is a confutation S' of $\Sigma, \phi, \neg\theta(x/t)$, where t is any term, $dp(S') \leq dp(S)$ and $S' \leq S \cdot t$.

We can now get the required confutation T' for $\Sigma, \neg\theta(x/t)$ by starting with ϕ at the first level and continuing as in S'. Clearly:

$$dp(T') = dp(S') + 1 \leq dp(S) + 1 = dp(T) \quad \text{and} \quad T' \leq T \cdot t.$$

This transformation is depicted in Figure 2.4.

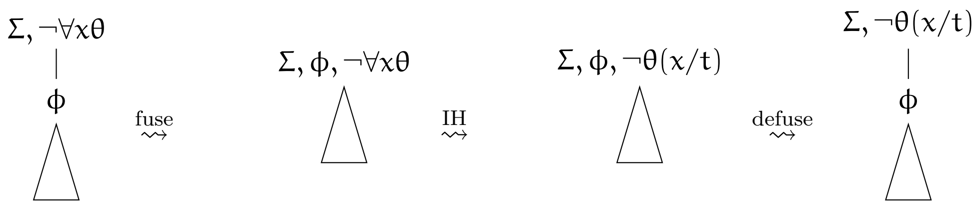

Figure 2.4: Transforming T to T', 1st case of Lemma 11

Next, we have to consider the case where the first level of T is obtained by applying rules 3.c) or d) to a formula $\chi \in \Sigma$. Let ϕ and ψ be the formulas obtained at the first level. If we add ϕ to the set $\Sigma, \neg\forall x\theta$ and erase ϕ and the subtree following ψ together with ψ we get a confutation T_1 for $\Sigma, \phi, \neg\forall x\theta$ such, that $dp(T_1) < dp(T)$ and $T_1 < T$. By the induction hypothesis there is a confutation T_1' of $\Sigma, \phi, \neg\theta(x/t)$, where t is any term, such, that

$$dp(T_1') \leq dp(T_1) < dp(T) \quad \text{and} \quad T_1' \cdot t < T_1 \cdot t < T \cdot t.$$

Similarly, we can get a confutation T_2' of $\Sigma, \psi, \neg\theta(x/t)$ for the same term t used for $\Sigma, \phi, \neg\theta(x/t)$ and such that

$$dp(T_2') \leq dp(T_2) < dp(T) \quad \text{and} \quad T_2' \cdot t < T_2 \cdot t < T \cdot t.$$

We can now get the required confutation T' for $\Sigma, \neg\theta(x/t)$ by starting with ϕ and ψ at the first level and continuing as in T_1' after ϕ and as in T_2' after ψ. Clearly:

$$dp(T') = \max\{dp(T_1'), dp(T_2')\} + 1 \leq \max\{dp(T_1), dp(T_2)\} + 1 = dp(T).$$

Furthermore, since $T \simeq T_1 + T_2$ and $T' \simeq T_1' + T_2'$, we get that

$$T' \simeq T_1' + T_2' < T_1 \cdot t + T_2 \cdot t \simeq T \cdot t.$$

This transformation is depicted in Figure 2.5.

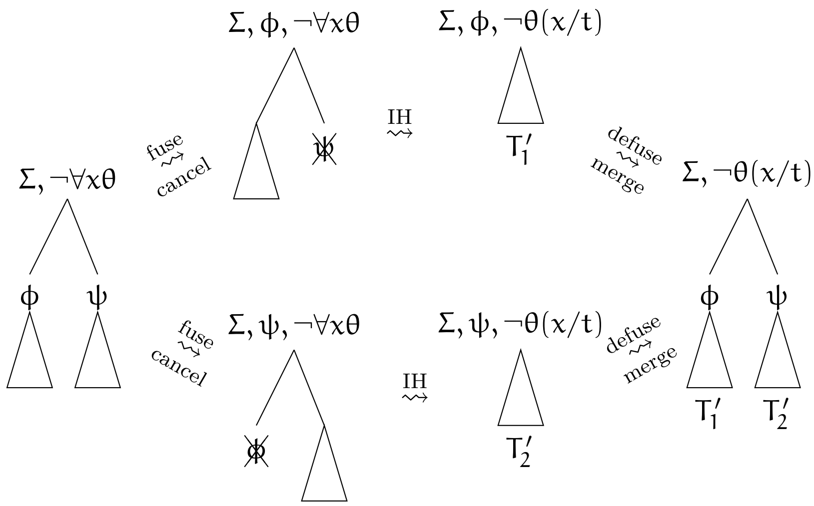

Figure 2.5: Transforming T to T', 2nd case of Lemma 11

For the final case, suppose that the first level of T is obtained by applying rule 3.f) to $\neg\forall x\theta$. Then the formula of the first level would be of the form $\neg\theta(x/y)$ where y is a variable which, by Lemma 5, we may assume does not occur at all in $\neg\theta(x/y)$. If we erase $\neg\theta(x/y)$ from T and add it to the initial set $\Sigma, \neg\forall x\theta$ we get a confutation T_1 for $\Sigma, \neg\theta(x/y), \neg\forall x\theta$ that has smaller depth and size than T i.,e. $dp(T_1) < dp(T)$ and $T_1 < T$.

By the induction hypothesis, for y instead of t, there is a confutation T_2 for:

$$\Sigma, \neg\theta(x/y), \neg\theta(x/y) \quad \text{or} \quad \Sigma, \neg\theta(x/y)$$

with $dp(T_2) \leq dp(T_1)$ and $T_2 < T_1 \cdot y \simeq T_1$.

By Lemma 6 there exists a confutation T' of $\Sigma(y/t), \neg\theta(x/y)(y/t)$, where t is any term, and T' is such that

$$dp(T') \leq dp(T_2) \quad \text{and} \quad T' < T_2 \cdot t.$$

However, the critical variable y does not appear in any of the tableaux T, T_1 and

T_2. Hence, y cannot be free in Σ and so $\Sigma(y/t) = \Sigma$. Furthermore, y was chosen so that it does not occur in θ at all. Hence,

$$\theta(x/y)(y/t) \equiv \theta(x/t)$$

and T' is a confutation for $\Sigma, \theta(x/t)$ and it is such that:

$$dp(T') \leq dp(T_2) \leq dp(T_1) < dp(T)$$

and

$$T' < T_2 \cdot t < T_1 \cdot t < T \cdot t.$$

See also Figure 2.6.

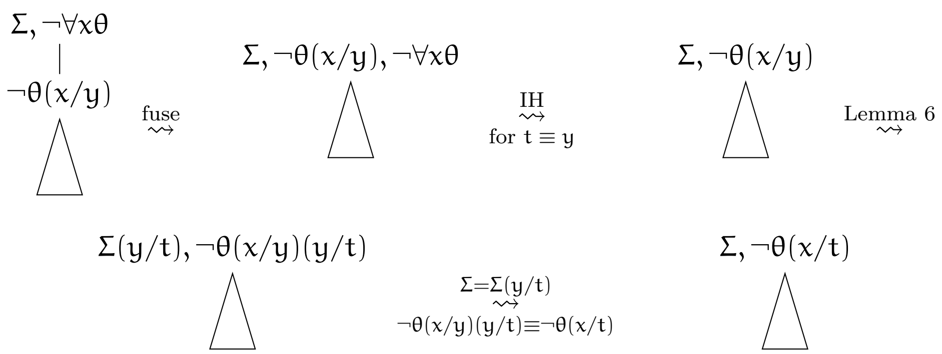

Figure 2.6: Transforming T to T', 3rd case of Lemma 11

□

Lemma 12. *Given confutations* T_1 *and* T_2 *of* $\Sigma + \theta$ *and* $\Sigma + \neg\theta$ *respectively, where* θ *is atomic, there is a confutation* T *of* Σ *such that:*

$$dp(T) \leq dp(T_1) + dp(T_2) \quad \textit{and} \quad T < T_1 + n_{T_1}(\theta)T_2.$$

Proof. If θ does not occur in T_1, then T_1 is a confutation of Σ and the Lemma holds for $T = T_1$. Similarly, if $\neg\theta$ does not occur in T_2, then T_2 is a confutation of Σ and the Lemma holds for $T = T_2$.

Suppose, now, that θ occurs in T_1 and $\neg\theta$ occurs in T_2. The only use that could have been made of θ was to close branches of T_1 in which $\neg\theta$ turned up and the only use that could have been made of $\neg\theta$ was to close branches of T_2 in which θ turned up. Let b be a branch of T_2 that $\neg\theta$ was used to close it. Let b' be the branch obtained by b by deleting $\neg\theta$ from it. T is the tableau obtained by T_1 by replacing each occurrence of θ by the branch b'. Since b' is smaller than b and b is a branch of T_2 we get that:

$$dp(T) \leq dp(T_1) + dp(T_2).$$

Since θ is a atomic, the number of subformulas of θ in T_1, $n_{T_1}(\theta)$, is equal to the to the number of the replacements of θ is T_1 by b'. Hence, T is derived by T_1 by "hanging" $n_{T_1}(\theta)$ branches which are of size at most T_2. Thus,

$$T < T_1 + n_{T_1}(\theta)T_2.$$

□

Lemma 13 (Elimination Lemma)**.** *For any model* M *of* $I\Delta_0 + \exp$*, any recursive theory* Σ *and any formula* θ*, given confutations* T_1 *and* T_2 *of* $\Sigma + \theta$ *and* $\Sigma + \neg\theta$ *respectively, there is a confutation* T *of* Σ *such that:*

$$\mathrm{dp}(T) \leq 2^{2^{f(\theta,T_1,T_2)}} \cdot \max\{\mathrm{dp}(T_1), \mathrm{dp}(T_2)\}$$

where $f(\theta, T_1, T_2) = \mathrm{cpl}(\theta) + n_{T_1}(\theta) + n_{T_2}(\neg\theta)$*.*

Syntactically if $M \models \mathrm{Tabinconseq}(\Sigma + \theta, T_1)$ *and* $M \models \mathrm{Tabinconseq}(\Sigma + \neg\theta, T_2)$*, then*

$$M \models \exists p \mathrm{Tabinconseq}(\Sigma, p) \wedge \mathrm{dp}(p) \leq 2^{2^{\mathrm{cpl}(\theta)+n_{T_1}(\theta)+n_{T_2}(\neg\theta)}} \cdot \max\{\mathrm{dp}(T_1), \mathrm{dp}(T_2)\}$$

Proof. If $\theta \in \Sigma$ or $\neg\theta \in \Sigma$ the Lemma holds trivially. The non trivial case where neither $\theta \in \Sigma$ nor $\neg\theta \in \Sigma$ will be proved by induction on the complexity of θ.

Base If θ is atomic it holds by Lemma 12.

IH Suppose that the hypothesis holds for all sets of sentences Σ and all formulas with complexity less than the complexity of θ.

IS For the inductive step we will consider all the cases for the formula θ.

If $\theta \equiv \neg\phi$, then we have confutations T_1 and T_2 of $\Sigma, \neg\phi$ and $\Sigma, \neg\neg\phi$ respectively. By Lemma 8 there is a confutation T_2' of Σ, ϕ. Hence, by the induction hypothesis, there is a confutation T of Σ such that:

$$\begin{aligned}
\mathrm{dp}(T) &\leq 2^{2^{\mathrm{cpl}(\phi)+n_{T_1}(\neg\phi)+n_{T_2'}(\phi)}} \cdot \max\{\mathrm{dp}(T_1), \mathrm{dp}(T_2')\} \\
&\leq 2^{2^{\mathrm{cpl}(\phi)+n_{T_1}(\neg\phi)+n_{T_2}(\neg\neg\phi)}} \cdot \max\{\mathrm{dp}(T_1), \mathrm{dp}(T_2)\} \\
&\leq 2^{2^{\mathrm{cpl}(\theta)+n_{T_1}(\theta)+n_{T_2}(\neg\theta)}} \cdot \max\{\mathrm{dp}(T_1), \mathrm{dp}(T_2)\}
\end{aligned}$$

since by Lemma 8 $\mathrm{dp}(T_2') \leq \mathrm{dp}(T_2)$ and

$$n_{T_2'}(\phi) \leq n_{T_2}(\neg\neg\phi) = n_{T_2}(\neg\theta).$$

If $\theta \equiv \phi \to \psi$, we have confutations T_1 and T_2 of $\Sigma, \phi \to \psi$ and $\Sigma, \neg(\phi \to \psi)$

respectively, see also Figure 2.7. By Lemmas 9 and 10 there are:

(2.1)	a confutation S_1 of $\Sigma, \neg\phi$	such that $\mathrm{dp}(S_1) \leq \mathrm{dp}(T_1)$,
(2.2)	a confutation S_2 of Σ, ψ	such that $\mathrm{dp}(S_2) \leq \mathrm{dp}(T_1)$ and
(2.3)	a confutation S_3 of $\Sigma, \phi, \neg\psi$	such that $\mathrm{dp}(S_3) \leq \mathrm{dp}(T_2)$.

By Lemma 7 and (2.2) we get

(2.4) a confutation S_2' of Σ, ϕ, ψ such that $\mathrm{dp}(S_2') = \mathrm{dp}(S_2) \leq \mathrm{dp}(T_1)$;

by (2.3) and (2.4) and the induction hypothesis for Σ, ϕ we get a confutation S_4 of Σ, ϕ such that

$$\mathrm{dp}(S_4) \leq 2^{2^{\mathrm{cpl}(\psi)+n_{S_2'}(\psi)+n_{S_3}(\neg\psi)}} \cdot \max\{\mathrm{dp}(T_1), \mathrm{dp}(T_2)\} \tag{2.5}$$

by (2.1) and (2.5) and the induction hypothesis for Σ we get a confutation T of Σ such that:

$$\begin{aligned}
\mathrm{dp}(T) &\leq 2^{2^{\mathrm{cpl}(\phi)+n_{S_4}(\phi)+n_{S_1}(\neg\phi)}} \cdot \max\{\mathrm{dp}(T_1), \mathrm{dp}(S_4)\} \\
&\leq 2^{2^{\mathrm{cpl}(\phi)+n_{S_4}(\phi)+n_{S_1}(\neg\phi)}} \cdot \\
&\qquad \max\left\{\mathrm{dp}(T_1), 2^{2^{\mathrm{cpl}(\psi)+n_{S_2'}(\psi)+n_{S_3}(\neg\psi)}} \cdot \max\{\mathrm{dp}(T_1), \mathrm{dp}(T_2)\}\right\} \\
&\leq 2^{2^{\mathrm{cpl}(\phi)+n_{S_4}(\phi)+n_{S_1}(\neg\phi)}} \cdot 2^{2^{\mathrm{cpl}(\psi)+n_{S_2'}(\psi)+n_{S_3}(\neg\psi)}} \cdot \max\{\mathrm{dp}(T_1), \mathrm{dp}(T_2)\} \\
&\leq 2^{2^{\mathrm{cpl}(\phi)+n_{S_4}(\phi)+n_{S_1}(\neg\phi)}+2^{\mathrm{cpl}(\psi)+n_{S_2'}(\psi)+n_{S_3}(\neg\psi)}} \cdot \max\{\mathrm{dp}(T_1), \mathrm{dp}(T_2)\}
\end{aligned}$$

$\mathrm{cpl}(\phi) \leq \max(\mathrm{cpl}(\phi), \mathrm{cpl}(\psi))$ and $\mathrm{cpl}(\psi) \leq \max(\mathrm{cpl}(\phi), \mathrm{cpl}(\psi))$, therefore

$$\mathrm{dp}(T) \leq 2^{2^{\max(\mathrm{cpl}(\phi),\mathrm{cpl}(\psi))+n_{S_4}(\phi)+n_{S_1}(\neg\phi)}+2^{\max(\mathrm{cpl}(\phi),\mathrm{cpl}(\psi))+n_{S_2'}(\psi)+n_{S_3}(\neg\psi)}} \cdot \max\{\mathrm{dp}(T_1), \mathrm{dp}(T_2)\}$$

$n_{S_2'}(\psi) + n_{S_3}(\neg\psi) \leq n_{T_1}(\theta) + n_{T_2}(\neg\theta)$, since the number of subformulas of ψ in S_2' is less than the number of subformulas of θ in T_1. Furthermore, the number of subformulas of $\neg\psi$ in S_3 is less than the number of subformulas of $\neg\theta$ in T_2. Also, $n_{S_4}(\phi)+n_{S_1}(\neg\phi) \leq n_{T_1}(\theta)+n_{T_2}(\neg\theta)$, since $n_{S_4}(\phi) \leq n_{S_3}(\phi) \leq n_{S_2}(\neg\theta)$ (notice that we extended only Σ, ψ with ϕ, so ϕ is not added to S_4). Finally, $n_{T_1}(\neg\phi) \leq n_{S_1}(\theta)$. Hence

$$\begin{aligned}
dp(T) &\leq 2^{2^{\max(cpl(\phi),cpl(\psi))+n_{T_1}(\theta)+n_{T_2}(\neg\theta)}+2^{\max(cpl(\phi),cpl(\psi))+n_{T_1}(\theta)+n_{T_2}(\neg\theta)}} \cdot \\
&\qquad \max\{dp(T_1), dp(T_2)\} \\
&\leq 2^{2\cdot 2^{\max(cpl(\phi),cpl(\psi))+n_{T_1}(\theta)+n_{T_2}(\neg\theta)}} \cdot \max\{dp(T_1), dp(T_2)\} \\
&\leq 2^{2^{\max(cpl(\phi),cpl(\psi))+n_{T_1}(\theta)+n_{T_2}(\neg\theta)}} \cdot \max\{dp(T_1), dp(T_2)\}
\end{aligned}$$

and since $cpl(\theta) = \max(cpl(\phi), cpl(\psi)) + 1$

$$dp(T) \leq 2^{2^{cpl(\theta)+n_{T_1}(\theta)+n_{T_2}(\neg\theta)}} \cdot \max\{dp(T_1), dp(T_2)\}$$

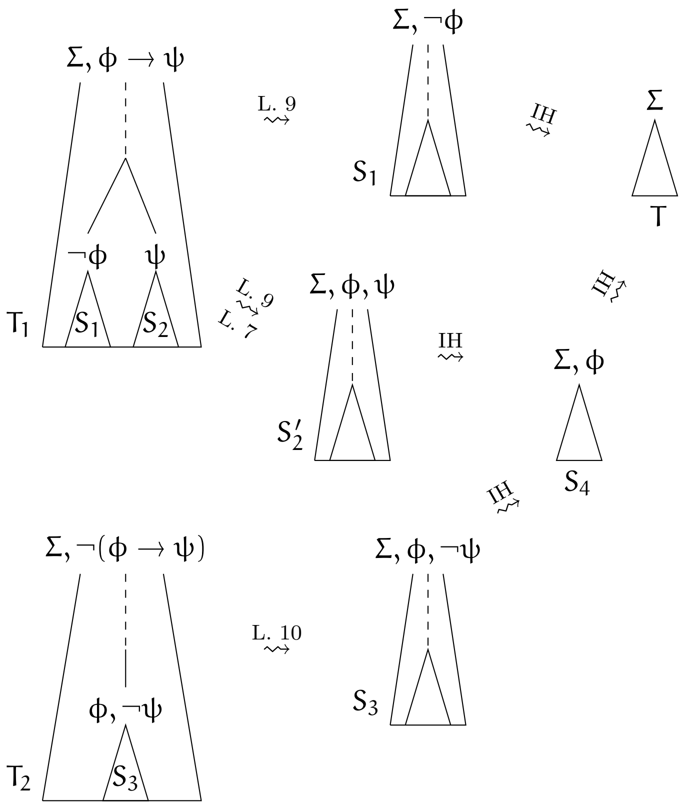

Figure 2.7: Elimination Lemma 13 implication

For the last case, suppose that $\theta \equiv \forall x\phi$ and we have confutations T_1 and T_2 of $\Sigma, \forall x\phi$ and $\Sigma, \neg\forall x\phi$ respectively.

Suppose that $\phi(x/t)$ appears in T_1 for some term t (see also Figure 2.8), if not, then T_1 is a confutation for Σ and the hypothesis holds trivially. Take a branch such that $\phi(x/t_1)$, for some term t_1, appears at the deepest possible level and let $\phi_{11}, \phi_{12}, \dots, \phi_{1p}$ be the formulas preceding $\phi(x/t_1)$ on the chosen branch. Clearly the subtree R_1 following $\phi(x/t_1)$ is a confutation for:

$$\Sigma, \phi_{11}, \dots, \phi_{1p}, \phi(x/t_1), \quad dp(R_1) \leq dp(T_1)$$

and the number of subformulas for ϕ in R_1 is less than the number of subformulas of θ in T_1. Hence,

$$n_{R_1}(\phi) \leq n_{T_1}(\theta).$$

Since T_2 is a confutation of $\Sigma, \neg\forall x\phi$ by Lemma 11 there is a confutation T_2' of $\Sigma, \neg\phi(x/t_1)$, for the same term t_1 used above and T_2' is such that $dp(T_2') \leq dp(T_2)$ and

$$n_{T_2'}(\neg\phi) \leq n_{T_2}(\neg\theta).$$

By Lemma 7 there is a confutation R_2 of: $\quad \Sigma, \phi_{11}, \dots, \phi_{1p}, \neg\phi(x/t_1) \quad$ such that

$$dp(R_2) \leq dp(T_2') \leq dp(T_2) \quad \text{and} \quad n_{R_2}(\neg\phi) = n_{T_2'}(\neg\phi) \leq n_{T_2}(\neg\theta).$$

Since the induction hypothesis holds for all sets of formulas there is a confutation S_1 of $\Sigma, \phi_{11}, \dots, \phi_{1p}$ such that:

$$\begin{aligned} dp(S_1) &\leq 2^{2^{cpl(\phi)+n_{R_1}(\phi)+n_{R_2}(\neg\phi)}} \cdot \max\{dp(R_1), dp(R_2)\} \\ &\leq 2^{2^{cpl(\phi)+n_{R_1}(\phi)+n_{R_2}(\neg\phi)}} \cdot \max\{dp(T_1), dp(T_2)\}. \end{aligned}$$

We repeat the process for all applications of rule 3.e) for $\forall x\phi$ at level $p+1$ and so the depth of each derived confutation S_{ip} is at most $dp(S_1)$.

We can simultaneously eliminate all $\phi(x/t_i)$ from each branch of T_1 at level $p+1$, by replacing them with the respective confutation S_{ip}. As we've seen above the branch $\phi_{i1}, \dots, \phi_{ip}$ followed by S_{ip} is closed. Let C_p be the tableau obtained by "hanging" S_{ip} after ϕ_{ip} in T_1 as described previously. Then:

$$\begin{aligned} dp(C_p) &\leq dp(S_1) + dp(T_1) \\ &\leq 2^{2^{cpl(\phi)+n_{R_1}(\phi)+n_{R_2}(\neg\phi)}} \cdot \max\{dp(T_1), dp(T_2)\} + dp(T_1) \\ &\leq 2 \cdot 2^{2^{cpl(\phi)+n_{R_1}(\phi)+n_{R_2}(\neg\phi)}} \cdot \max\{dp(T_1), dp(T_2)\} \\ &\leq 2^{2^{cpl(\phi)+n_{R_1}(\phi)+n_{R_2}(\neg\phi)}+1} \cdot \max\{dp(T_1), dp(T_2)\}. \end{aligned}$$

Hence, if we set $R_1 = R_{p+1}$,

$$\text{(2.6)} \qquad dp(C_p) \leq 2^{2^{cpl(\phi)+n_{R_{p+1}}(\phi)+n_{R_2}(\neg\phi)}+1} \cdot \max\{dp(T_1), dp(T_2)\}.$$

Repeating the process for level p in C_p, if an application of rule 3.f) for $\forall x\phi$ exists at level $p-1$, we get a confutation C_{p-1} such that

$$\begin{aligned} dp(C_{p-1}) &\leq dp(S_2) + dp(C_p) \\ &\leq 2^{2^{cpl(\phi)+n_{R_p}(\phi)+n_{R_2}(\neg\phi)}} \cdot \max\{dp(C_p), dp(T_2)\} + dp(C_p) \\ &\leq 2 \cdot 2^{2^{cpl(\phi)+n_{R_p}(\phi)+n_{R_2}(\neg\phi)}} \cdot \max\{dp(C_p), dp(T_2)\} \\ &\leq 2^{2^{cpl(\phi)+n_{R_p}(\phi)+n_{R_2}(\neg\phi)}+1} \cdot \max\{dp(C_p), dp(T_2)\} \end{aligned}$$

by (2.6)

$$\begin{aligned} &\leq 2^{2^{cpl(\phi)+n_{R_p}(\phi)+n_{R_2}(\neg\phi)}+1} \\ &\quad \cdot \max\left\{2^{2^{cpl(\phi)+n_{R_{p+1}}(\phi)+n_{R_2}(\neg\phi)}+1} \cdot \max\{dp(T_1), dp(T_2)\}, dp(T_2)\right\} \end{aligned}$$

since $\max\{\alpha \cdot \beta, \gamma\} \leq \alpha \cdot \max\{\beta, \gamma\}$ and $\max\{\max\{\alpha, \beta, \}, \beta\} = \max\{\alpha, \beta\}$

$$\begin{aligned} &\leq 2^{2^{cpl(\phi)+n_{R_p}(\phi)+n_{R_2}(\neg\phi)}+1} \cdot 2^{2^{cpl(\phi)+n_{R_{p+1}}(\phi)+n_{R_2}(\neg\phi)}+1} \cdot \max\{dp(T_1), dp(T_2)\} \\ &\leq 2^{2^{cpl(\phi)+n_{R_p}(\phi)+n_{R_2}(\neg\phi)}+1+2^{cpl(\phi)+n_{R_{p+1}}(\phi)+n_{R_2}(\neg\phi)}+1} \cdot \max\{dp(T_1), dp(T_2)\} \end{aligned}$$

and $2^{2^n+1} \leq 2^{2^{n+1}}$, hence

$$\leq 2^{2^{cpl(\phi)+n_{R_p}(\phi)+n_{R_2}(\neg\phi)+1}+2^{cpl(\phi)+n_{R_{p+1}}(\phi)+n_{R_2}(\neg\phi)+1}} \cdot \max\{dp(T_1), dp(T_2)\}$$

but $cpl(\phi) + 1 = cpl(\theta)$, so

$$\begin{aligned} &= 2^{2^{cpl(\theta)+n_{R_p}(\phi)+n_{R_2}(\neg\phi)}+2^{cpl(\theta)+n_{R_{p+1}}(\phi)+n_{R_2}(\neg\phi)}} \cdot \max\{dp(T_1), dp(T_2)\} \\ &= 2^{2^{cpl(\theta)+n_{R_2}(\neg\phi)}\left(2^{n_{R_p}(\phi)}+2^{n_{R_{p+1}}(\phi)}\right)} \cdot \max\{dp(T_1), dp(T_2)\}. \end{aligned}$$

Hence,

$$dp(C_{p-1}) \leq 2^{2^{cpl(\theta)+n_{R_2}(\neg\phi)}\left(2^{n_{R_p}(\phi)}+2^{n_{R_{p+1}}(\phi)}\right)} \cdot \max\{dp(T_1), dp(T_2)\}.$$

Continuing this way after at most p steps we get a confutation $T = C_0$ for Σ such that:

$$dp(T) \leq 2^{2^{cpl(\theta)+n_{R_2}(\neg\phi)} \cdot \left(2^{n_{R_1}(\phi)}+\dots+2^{n_{R_p}(\phi)}+2^{n_{R_{p+1}}(\phi)}\right)} \cdot \max\{dp(T_1), dp(T_2)\}$$

since the number of subformulas of ϕ is decreasing when we go from C_{i+1} to C_i we have that $2^{n_{R_1}(\phi)} + \dots + 2^{n_{R_p}(\phi)} + 2^{n_{R_{p+1}}(\phi)} \leq 2^1 + \dots + 2^{n_{R_p}(\phi)} + 2^{n_{R_{p+1}}(\phi)}$, hence

$$dp(T) \leq 2^{2^{cpl(\theta)+n_{R_2}(\neg\phi)} \cdot \left(2^1+\dots+2^{n_{R_p}(\phi)}+2^{n_{R_{p+1}}(\phi)}\right)} \cdot \max\{dp(T_1), dp(T_2)\}$$

and $2^1 + \dots + 2^{n_{R_p}(\phi)} + 2^{n_{R_{p+1}}(\phi)} = 2 \cdot \frac{2^{n_{R_{p+1}}(\phi)}-1}{2-1}$ and $R_{p+1} = R_1$

$$= 2^{2^{cpl(\theta)+n_{R_2}(\neg\phi)} \cdot \left(2^{n_{R_1}(\phi)+1}-2\right)} \cdot \max\{dp(T_1), dp(T_2)\}$$

$n_{R_1}(\phi) + 1 \leq n_{T_1}(\theta)$ and $n_{R_2}(\neg\phi) \leq n_{T_2}(\neg\theta)$

$$= 2^{2^{cpl(\theta)+n_{T_2}(\neg\theta)} \cdot \left(2^{n_{T_1}(\theta)}-2\right)} \cdot \max\{dp(T_1), dp(T_2)\}$$
$$\leq 2^{2^{cpl(\theta)+n_{T_1}(\theta)+n_{T_2}(\neg\theta)}} \cdot \max\{dp(T_1), dp(T_2)\}.$$

Hence,

$$dp(T) \leq 2^{2^{cpl(\theta)+n_{T_1}(\theta)+n_{T_2}(\neg\theta)}} \cdot \max\{dp(T_1), dp(T_2)\}.$$

as required and the induction is completed.

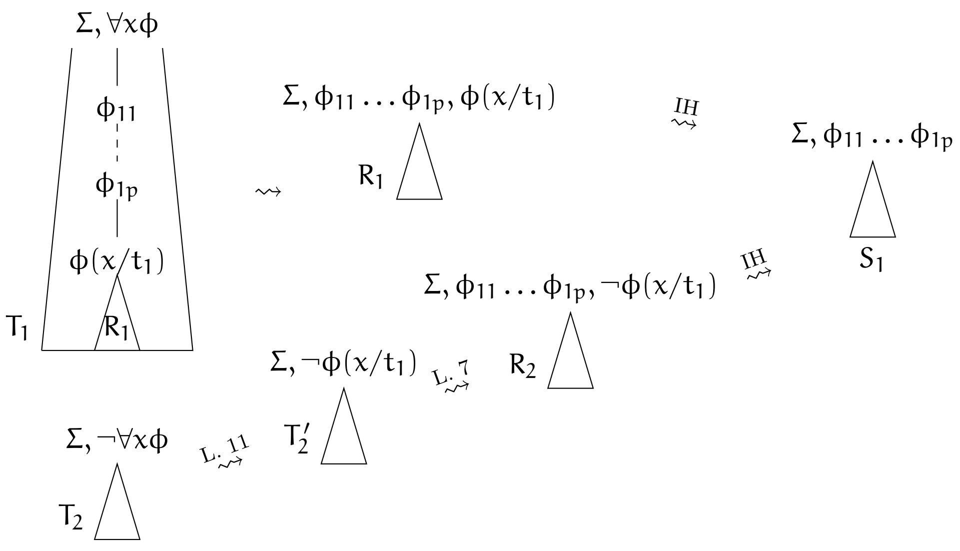

Figure 2.8: Elimination Lemma 13 quantifier

□

The following lemma is trivial semantically. However, it is useful and it simplifies the merging of two tableau proofs.

Lemma 14. *Let* M *be a model of* $I\Delta_0 + \Omega_1$, Σ *is a set of formulas. Then for all* ϕ

$$M \models \neg\text{Tabcon}(\Sigma) \quad \Longleftrightarrow \quad M \models \neg\text{Tabcon}(\Sigma, \phi \to \phi)$$

Proof. If $\phi \to \phi \in \Sigma$, then the lemma holds.

Suppose that $\phi \to \phi \notin \Sigma$ and that T is the tableau proof of a contradiction from $\Sigma, \phi \to \phi$. We will show by induction, on the number of applications of rule 3(c) for $\phi \to \phi$ in T, that

$$M \models \neg\text{Tabcon}(\Sigma, \phi \to \phi) \quad \Rightarrow \quad M \models \neg\text{Tabcon}(\Sigma)$$

Base If rule 3(c) is not used for $\phi \to \phi$ in T, then T is a confutation of Σ.

I.H. If rule 3(c) is used less than $n + 1$ times for $\phi \to \phi$ in T and T confutes $\Sigma, \phi \to \phi$, then there is a tableau T' that confutes Σ.

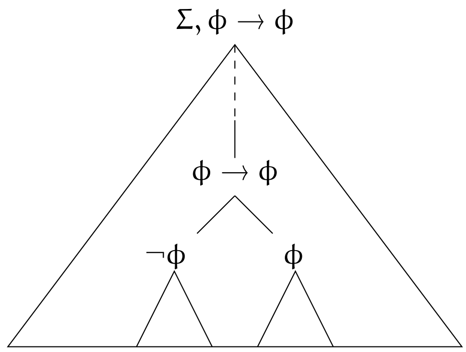

Figure 2.9: Confutation of $\Sigma, \phi \to \phi$

I.S.

Suppose that rule 3(c) is used $n + 1$ times for $\phi \to \phi$ in T and T confutes $\Sigma, \phi \to \phi$. In T find a node N in the deepest level at which $\phi \to \phi$ is used. Let Σ' be the set of formulas that occur in the node except from $\phi \to \phi$ and let T_1 and T_2 be the tableau we get when we apply the rule 3(c). Rule 3(c) is not used again for $\phi \to \phi$ in T_1 and T_2. Let T_1' and T_2' be the tableaux we get from T_1 and T_2 respectively, if we erase every $\phi \to \phi$ from them. Both T_1' and T_2' remain closed when we eliminate $\phi \to \phi$ from T_1 and T_2 because rule 3(c) is not used for $\phi \to \phi$. Hence, T_1' is a confutation for $\Sigma' + \neg\phi$ and T_2' is a confutation for $\Sigma' + \phi$ i.e.

$$M \models \neg\text{Tabcon}(\Sigma' + \neg\phi) \quad \text{and} \quad M \models \neg\text{Tabcon}(\Sigma' + \phi).$$

By the Elimination Lemma 13

$$M \models \neg\text{Tabcon}(\Sigma').$$

Hence, there is a tableau T_3' which confutes Σ'. Add $\phi \to \phi$ to every node of T_3' to get T_3. Replace node N and the tree under node N in T to get T''. Then T'' is a confutation of $\Sigma, \phi \to \phi$ and rule 3(c) is used n times for $\phi \to \phi$ and by the induction hypothesis, there is a tableau T' that confutes Σ. □

Proposition 1. *Let M be a model of bounded induction with exponentiation. Let T be a recursive theory in the sense of M and let θ be a sentence such that $T \vdash \theta$. Then*

$$M \models \text{Tabcon}(T) \quad \Longleftrightarrow \quad M \models \text{Tabcon}(T + \theta).$$

Proof. If T is inconsistent, the equivalence is trivially true.

Suppose, towards a contradiction, that

$$M \models \text{Tabcon}(T) \tag{2.7}$$

and

$$M \models \neg\text{Tabcon}(T + \theta). \tag{2.8}$$

Since $T \vdash \theta$ we get that $T + \neg\theta$ is inconsistent hence

$$M \models \neg\text{Tabcon}(T + \neg\theta). \tag{2.9}$$

By (2.8), (2.9) and the Elimination Lemma 13

$$M \models \neg\text{Tabcon}(T)$$

which contradicts (2.7). Hence,

$$M \models \text{Tabcon}(T + \theta).$$

Conversely, if

$$M \models \text{Tabcon}(T + \theta),$$

then it cannot be the case that

$$M \models \neg\text{Tabcon}(T)$$

because then, by Lemma 7, we would have that the superset $T + \theta$ is tableau inconsistent contradicting the hypothesis.

□

3 End extensions of countable models of weak arithmetic

3.1 End extensions

As it was the case with other fundamental theorems that were known to hold for PA, there were attempts to miniaturize the results presented at the end of chapter 1, i.e. prove their counterparts for fragments of PA. Such a result, described as "a mild refinement of the arithmetized completeness theorem", was proved by J. Paris ([13]) and is essentially the following.

Theorem 7. *Let M be a model of $B\Sigma_n$, $n \geq 2$, and $T \supseteq I\Delta_0$ be a theory Δ_{n-1} definable in M such that $M \models \mathsf{Con}(T)$. Then there exists a model K of T which is Δ_n definable in M and M is isomorphic to a proper initial segment of K.*

By applying this result, the same author showed that (see Theorems 2 and 5 in [13])

(i) every model of $B\Sigma_n$, $n \geq 2$, has a proper end extension $J \models B\Sigma_n$

(ii) every model of $I\Sigma_n$, $n \geq 2$, has a proper end extension $J \models I\Sigma_n$

(in fact, the author proved stronger results, but we are restricting our attention to versions relevant to our work).

In order to obtain a model K that has a nicer relationship to M in theorem 7 we need an extra assumption on M as the following result obtained by J. Paris and L. Kirby in their classic paper [15] shows.

Theorem 8. *For any countable model M of $I\Delta_0$ and $n \geq 2$,*

(a) $M \models B\Sigma_n \Leftrightarrow$ *there exists* $K \models I\Delta_0$ *such that* $M \prec_{n,e} K$.

(b) if M has a proper Σ_1 elementary end extension, then $M \models B\Sigma_2$.

Note that the previous result is also a miniaturization of the following well-known theorem that was proved by R. MacDowell and E. Specker (see [11]).

Theorem 9. *Every nonstandard model of PA has a proper elementary end extension.*

Concerning part (a) of the previous Theorem, let us note that the proof of ($\Leftarrow$) does not rely on the countability of M, while the proof of the converse implication relies heavily on this assumption. Despite attempts to show that *any* model M of $B\Sigma_n$, $n \geq 2$, is extendable to a model K of $I\Delta_0$ such that $M \prec_{n,e} K$, this question still remains open (see, e.g., [3] and [4]).

In view of Theorem 8, a natural question that arises is whether (a) holds for $n = 0, 1$. Concerning the implication ($\Leftarrow$), it holds for both $n = 0$ and $n = 1$, by the fact that $B\Sigma_0 \Leftrightarrow B\Sigma_1$ and the fact that if $M \subset_e K \models I\Delta_0$, then $M \models B\Sigma_1$. Concerning the converse implication, it does not hold for $n = 1$, by part (b) of Theorem 8 and the existence of models of the theory $B\Sigma_1 + \neg B\Sigma_2$ (see Theorem 1 on page 6). Therefore, the only remaining question is

Problem 1. Is every countable model M of $B\Sigma_1$ extendable to a model K of $I\Delta_0$ such that $M \prec_{0,e} K$?

Since for any structures M, K for $\mathcal{LA}$, $M \subset_e K$ implies $M \prec_0 K$ (see Theorem 2.7 in [9]), it follows that Problem 1 is equivalent to

Problem 2. Is every countable model M of $B\Sigma_1$ extendable to a model K of $I\Delta_0$ such that $M \subset_e K$?

Problem 2 is considered one of the main problems concerning fragments of PA (see "Fundamental problem F" in [5]). This problem was examined exhaustively by J. Paris and A. Wilkie in [18]. These authors introduced the notion of Γ-*fullness*, Γ being a set of sentences, and showed that this problem has a positive answer, provided that M is $I\Delta_0$-full, i.e. they proved the following result.

Theorem 10. *For any countable model M of $B\Sigma_1$, if M is $I\Delta_0$-full, then there exists $K \models I\Delta_0$ such that $M \subset_e K$.*

Moreover, Paris and Wilkie proved that certain natural conditions on M imply $I\Delta_0$-fullness. In order to be able to state their result precisely, we need to recall the definition of a notion and some notation.

Definition 23. Let M be a structure for $\mathcal{LA}$. M is said to be *short Π_1-recursively saturated*, if whenever $\Phi = \{x < a \wedge \varphi_i(x, \vec{b}) : i \in \mathbb{N}\}$ is a recursive set of Π_1 formulas (with parameters from M) finitely satisfiable in M, then Φ is satisfiable in M.

Notation 1. $I\Delta_0 \vdash \neg\Delta_0 H$ stands for the hypothesis that the Δ_0 hierarchy provably collapses in $I\Delta_0$, i.e. there is a fixed n such that for any formula $\theta \in \Delta_0$ there is a formula $\chi \in \Delta_0$ in prenex normal form with at most n alternations of bounded quantifiers such that $I\Delta_0 \vdash \theta \leftrightarrow \chi$.

Now we can state the result in [18] which concerns sufficient conditions for a model of $B\Sigma_1$ to be $I\Delta_0$-full.

Theorem 11. *For any countable nonstandard model M of $B\Sigma_1$, each of the following conditions imply that M is $I\Delta_0$-full:*

(1) *M is short Π_1-recursively saturated*

(2) $M \models exp$

(3) $I\Delta_0 \vdash \neg\Delta_0 H$ *and* $\exists \mathbb{N} < \gamma \in M, M \models \forall x \exists y (y = x^\gamma)$

(4) $I\Delta_0 \vdash \neg\Delta_0 H$ *and* $\exists a \in M \forall b \in M \exists n \in \mathbb{N}, b \leq a^n$

(5) $I\Delta_0 \vdash \neg\Delta_0 H$ *and* $\exists M \subset_e K \models B\Sigma_1$.

However interesting the notion of Γ-fullness may be, it is highly technical and, therefore, not very intuitive. For this reason, we found it worthwhile to look for an alternative approach, which would avoid use of this notion and would be easier to grasp. Actually, the answer lies in a remark in [18], made just after the end of the proof of Theorem 5(2) (page 154), which reads as follows.

Remark. A direct proof that any countable model of $I\Delta_0 + B\Sigma_1$ *which is closed under exponentiation has a proper end extension to a model of* $I\Delta_0$ *may be obtained by mimicking the proof of Theorem 4 but with "Semantic Tableau consistency of* Γ*" in place of "*Γ*-full" and adding a new constant symbol* $\pi > M$ *to ensure that the end extension is proper.*

This chapter is dedicated to showing how one can apply variants of the ACT to prove in an alternative way that if a countable nonstandard model M of $B\Sigma_1$ satisfies any of conditions (1) – (4) of Theorem 11, then it is properly end extendable to a model of $I\Delta_0$. Note that working with condition (5) makes no sense in our context, as it presupposes the proper end extendability of M. Although we have obtained no new results, we feel our undertaking is interesting from a methodological point of view, as it connects Problem 2 with the approach suggested by the ACT.

3.2 Exponentiation and end extensions

In this section, we will prove that every countable model of $B\Sigma_1 + \exp$ has a proper end extension satisfying $I\Delta_o$. We begin with the definition of the diagram of a model.

Let $\mathcal{A}$ be a model for a language $\mathcal{L}$. Let:

$$\mathcal{L}_A = \mathcal{L} \cup \{c_a \mid a \in A\}$$

where, for each $a \in A$, c_a is a new constant symbol. So the new language $\mathcal{L}_A$ is an expansion of $\mathcal{L}$. The model $\mathcal{A}$ can be expanded to a model for $\mathcal{L}_A$, denoted by $\mathcal{A}_A$, where each new constant c_a is interpreted by the element a.

Definition 24. The *diagram* of $\mathcal{A}$, denoted by $\Delta_{\mathcal{A}}$ or Δ when there is no doubt about the model $\mathcal{A}$, is the set of all atomic sentences and negations of atomic sentences of $\mathcal{L}_A$ which hold in the model $\mathcal{A}_A$.

$$\Delta_{\mathcal{A}} = \{\phi \in S_{\mathcal{L}_A} \mid \mathcal{A}_A \models \phi, \quad \phi \equiv \theta \quad \text{or} \quad \phi \equiv \neg\theta, \quad \theta \in AS_{\mathcal{L}_A}\}$$

In order to have a simpler notation for the expanded language, for the rest of this chapter we will denote $\mathcal{L}\mathcal{A}$ by $\mathcal{L}$.

We will now start with a couple of lemmas that provide the means to implement the idea of the Arithmetized Completeness Theorem.

In [17] lemma 8.10 states that if $r \in \omega$ and σ is any Σ_2 sentence, then $I\Delta_0 + \sigma + \exp$ proves the tableau consistency of $I\Delta_0 + \sigma + \Omega_r$. We will modify this proof in order to get the tableau consistency of $I\Delta_0 + \Delta + \bar{c} > M$, which will later be the base theory in the completeness argument.

Lemma 15. *Let* M *be a countable model for* $\mathcal{L}^*_M = \mathcal{L}_M \cup \{\bar{c}\}$ *where* $\bar{c}$ *is a new constant symbol. If* M *is a model of* $I\Delta_0 + \exp$*, then* M *is a model of the tableau consistency of the* Π_1 *theory* $I\Delta_0 + \Delta + \bar{c} > M$*, where* $\bar{c} > M$ *denotes the set of* $\mathcal{L}^*_M$ *sentences* $\{\bar{c} > c_a \mid a \in M\}$.

Proof. Suppose, towards a contradiction, that $M \models \neg\,\mathrm{Tabcon}(I\Delta_0 + \Delta + \bar{c} > M)$. Since $M \models \exp$ for any b, c, $t \in M$, there is a Δ_0 formula $\mathrm{Sat}_{b,c}(x, y)$ asserting that "If $x = \ulcorner\theta(x_1, \ldots, x_t)\urcorner$ is any $\mathcal{L}^*_M$ formula with $x < c$ and $y = \langle b_1, \ldots, b_t\rangle$ is a sequence of elements less than b, then $\theta(b_1, \ldots, b_t)$ is true in M" (see paragraph 1.4).

Suppose $\Gamma_0, \Gamma_1, \ldots, \Gamma_s$ is a tableau proof (in M) from $I\Delta_0+\Delta+\bar{c} > M$ of a contradiction. Let $c \in M$ be larger than the Gödel number of any formula occurring in any Γ_i, let $d - 1 \in M$ be larger that the Gödel number of any constant occurring in any Γ_i and set $b > d^{\log^{s+1} c}$.

For all $i < s$ and $X \in \Gamma_i$ we define in M a function $F_{i,X}$ with domain the set of variables occurring in formulas in X and range bounded by b, by recursion on i as follows:

- If $i = 0$, $F_{i,X}$ is empty
- For x a variable in (some formula in) $Y \in \Gamma_{i+1}$ pick $X \in \Gamma_i$ such that Y is derived from X by one of the tableau rules.
  - If x appears in X, set $F_{i+1,Y}(x) = F_{i,X}(x)$.
  - If $Y = X \cup \{\neg\theta(x, x_1, \ldots, x_p, \bar{c}, \vec{c}_{\vec{a}})\}$ where $\neg\forall x\theta(x, x_1, \ldots, x_p, \bar{c}, \vec{c}_{\vec{a}}) \in X$, set

    $$F_{i+1,Y}(x) = \begin{cases} m_0 & \text{,if } M \models m_0 = (\mu m < b)[\mathrm{Sat}_{b,c}(e, \langle m, \vec{F}_{i,X}(\vec{x}, d, \vec{a})\rangle)] \\ 0 & \text{,otherwise,} \end{cases}$$

    where

    $$e = \ulcorner\neg\theta(x, \vec{x}, z, \vec{y})\urcorner, \quad \vec{F}_{i,X}(\vec{x}) = F_{i,X}(x_1), \ldots, F_{i,X}(x_p)$$

    and $(\mu m < b)[\phi(m)]$ is the least $m \in M$ which is less than b and it is such that $M \models \phi(m)$.
  - In all other cases set $F_{i+1,Y}(x) = 0$.

Lemma 16. *For all* $i \leq s$ *and for all* $X \in \Gamma_i$*,*

$$\mathrm{Range}(F_{i,X}) \subseteq \{m \in M \mid M \models m < d^{\log^{i+1} c}\}.$$

Proof of lemma. By induction on i.

Base It holds for all $X \in \Gamma_0$ trivially since $F_{0,X}$ is empty.

IH For all $X \in \Gamma_i$, $\mathit{Range}(F_{i,X}) \subseteq \{m \in M \mid M \models m < d^{\log^i c}\}$.

IS For each $X \in \Gamma_{i+1}$ that is obtained from $Y \in \Gamma_i$ using rules 3.(a)-(d) no new variables are introduced. Rule 3.(e) may introduce new variables which will be evaluated by 0. Finally, when rule 3.(f) is used to eliminate an unbounded quantifier the worst case would be to have $\log c$ multiplications. Each multiplication factor is less than d. Hence, in the worst case we substitute i times and we get that the number assigned to the fresh variable y would have to be less than $d^{\log^i c}$.

□

Lemma 17. *For all $i \leq s$ there is an $X \in \Gamma_i$ such that for all formulas:*

$$\theta(x_1, \ldots, x_p, c_{\alpha_1}, \ldots, c_{\alpha_q})$$

in X which are either Σ_1 or Π_1

$$M \models \mathrm{Sat}_{b,c}(\ulcorner \theta(x_1, \ldots, x_p, \vec{c}) \urcorner, \langle F_{i,X}(x_1), \ldots, F_{i,X}(x_p) \rangle).$$

Proof of lemma. By induction on i.

Base It holds for Γ_0 trivially since M satisfies $I\Delta_0$, M satisfies its diagram Δ and for every valuation σ for M such that $\sigma(\bar{c}) >^M \sigma(c_\alpha)$ for all c_α used in the tableau proof M satisfies $\bar{c} > M$.

IH Suppose that for all $i < s$ there is an $X \in \Gamma_i$ such that for all formulas θ in X which are either Σ_1 or Π_1

$$M \models \mathrm{Sat}_{b,c}(\ulcorner \theta(x_1, \ldots, x_p, \vec{c}) \urcorner, \langle F_{i,X}(x_1), \ldots, F_{i,X}(x_p) \rangle).$$

IS We will show that for $i + 1 \leq s$ there is an $X \in \Gamma_{i+1}$ such that for all formulas θ in X which are either Σ_1 or Π_1

$$M \models \mathrm{Sat}_{b,c}(\ulcorner \theta(x_1, \ldots, x_p, \vec{c}) \urcorner, \langle F_{i+1,X}(x_1), \ldots, F_{i+1,X}(x_p) \rangle).$$

By the induction hypothesis $i < s$ there is an $X \in \Gamma_i$ such that for all formulas $\theta(x_1, \ldots, x_p)$ in X which are either Σ_1 or Π_1

$$M \models \mathrm{Sat}_{b,c}(\ulcorner \theta(x_1, \ldots, x_p, \vec{c}) \urcorner, \langle F_{i,X}(x_1), \ldots, F_{i,X}(x_p) \rangle)$$

and there is a Y in Γ_{i+1} which is derived from X by one of the rules 3.(a)-(f). Hence, we have that:

- if rule 3.(a) is the case, $X \in \Gamma_{i+1}$ and the hypothesis for $i+1$ holds for $Y = X$.
- If rule 3.(b) is the case, we have that

$$M \models \mathrm{Sat}_{b,c}(\ulcorner \neg\neg\theta(x_1, \ldots, x_p, \vec{c})\urcorner, \langle F_{i,X}(x_1), \ldots, F_{i,X}(x_p)\rangle)$$

and by the properties of $\mathrm{Sat}_{b,c}(x, y)$

$$M \models \mathrm{Sat}_{b,c}(\ulcorner \theta(x_1, \ldots, x_p, \vec{c})\urcorner, \langle F_{i,X}(x_1), \ldots, F_{i,X}(x_p)\rangle)$$

and the hypothesis for $i+1$ holds for

$$Y = X \cup \{\theta(x_1, \ldots, x_p, \vec{c})\}.$$

- If rule 3.(c) is the case, we have that

$$M \models \mathrm{Sat}_{b,c}(\ulcorner \theta_1(x_1, \ldots, x_p, \vec{c}) \to \theta_2(x_1, \ldots, x_p, \vec{c})\urcorner, \langle F_{i,X}(x_1), \ldots, F_{i,X}(x_p)\rangle)$$

and, by the properties of $\mathrm{Sat}_{b,c}(x, y)$, either

$$M \models \mathrm{Sat}_{b,c}(\ulcorner \neg\theta_1(x_1, \ldots, x_p, \vec{c})\urcorner, \langle F_{i,X}(x_1), \ldots, F_{i,X}(x_p)\rangle)$$

or

$$M \models \mathrm{Sat}_{b,c}(\ulcorner \theta_2(x_1, \ldots, x_p, \vec{c})\urcorner, \langle F_{i,X}(x_1), \ldots, F_{i,X}(x_p)\rangle)$$

and the hypothesis for $i+1$ holds for

$$Y = X \cup \{\neg\theta_1(x_1, \ldots, x_p, \vec{c})\} \quad \text{or} \quad Y = X \cup \{\theta_2(x_1, \ldots, x_p, \vec{c})\}.$$

- If rule 3.(d) is the case, we have that

$$M \models \mathrm{Sat}_{b,c}(\ulcorner \neg(\theta_1(x_1, \ldots, x_p, \vec{c}) \to \theta_2(x_1, \ldots, x_p, \vec{c}))\urcorner, \langle F_{i,X}(x_1), \ldots, F_{i,X}(x_p)\rangle)$$

and, by the properties of $\mathrm{Sat}_{b,c}(x, y)$,

$$M \models \mathrm{Sat}_{b,c}(\ulcorner \theta_1(x_1, \ldots, x_p, \vec{c})\urcorner, \langle F_{i,X}(x_1), \ldots, F_{i,X}(x_p)\rangle)$$

and

$$M \models \mathrm{Sat}_{b,c}(\ulcorner \neg\theta_2(x_1, \ldots, x_p, \vec{c})\urcorner, \langle F_{i,X}(x_1), \ldots, F_{i,X}(x_p)\rangle)$$

and the hypothesis for $i+1$ holds for

$$Y = X \cup \{\theta_1(\vec{x}, \vec{c}), \neg\theta_2(\vec{x}, \vec{c})\}.$$

- If rule 3.(e) is the case, we have that

$$M \models \mathrm{Sat}_{b,c}(\ulcorner \forall x\theta(x, x_1, \ldots, x_p, \vec{c})\urcorner, \langle F_{i,X}(x_1), \ldots, F_{i,X}(x_p)\rangle)$$

and, by the properties of $\mathrm{Sat}_{b,c}(x,y)$,

$$M \models \mathrm{Sat}_{b,c}(\ulcorner\theta(t(\vec{x}), x_1, \ldots, x_p, \vec{c})\urcorner, \\ \langle F_{i,X}(x_1), \ldots, F_{i,X}(x_p), F_{i+1,Y}(x_{p+1}), \ldots, F_{i+1,Y}(x_l)\rangle)$$

for all terms $t(\vec{x})$ freely substitutable for x in θ with $\ulcorner t\urcorner < c$ and where

$$Y = X \cup \{\theta(t(\vec{x}), x_1, \ldots, x_p, \vec{c})\}$$

and $\vec{x} = x_{p+1}, \ldots, x_l$ are fresh variables introduced by $t(\vec{x})$. Clearly, the hypothesis for $i+1$ holds for Y.

- If rule 3.(f) is the case, we have that

$$M \models \mathrm{Sat}_{b,c}(\ulcorner\neg\forall x\theta(x, x_1, \ldots, x_p, \vec{c})\urcorner, \langle F_{i,X}(x_1), \ldots, F_{i,X}(x_p)\rangle)$$

and, by the properties of $\mathrm{Sat}_{b,c}(x,y)$,

$$M \models \mathrm{Sat}_{b,c}(\ulcorner\neg\theta(y, x_1, \ldots, x_p, \vec{c})\urcorner, \\ \langle F_{i+1,X\cup\{\neg\theta(y,x_1,\ldots,x_p,\vec{c})\}}(y), F_{i,X}(x_1), \ldots, F_{i,X}(x_p)\rangle)$$

for some variable y not occurring in any formula in X, and the hypothesis holds for $i+1$ for $Y = X \cup \{\neg\theta(y, x_1, \ldots, x_p, \vec{c})\}$.

□

By Lemma 17, there exists $X \in \Gamma_s$ such that for all $\theta \in X$ which are either Σ_1 or Π_1

$$M \models \mathrm{Sat}_{b,c}(\ulcorner\theta(x_1, \ldots, x_p)\urcorner, \langle F_{i,X}(x_1), \ldots, F_{i,X}(x_p)\rangle).$$

But 1. of definition 15 implies that X contains θ and $\neg\theta$, for some atomic formula θ. Therefore,

$$M \models \mathrm{Sat}_{b,c}(\ulcorner\theta(x_1, \ldots, x_p)\urcorner, \langle F_{i,X}(x_1), \ldots, F_{i,X}(x_p)\rangle)$$

and

$$M \models \mathrm{Sat}_{b,c}(\ulcorner\neg\theta(x_1, \ldots, x_p)\urcorner, \langle F_{i,X}(x_1), \ldots, F_{i,X}(x_p)\rangle)$$

which is a contradiction. □

The following lemma will enable us to prove that the ground model is an initial segment of any model satisfying the theory constructed in the completeness argument. Note that this is the point where we need Σ_1-collection in order to bound tableaux proofs uniformly.

Lemma 18. *Let M be a countable model for $\mathcal{L}_M^* = \mathcal{L}_M \cup \{\bar{c}\}$ where $\bar{c}$ is a new constant symbol, such that $M \models \mathrm{I}\Delta_0 + \exp$. If $\theta(y, \bar{c}, \vec{c})$ is a formula of $\mathcal{L}_M^*$, $a \in M$ and T is a finite extension of $\mathrm{I}\Delta_0$ such that:*

$$M \models \mathrm{Tabcon}(T + \Delta + \bar{c} > M), \tag{3.1}$$

then either:

$$M \models \mathrm{Tabcon}(T + \Delta + \bar{c} > M + \neg\exists y \le c_a \theta(y, \bar{c}, \vec{c})) \tag{3.2}$$

or for some $b \in M$ *such that* $M \models b \le a$*:*

$$M \models \mathrm{Tabcon}(T + \Delta + \bar{c} > M + \theta(c_b, \bar{c}, \vec{c})) \tag{3.3}$$

Proof. Suppose, towards a contradiction, that:

$$M \models \neg\,\mathrm{Tabcon}(T + \Delta + \bar{c} > M + \neg\exists y \le c_a \theta(y, \bar{c}, \vec{c})) \tag{3.4}$$

and

$$M \models \forall b \le a \neg\,\mathrm{Tabcon}(T + \Delta + \bar{c} > M + \theta(c_b, \bar{c}, \vec{c})) \tag{3.5}$$

For simplicity let $T' = T + \Delta + \bar{c} > M$ and $\theta(x) = \theta(x, \bar{c}, \vec{c})$. By Definition 17 of $\mathrm{Tabcon}(T)$ and (3.5)

$$M \models \forall b \le a \exists p \mathrm{Tabinconseq}(T' + \theta(c_b), p).$$

Since $M \models B\Sigma_1$, we can bound p. So we get that

$$M \models \exists t_0 \forall b \le a \exists p \le t_0 \mathrm{Tabinconseq}(T' + \theta(c_b), p) \tag{3.6}$$

and $\forall b \le a \exists p \le t_0 \mathrm{Tabinconseq}(T' + \theta(c_b), p)$ is a Δ_0 formula.

By Δ_0 induction and the fact that M is closed under exponentiation we will show that:

$$M \models \forall y \le a \exists p \le t \mathrm{Tabinconseq}(T' + \exists x \le c_y \theta(x), p), \tag{3.7}$$

where, for the t_0 that were found in (3.6), $t = (\log t_0)^{t_0^{2^y}}$, each set of formulas that appears on a node of the tableau proof, say S_1', has g.n. less than $\log\log t_0$ and

$$\mathrm{dp}(S_1') \le 2^y \log t_0.$$

Base By (3.5) and (3.6) it holds for $y = 0$.

IH Suppose that (3.7) holds for all $y = b \le a$, i.e. if $b \le a$, then there is a $t_1 \in M$ such that:

$$M \models \forall b \le a \exists p \le t_1 \mathrm{Tabinconseq}(T' + \exists x \le c_b \theta(x), S_1)$$

where $t_1 = (\log t_0)^{t_0^{2^b}}$, each set of formulas that appears on a node of the tableau proof S_1 has g.n. less than $\log\log t_0$ and

$$\mathrm{dp}(S_1) \le 2^b \log t_0.$$

IS Let $y = b + 1 \le a$. The implication $x < c_{b+1} \to x \le c_b \vee x = c_{b+1}$ is provable from $I\Delta_0$ hence,

$$M \models x < c_{b+1} \to x \le c_b \vee x = c_{b+1}. \tag{3.8}$$

Hence, in order to get a confutation in M for $T' + \exists x \le c_{b+1}\theta(x)$, it suffices to construct a confutation for $T' + \exists x(x \le c_b \wedge \theta(x) \vee x = c_{b+1} \wedge \theta(x))$.

By (3.6) we get that:

$$M \models \exists p \le t_0 \text{Tabinconseq}(T' + \theta(c_{b+1}), S_2)$$

and by the induction hypothesis for $b < b + 1 \le a$

$$M \models \forall b \le a \exists p \le t_1 \text{Tabinconseq}(T' + \exists x \le c_b\theta(x), S_1)$$

where $t_1 = (\log t_0)^{t_0^{2^b}}$, each set of formulas that appears on a node of the tableau proof S_1 has g.n. less than $\log\log t_0$ and

$$\text{dp}(S_1) \le 2^b \log t_0.$$

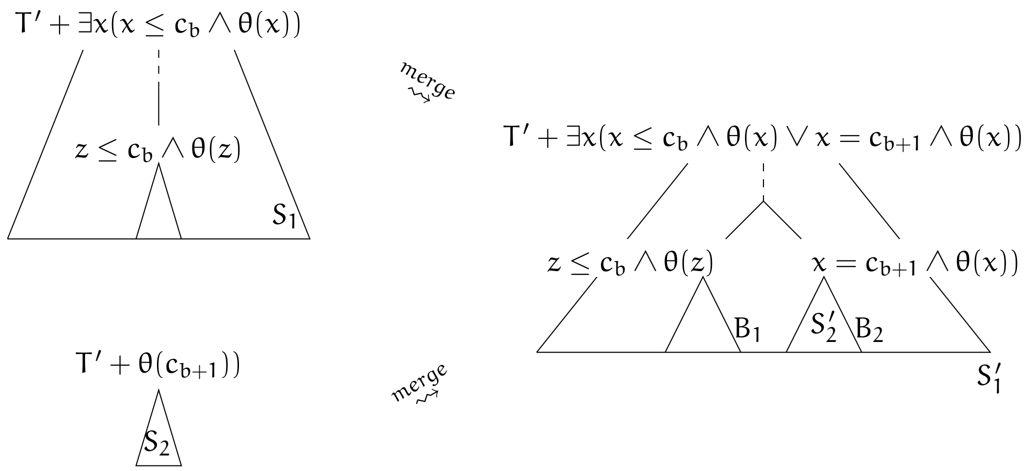

Figure 3.1: Combining proofs

Combining the above two proofs, as in Figure 3.1 we can get a tableaux proof of a contradiction for $T' + \exists x \le c_{b+1}\theta(x)$. To see this let S_1 be the confutation of $T' + \exists x \le c_b\theta(x)$ and let S_2 be the confutation of $T' + \theta(c_{b+1})$. In S_1 find a application of rule 3.f) for $\exists x \le c_b\theta(x)$ as close to the root of the tableau tree as possible and let z be the critical variable. Replace it by two subtrees B_1 and B_2. The subtree B_1 has $z \le c_b \wedge \theta(z)$ for initial node and the rest of the tree is as in S_1. The other subtree, namely B_2 has $z = c_{b+1} \wedge \theta(z))$ for initial node and the rest of the tree is as in S_2.

Repeat the process for another branch where rule 3.f) is applied for $\exists x \leq c_b \theta(x)$, if such a branch exists. Note that this process needs to be carried out only once for each branch containing the application of rule 3.f) for $\exists x \leq c_b \theta(x)$.

Continuing this way in a finite number of repetitions of the above described process, we can replace all the applications of rule 3.f) for $\exists x \leq c_b \theta(x)$ and get the desired confutation for $T' + \exists x \leq c_{b+1} \theta(x)$.

For every such replacement we "hang" in S_1 a tree of depth $\max\{dp(S_1), dp(S_2)\}$. Since none of these replacements can happen in the same branch of S_1, we can hang at most 1 tree of depth $\max\{dp(S_1), dp(S_2)\}$ on every branch of the derived confutation S_1'. Hence,

$$\begin{aligned} dp(S_1') &\leq dp(S_1) + \max\{dp(S_1), dp(S_2)\} \\ &\leq 2\max\{dp(S_1), dp(S_2)\} \\ \text{(IH)} \qquad &\leq 2\max\{2^b \log t_0, \log t_0\} \\ &= 2 \cdot 2^b \log t_0 \\ &= 2^{b+1} \log t_0. \end{aligned}$$

Furthermore, by the inductive hypothesis, the codes of the sets of formulas of S_1 are all bounded by $\log\log t_0$. By (3.6) the later is also true for all the sets of formulas of S_2, i.e., the codes of all sets of formulas of S_2 are bounded by $\log\log t_0$. In S_1' there are $2^{dp(S_1)}$ sets of formulas each of which has a code less than $\log\log t_0$. Hence,

$$\begin{aligned} \ulcorner S_1' \urcorner &\leq 2^{2^{dp(S_1)} \cdot \log\log t_0} \\ &= 2^{t_0^{2^{b+1}} \cdot \log\log t_0} \\ &= (\log t_0)^{t_0^{2^{b+1}}}. \end{aligned}$$

Hence, there is a $t = (\log t_0)^{t_0^{2^{b+1}}}$ such that:

$$M \models \forall b \leq a \exists p \leq t \mathrm{Tabinconseq}(T' + \exists x \leq c_b \theta(x), p)$$

and so the proof of the inductive step is complete.

Setting $y = a$ in (3.7), we deduce that:

$$M \models \neg \mathrm{Tabcon}(T' + \exists x \leq c_a \theta(x)) \tag{3.9}$$

By (3.4), (3.9) and the Elimination Lemma 13 we get that:

$$M \models \neg \mathrm{Tabcon}(T')$$

which contradicts (3.1). □

2nd proof of Lemma 18. We can also prove the lemma by an indirect appeal to the Elimination Lemma 13. Suppose, towards a contradiction, that the assumption doesn't hold. Then

$$M \models \neg \operatorname{Tabcon}(T' + \neg \exists y \leq c_a \theta(y)) \tag{3.10}$$

and

$$M \models \forall b \leq a \neg \operatorname{Tabcon}(T' + \{\theta(c_b)\}). \tag{3.11}$$

We can now confute

$$T' + \exists y \leq c_a \theta(y) \to \exists y \leq c_a \theta(y)$$

Start by applying rule 3.c) for $\exists y \leq c_a \theta(y) \to \exists y \leq c_a \theta(y)$ at the first level. Then the resulting branches are as good as closed by (3.10) and (3.11).

By Lemma 14 for $\phi \equiv \exists y \leq c_a \theta(y) \to \exists y \leq c_a \theta(y)$ there exists a confutation for T' which contradicts (3.1). ☐

Theorem 12. *If M is a countable model of $B\Sigma_1 + \exp$, then there exists a proper end extension $K \models I\Delta_0$ of M.*

Proof. Let $\mathcal{L}^*_M = \mathcal{L} \cup \{c_m | m \in M\} \cup \{\bar{c}\}$ where $\bar{c}$ is a new constant symbol.

We enumerate recursively all sentences of $\mathcal{L}^*_M$

$$\phi_1, \phi_2, \ldots.$$

Note that this is a countable list, since M is countable.

By Lemma 15 there is a countable model M for $\mathcal{L}^*_M$ such that:

$$M \models \operatorname{Tabcon}(I\Delta_0 + \Delta + \bar{c} > M).$$

Starting with:

$$T_0 = I\Delta_0 + \Delta + \bar{c} > M$$

we will construct a sequence of consistent theories T_i such that:

1. for all $i \in \mathbb{N}$ $T_i \subseteq T_{i+1}$
2. for all $i \in \mathbb{N}$ $M \models \operatorname{Tabcon}(T_i + \Delta + \bar{c} > M)$
3. either $\phi_i \in T_i$ or $\neg\phi_i \in T_i$, for all $i \in \mathbb{N}^*$, and
4. if $\phi_{i+1} \equiv \exists x \leq c_a \theta(x)$ for some $a \in M$ and $\phi_{i+1} \in T_{i+1}$, then:

   $$\theta(c_k) \in T_{i+1} \tag{3.12}$$

   where c_k is a constant such that $M \models k \leq a$.

For the base of the definition set by Lemma 15 we have that:

$$M \models \mathrm{Tabcon}(I\Delta_0 + \Delta + \bar{c} > M)$$

which satisfies conditions 1.-4.

For the inductive step suppose T_i has been defined. Then:

$$M \models \mathrm{Tabcon}(T_i + \Delta + \bar{c} > M).$$

Hence,

$$M \models \mathrm{Tabcon}(T_i). \tag{3.13}$$

Suppose, towards a contradiction that:

$$M \models \neg\,\mathrm{Tabcon}(T_i + \phi_{i+1}) \quad \text{and} \quad M \models \neg\,\mathrm{Tabcon}(T_i + \neg\phi_{i+1}).$$

Then by the Elimination Lemma 13 we have that:

$$M \models \neg\,\mathrm{Tabcon}(T_i)$$

which contradicts (3.13). Hence,

$$M \models \mathrm{Tabcon}(T_i + \phi_{i+1}) \quad \text{or} \quad M \models \mathrm{Tabcon}(T_i + \neg\phi_{i+1})$$

Set

$$\psi_{i+1} \equiv \begin{cases} \phi_{i+1} & \text{if } M \models \mathrm{Tabcon}(T_i + \phi_{i+1}) \\ \neg\phi_{i+1} & \text{otherwise} \end{cases}$$

If ψ_{i+1} is of the form $\exists x \leq c_\alpha \theta(x)$ by Lemma 18 set:

$$T_{i+1} = T_i \cup \{\psi_{i+1}, \theta(c_k)\}$$

where c_k is such that $k \leq^M \alpha$. Otherwise, set:

$$T_{i+1} = T_i \cup \{\psi_{i+1}\}.$$

Thus, the recursive definition is complete.

Set $T_\infty = \bigcup_{i \in N} T_i$. Then T_∞ is tableau consistent in M for, otherwise, there would be tableau proof S of a contradiction from T_∞. There is, however, an $n \in \mathbb{N}$ such that $\chi \in T_n$ for each formula χ that appears in S. Hence, S is a also a tableau proof of a contradiction for T_n which contradicts the construction of the T_is.

Thus, T_∞ is a complete and consistent theory in $\mathcal{L}^*_M$ extending $I\Delta_0$, containing the diagram of M and all sentences of the form $\bar{c} > c_\alpha, \alpha \in M$ and whenever:

$$\exists x \leq c_\alpha \theta(x, \bar{c}, \vec{c}) \in T_\infty,$$

then there exists $b \leq a$ in M such that $\theta(c_b, \bar{c}, \vec{c}) \in T_\infty$.

We will now apply the omitting types theorem to obtain a model K^* of T_∞ in which the interpretation of the constant symbols $\{c_\alpha \mid \alpha \in M\}$ form an initial segment.

For each $\alpha \in M$ let $\Sigma_\alpha(x)$ be the type:

$$\Sigma_\alpha(x) = \{x \neq c_m \mid M \models m \leq a\} \cup \{x \leq c_\alpha\}$$

And let

$$\Sigma(x) = \bigcup_{\alpha \in M} \Sigma_\alpha(x).$$

Let $\theta(x)$ be a formula consistent with T_∞. Then there is a model $\mathcal{A}$ of T_∞ such that $\mathcal{A} \models \theta[i]$ for some $i \in \mathcal{A}$. For all $\alpha \in M$ it holds that

$$\mathcal{A} \models i \leq c_\alpha^{\mathcal{A}} \vee i > c_\alpha^{\mathcal{A}}.$$

- If for all $\alpha \in M$ it holds that: $\mathcal{A} \models i > c_\alpha^{\mathcal{A}}$, then for all $\alpha \in M$

  $$\mathcal{A} \models \exists x(x > c_\alpha \wedge \theta(x))$$

  or equivalently

  $$\mathcal{A} \models \exists x(\neg(x \leq c_\alpha) \wedge \theta(x)).$$

  Hence, since $\mathcal{A}$ is a model of T_∞, $\neg(x \leq c_\alpha) \wedge \theta(x)$ is consistent with T_∞. Thus, T_∞ locally omits Σ.

- If for some $\alpha \in M$ it holds that $\mathcal{A} \models i \leq c_\alpha^{\mathcal{A}}$, then

  $$\mathcal{A} \models i \leq c_\alpha^{\mathcal{A}} \wedge \theta[i]$$

  or equivalently

  $$\mathcal{A} \models \exists x(x \leq c_\alpha \wedge \theta(x)).$$

  Then

  $$\mathcal{A} \models \exists x \leq c_\alpha \theta(x).$$

$\mathcal{A}$ is a model of T_∞ and T_∞ is complete, thus $\exists x \leq c_\alpha \theta(x) \in T_\infty$. By 3.12 there is a $k < \alpha$ in M such that $\theta(c_k) \in T_\infty$. Hence, since $\mathcal{A}$ is a model for T_∞

$$\mathcal{A} \models \exists x(\theta(x) \wedge x = c_k).$$

or

$$\mathcal{A} \models \exists x(\theta(x) \wedge \neg(x \neq c_k)).$$

Hence, $\theta(x) \wedge \neg(x \neq c_k)$ is consistent with T_∞ therefore T_∞ locally omits Σ.

By the Omitting Types Theorem, T_∞ has a model K^* which omits $\Sigma(x)$.

We go on to show that M is a proper initial segment of K^*. We will show that M is a submodel of K^* first. Indeed, let:

$$f : M \to K^*$$

be a function such that for all $\alpha \in M$

$$f(c_\alpha^M) = c_\alpha^K.$$

Clearly, since T_∞ contains the Δ_0 diagram of M, f is an homomorphic embedding of M into K^*, i.e.

$$M \subseteq_f K^*.$$

Let $M^* = f(M)$. We will show that M^* is an initial segment of K^*. Indeed, suppose that:

$$K^* \models l < c_\alpha^{K^*}$$

for some $\alpha \in M$. Furthermore, for every $\alpha \in M$ by the definition of M^* we get that $c_\alpha^{K^*} \in M^*$. Let $\phi(x) \equiv x < c_\alpha$. K^* omits $\Sigma(x)$ therefore for some $k \in M$:

$$K^* \models \phi(l) \wedge (l = c_k)$$

which implies that

$$K^* \models l = c_k^{K^*} = f(c_k^M),$$

which by the definition of f and M^* we get that $l \in M^*$.

Thus, the interpretations of the c_α for $\alpha \in M$ form an initial segment of K^*.

Furthermore, K^* is a proper end extension of M^* since for all $\alpha \in M$

$$K^* \models \bar{c} > c_\alpha.$$

Finally, since the base theory T_0 contains $I\Delta_0$ we get that K^* is a model of $I\Delta_0$ as well. The reduct K of K^* to $\mathcal{L}$ has the required properties, i.e.

$$M \subsetneq_e K \models I\Delta_0.$$

□

3.3 Other conditions

Our aim in this section is to show that Theorem 12 holds, if we replace the assumption that M is closed under exponentiation by each of conditions (1), (3), (4) of Theorem 11. Note that conditions (3) – (4) contain the assumption that $I\Delta_0 \vdash \neg\Delta_0 H$, defined at Notation 1, which may well be false. However, following [18], we consider it worthwhile to study how it affects Problem 2.

The argument when we adopt one of conditions (1), (3), (4) is basically similar to that employed when $M \models \mathit{exp}$. The main difference between the approach in 3.2 and the one taken in this section is that here we have to pay more attention to the behaviour of the satisfaction formula Sat_0, so that we can keep on working with (modifications of) it even when M satisfies properties other than being a model of exp. In fact, assuming either one of conditions (3) and (4), Sat_0 works with b significantly smaller than in Theorem 3; this is due to the following result from [14].

Theorem 13. *Assuming* $I\Delta_0 \vdash \neg\Delta_0 H$, *the bound* $2^{(\max(\vec{a})+2)^{\ulcorner\varphi\urcorner}}$ *in Theorem 3 can be replaced by* $(\max(\vec{a})+2)^{\ulcorner\varphi\urcorner}$.

Remark 4. The assumption $I\Delta_0 \vdash \neg\Delta_0 H$ is necessary only if we need to be able to talk about satisfiability of all standard formulas. So, if we need to talk about the satisfiability of formulas with Gödel number less than a natural number k, it suffices to know that $(\max(\vec{a})+2)^k$ exists, which is guaranteed in any model of $I\Delta_0$.

Independently of which condition we will be assuming, we have to work with a restricted form of the formula $\mathsf{Tabcon}(T)$. Indeed, we will be using the formula k-$\mathsf{Tabcon}(T)$ defined in section 1.3.

First, we need to check that

Lemma 13 holds for the restricted form of the formula expressing the tableau consistency of a theory. This is due to the fact that substitution in restricted formulas can be performed in $I\Delta_0$ alone. For unrestricted formulas we need at least $I\Delta_0 + \Omega_1$ (see the discussion after Lemma 5.1.5 in [2])

Lemma 19. *For any model* M *of* $I\Delta_0$ *and* $i \in M$, *any theory* T *coded in* M *and any sentence* θ, *if* $M \models \neg$ i-$\mathsf{Tabcon}(T+\theta)$ *and* $M \models \neg$ i-$\mathsf{Tabcon}(T+\neg\theta)$, *then* $M \models \neg$ i-$\mathsf{Tabcon}(T)$.

Now we can proceed to proving the following variant of Lemma 15.

Lemma 20. *(a) If* M *satisfies condition (1) or (4), then for all* $k \in \mathbb{N}$

$$M \models k\text{-}\mathsf{Tabcon}(I\Delta_0 + \Delta + \bar{c} > M).$$

(b) If M *satisfies condition (3), then there exists* $j \in M - \mathbb{N}$ *such that*

$$M \models j\text{-}\mathsf{Tabcon}(I\Delta_0 + \Delta + \bar{c} > M).$$

Proof of Lemma 20. (a) We essentially repeat the proof of Lemma 15, noting that the formula Sat_0 is still at our disposal, in view of Remark 4.

(b) In this case, we can do better than when M satisfies condition (1) or (4). Indeed, one can mimic the proof of Lemma 15, working with j-tableau proofs, for any nonstandard j much smaller than the γ of condition (3).

□

Now we can proceed to proving the following variant of Theorem 12.

Theorem 14. *If* M *is a countable model of* $B\Sigma_1$ *satisfying one of conditions (1), (3), (4) of Theorem 11, then there exists* $K \models I\Delta_0$ *such that* $M \subset_e K$.

Proof. Letting M be as in the hypothesis, we use the same notation as in the proof of Theorem 12. Clearly, what we have to prove is modification of Lemma 18.

Let us now proceed to the counterpart of Lemma 18.

Lemma 21. *(a) Assume M satisfies condition (1) or (4). If $\theta(y, \bar{c}, \vec{c})$ is a formula of LA^*, $a \in M$ and T is a finite extension of $I\Delta_0$ such that*

$$M \models k\text{-Tabcon}(T + \Delta + \bar{c}{>}M), \textit{ for all } k \in \mathbb{N},$$

then either

$$M \models k\text{-Tabcon}(T + \Delta + \bar{c}{>}M + \neg\exists y{\le}c_a\theta(y, \bar{c}, \vec{c})), \textit{ for all } k \in \mathbb{N},$$

or there exists $b{\le}^M a$ such that

$$M \models k\text{-Tabcon}(T + \Delta + \bar{c}{>}M + \theta(c_b, \bar{c}, \vec{c})), \textit{ for all } k \in \mathbb{N}.$$

(b) Assume M satisfies condition (3). If $\theta(y, \bar{c}, \vec{c})$ is a formula of LA^*, $a \in M$ and T is a finite extension of $I\Delta_0$ such that

$$M \models j\text{-Tabcon}(T + \Delta + \bar{c}{>}M), \textit{ for some } j \in M - \mathbb{N},$$

then either

$$M \models j\text{-Tabcon}(T + \Delta + \bar{c}{>}M + \neg\exists y{\le}c_a\theta(y, \bar{c}, \vec{c})), \textit{ for some } j \in M - \mathbb{N},$$

or there exists $b{\le}^M a$ such that

$$M \models j\text{-Tabcon}(T + \Delta + \bar{c}{>}M + \theta(c_b, \bar{c}, \vec{c})), \textit{ for some } j \in M - \mathbb{N}.$$

Proof of Lemma 21. (a) First, we note that, as shown in [18], if M satisfies condition (4), then it satisfies condition (1). Hence, it suffices to work with M satisfying condition (1). So let us assume M is short Π_1-recursively saturated and T is a finite extension of $I\Delta_0$ such that

(i) $M \models k\text{-Tabcon}(T + \Delta + \bar{c}{>}M)$, for all $k \in \mathbb{N}$, and

(ii) $M \models \neg k_0\text{-Tabcon}(T + \Delta + \bar{c}{>}M + \neg\exists y{\le}c_a\theta(y, \bar{c}, \vec{c}))$, for some $k_0 \in \mathbb{N}$.

We will show that there exists $b \le^M a$ such that

$$M \models k - \text{Tabcon}(T + \Delta + \bar{c}{>}M + \theta(c_b, \bar{c}, \vec{c})), \text{ for all } k \in \mathbb{N}. \tag{3.14}$$

Observe that the set

$$Z = \{z \le a \wedge k\text{-Tabcon}(T + \Delta + \bar{c}{>}M + \theta(c_z, \bar{c}, \vec{c})) | k \in \mathbb{N}\}$$

is a recursive set of Π_1 formulas. We claim that Z is finitely satisfiable in M. Supposing not, there would be some $k_1, \ldots, k_m \in \mathbb{N}$ such that

$$M \models \neg\exists z \le a \bigwedge_{1\le i\le m} k_i\text{-Tabcon}(T + \Delta + \bar{c}{>}M + \theta(c_z, \bar{c}, \vec{c})).$$

Letting $K = \max\{k_1, \ldots, k_m\}$, we see that

$$M \models \forall z \le a \exists t \neg K\text{-Tabcon}(T + \Delta_t + \bar{c}{>}t + \theta(c_z, \bar{c}, \vec{c})),$$

where Δ_t denotes the restriction of the diagram to sentences involving constants with index less than t.
Since M satisfies $B\Sigma_1$, there exists $b \in M$ such that

$$M \models \forall z \leq a \exists t \leq b \neg \text{K-Tabcon}(T + \Delta_t + \bar{c}{>}t + \theta(c_z, \bar{c}, \vec{c})).$$

But now note that the size of a K-tableau proof from $T + \Delta_t + \bar{c}{>}t + \theta(c_z, \bar{c}, \vec{c})$ of a contradiction cannot exceed $max(a, b)^L$, for some natural number L depending on K. This is because the depth of the tree depends and hence, the number of the nodes of the tree depend on the complexity of the formulas that appear in the initial set and all of these formulas are bounded by K. The formulas that appear in the K-tableau are substitution instances of K-formulas and the terms substituted are all bounded by standard powers of the parameters a and b. Therefore, by an inductive argument similar to that used in the proof of Lemma 15, we can show that

$$M \models \neg \text{K-Tabcon}(T + \Delta + \bar{c}{>}M + \exists y{\leq}c_a \theta(y, \bar{c}, \vec{c})).$$

But then, by (ii) and Lemma 19, it follows that

$$M \models \neg \text{L-Tabcon}(T + \Delta + \bar{c}{>}M),$$

with $L = max(k_0, K)$, which contradicts (i).
It follows that Z is finitely satisfiable in M and so it is satisfied in M, by the saturation hypothesis about M. Therefore, there exists $b \leq^M a$ such that (3.14) holds, as required.
(b) Suppose that M satisfies condition (3) and T is a finite extension of $I\Delta_0$ such that

(i) $M \models j_0\text{-Tabcon}(T + \Delta + \bar{c}{>}M)$, for some $j_0 \in M - \mathbb{N}$

(ii) $M \models \neg \text{j-Tabcon}(T + \Delta + \bar{c}{>}M + \neg \exists y{\leq}c_a \theta(y, \bar{c}, \vec{c}))$, for all $j \in M - \mathbb{N}$

(iii) for all $b \leq^M a$,

$$M \models \neg \text{j-Tabcon}(T + \Delta + \bar{c}{>}M + \theta(c_b, \bar{c}, \vec{c})), \text{ for all } j \in M - \mathbb{N}.$$

Clearly, (iii) implies that

$$M \models \forall z \leq a \exists t \neg j_0\text{-Tabcon}(T + \Delta_t + \bar{c}{>}t + \theta(c_z, \bar{c}, \vec{c})).$$

Since M satisfies $B\Sigma_1$, there exists $b \in M$ such that

$$M \models \forall z \leq a \exists t \leq b \neg j_0\text{-Tabcon}(T + \Delta_t + \bar{c}{>}t + \theta(c_z, \bar{c}, \vec{c})).$$

As in the first part of the proof, we observe that the size of a j_0-tableau proof of a contradiction from $T + \Delta_t + \bar{c}{>}t + \theta(c_z, \bar{c}, \vec{c})$ cannot exceed $max(a, b)^{j_0}$. Therefore, one can use induction on z to prove that

$$M \models \neg j_0 - \text{Tabcon}(T + \Delta + \bar{c}{>}M + \exists y{\leq}c_a \theta(y, \bar{c}, \vec{c})). \tag{3.15}$$

But now, combining (3.15) with (ii) and Lemma 19, it follows that

$$M \models \neg j_0\text{-Tabcon}(T + \Delta + \bar{c}{>}M),$$

which contradicts (i). □

Returning to the proof of Theorem 14, we see that Lemma 20 and Lemma 21 enable us to construct a sequence of theories in LA^* satisfying conditions 1. − 4. of Theorem 12, the only difference being that the formula $\mathtt{Tabcon}(\ldots)$ has to be replaced by its restricted version. Then we can apply the omitting types theorem as before, to obtain a proper end extension $\mathsf{K} \models \mathrm{I}\Delta_0$ of M. □

Remark 5. For an alternative proof, we can employ the the fact that

$$\mathrm{I}\Delta_0 + \neg\Delta_0\mathsf{H} \Longrightarrow \mathrm{I}\Delta_0 \text{ is finitely axiomatizable}$$

(See the discussion before lemma 7 in [17])

4 End extensions of models of weak arithmetic

To obtain both Theorem 8 and Theorem 10, one has to use very strongly the countability of the model. The countability of the model is again present, if we want to show that there is a proper Σ_n-elementary end extension of a model of Σ_n-collection. Indeed in [15] J. Paris and L. Kirby prove that:

Theorem 15. *If M is a countable model of Σ_n-collection, $n \geq 2$, then M has a proper Σ_n-elementary end extension.*

In fact, J. Paris and L. Kirby prove the converse result as well, but we will focus our attention on the one direction only.

In this chapter, we will use the methods of chapter 3 to prove a generalization of Theorem 12. It is interesting that the Arithmetized Completeness Theorem can be used in a (sort-of) uniform manner to prove the proper Σ_n-elementary end extendability of countable models of $B\Sigma_n(+\exp$ when $n = 1)$ for all $n \geq 1$.

We will begin with a theorem that will allow us to bypass Lemma 15.

Theorem 16. *For each $k > 1$, $I\Sigma_k$ proves the consistency of the set of all true Π_{k+1} sentences, i.e. if $\mathsf{Tr}(\Pi_{k+1})$ is the Π_{k+1}-set of all true Π_{k+1}-sentences then*

$$I\Sigma_k \vdash \mathrm{Con}(\mathsf{Tr}(\Pi_{k+1})).$$

For a proof see Theorem 4.33 in chapter I of [7]. One way of seeing $\mathsf{Tr}(\Pi_{n+1})$ is that it is a formula having only one free variable x saying "x is a closed formula and $\mathrm{Sat}_{\Pi_{k+1}}(x)$". Since the complete formulas $\mathrm{Sat}_{\Pi_{k+1}(x)}$ are model dependent, $\mathrm{Con}(\mathsf{Tr}_{\Pi_{k+1}}(M))$ is an abbreviation for

$$M \models \forall \phi[\mathrm{Sat}_{\Pi_{k+1}}(\varphi) \to \mathrm{Con}(\varphi)],$$

where ϕ is the code (in M) of a Π_{k+1} sentence.

We will also make use of the following proposition.

Proposition 2. *If $M \subseteq_e N$ and $N \models \mathsf{Tr}_{\Pi_n}(M)$, $n \geq 0$, then $M \prec_{n,e} N$.*

Proof. We will show by induction on m that If $M \subseteq_e N$ and $N \models \mathsf{Tr}_{\Pi_n}(M)$, $n \geq 0$, then $\forall m \leq n\ M \prec_{m,e} N$.

Base It holds by Theorem 2.

IH Suppose that it holds for $m < n$, i.e. $M \prec_{m,e} N$.

IS Let $\bar{a} \in M$, $\phi(x) \in \Sigma_{m+1}$ such that $\phi(x) \equiv \exists y \theta(y, x)$ and $\theta(y, x) \in \Pi_m$. Then

$$N \models \phi(\bar{a}) \iff N \models \exists y \theta(y, \bar{a}). \tag{4.1}$$

Suppose, towards a contradiction, that

$$\forall b \in M \models \neg\theta(b, \bar{a}) \Rightarrow M \models \forall y \neg\theta(y, \bar{a}).$$

However, $\forall y \neg\theta(y, x) \in \Pi_{m+1}$, $m + 1 \leq n$ and $N \models Tr_{\Pi_n}(M)$, hence,

$$N \models \forall y \neg\theta(y, \bar{a})$$

which contradicts (4.1).

For the other direction, let $\bar{a} \in M$, $\phi(x) \in \Sigma_{m+1}$ such that $\phi(x) \equiv \exists y \theta(y, x)$ and $\theta(y, x) \in \Pi_m$. Then

$$\begin{aligned} M \models \phi(\bar{a}) &\iff M \models \exists y \theta(y, \bar{a}) \\ &\iff \exists b \in M \models \theta(b, \bar{a}). \end{aligned}$$

$\theta(y, x) \in \Pi_m$ and, by the inductive hypothesis, $M \prec_{m,e} N$, hence,

$$N \models \theta(b, \bar{a}).$$

□

Actually, for $n = 0$, $M \subseteq_e N$ is enough as we have seen in Theorem 2.

Remark 6. There is another significant difference from the proofs of chapter 3. In the presence of $I\Delta_0 + \text{supexp}$ the formalized consistency $\text{Con}(X)$ and the formalized tableau consistency $\text{Tabcon}(X)$ are equivalent. By Theorem 1 and (1.1) we have that:

$$B\Sigma_2 \Rightarrow I\Sigma_1 \Rightarrow \text{supexp}\,. \tag{4.2}$$

Hence, we may replace $\text{Con}(X)$ with $\text{Tabcon}(X)$ and vice versa.

Let $\mathcal{LA}^*$ be the language obtained from $\mathcal{LA}$ by adding new constant symbols c and $\{c_a : a \in M\}$. Let $\mathcal{LA}^{*,H}$ be the language obtained by adding to $\mathcal{LA}^*$ the Henkin constants $\{d_a : a \in M\}$.

Lemma 22. *Let M be a structure for $\mathcal{LA}^{*,H}$ such that $M \models B\Sigma_{k+1}$ for $k \geq 1$ and*

$$M \models \text{Con}(I\Delta_0 + Tr_{\Pi_{k+1}}(M) + c > M).$$

Then for every consistent extension T of $I\Delta_0 + Tr_{\Pi_{k+1}}(M) + c > M$ and for every sentence $\exists x \varphi(x)$ of $LA^{*,H}$ there exists a constant d_φ such that:

$$M \models \text{Con}(T + \exists x \varphi(x) \to \varphi(d_\varphi)).$$

Proof. Let T be any finite consistent extension of $I\Delta_0 + Tr_{\Pi_{k+1}}(M) + c > M$ and let $\exists x\varphi(x)$ be a sentence of $LA^{*,H}$. Choose a $d_\phi \in LA^{*,H}$ that doesn't appear in T and $\exists x\varphi(x)$ and suppose, towards a contradiction, that:

$$M \models \neg \mathrm{Con}(T + \exists x\varphi(x) \to \varphi(d_\varphi)).$$

Then by the properties of Con(X)

$$M \models \neg \mathrm{Con}(T + \exists x\varphi(x) \to \exists x\varphi(x)).$$

By Remark 6 we have that:

$$M \models \neg \mathrm{Tabcon}(T + \exists x\varphi(x) \to \exists x\varphi(x))$$

and by Lemma 14 we get that:

$$M \models \neg \mathrm{Tabcon}(T).$$

Hence, by Remark 6:

$$M \models \neg \mathrm{Con}(T)$$

which contradicts the assumption about T. □

We will also employ the notion of the definable elements of a model from a set of parameters.

Definition 25. Let $M \models PA$, let $n \geq 1$ and let $A \subseteq M$. Then $K^n(M, A)$ is the substructure of M consisting of all $b \in M$ such that

$$M \models \theta(b, \vec{a}) \wedge \forall x(\theta(x, \vec{a}) \to x = b)$$

for some $\theta(x, \vec{y}) \in \Sigma_n$ and some $\vec{a} \in A$.

The relation between the Σ_n-definable elements and the initial model is described by the following Theorem.

Theorem 17. *Let* $n \geq k \geq 1$ *and suppose* $A \subseteq M \models I\Sigma_{k-1}$. *Then* $A \subseteq K^n(M, A) \prec_k M$.

For a proof see Theorem 10.2 in [9].

4.1 Σ_2-collection

The goal of this section is to prove that every countable model of Σ_2 collection is properly and Σ_2-elementarily end extendable to a model of bounded induction. The proof resembles that given in the previous chapter for Theorem 12; the main modifications made are:

1. replacing the diagram with the set of true Π_2 sentences in the model and

2. employing the usual formula $\mathrm{Con}(X)$ expressing the consistency of the theory X instead of the tableau consistency formula $\mathrm{Tabcon}(X)$.

More specifically, instead of starting with the formula $\mathrm{Tabcon}(I\Delta_0 + \Delta + c{>}M)$ we will start with the formula:

$$\forall\varphi[\mathrm{Sat}_{\Pi_2}(\varphi) \to \mathrm{Con}(I\Delta_0 + \varphi + c{>}M)],$$

where $\mathrm{Sat}_{\Pi_2}(\dots)$ denotes a complete formula for Π_2 truth in M.

Theorem 18. *For any countable model M of $B\Sigma_2$ there exists a model K of $I\Delta_0$ such that $M \prec_{2,e} K$.*

Proof. Let M be a model of $B\Sigma_2$ and LA^* be the language obtained from LA by adding new constant symbols c and $\{c_a : a \in M\}$. As before, the proof will be based on a couple of lemmas, the first of which is as follows.

Lemma 23. $M \models \forall\varphi[\mathrm{Sat}_{\Pi_2}(\varphi) \to \mathrm{Con}(I\Delta_0 + \varphi + c{>}M)]$.

Proof of Lemma 23. By Theorem 1 we have that:

$$B\Sigma_2 \Rightarrow I\Sigma_1.$$

Recall that, by theorem 16

$$I\Sigma_1 \vdash \mathrm{Con}(\mathrm{Tr}(\Pi_2)).$$

Since $I\Delta_0$ is Π_2-axiomatized and $c > M$ is a set of Δ_0-sentences we get that

$$M \models \mathrm{Con}(I\Delta_0 + \mathrm{Tr}_{\Pi_2}(M) + c > M).$$

□

Now let $LA^{*,H}$ be the extension of LA^* obtained by adding Henkin constants $\{d_a : a \in M\}$ and H the corresponding set of Henkin sentences.

Lemma 24. *If $a \in M$, $\theta(y, c, \vec{c}) \in \Delta_1(M)$ is an $LA^{*,H}$-formula where $\vec{c} = c_{a_1}, \dots, c_{a_l}$ for $a_1, \dots a_l \in M$ and T is a finite extension of*

$$I\Delta_0 + \mathrm{Tr}_{\Pi_2}(M) + c > M,$$

then either

$$M \models \mathrm{Con}(T + \forall z \exists x > z \neg \exists y \le c_a \theta(y, x, \vec{c}))$$

or

$$M \models \mathrm{Con}(T + \forall z \exists x > z \theta(c_b, x, \vec{c})), \quad \textit{for some } b \le a,\ b \in M.$$

Proof. First, notice that only one of the following holds:

$$M \models \forall z \exists x > z \neg \exists y \le c_a \theta(y, x, \vec{c}) \tag{4.3}$$

or for some $b \leq a$, $b \in M$

$$M \models \forall z \exists x > z \theta(c_b, x, \vec{c}) \tag{4.4}$$

Indeed, suppose that both (4.3) and (4.4) fail, i.e.:

$$M \models \exists z \forall x > z \exists y \leq c_a \theta(y, x, \vec{c}) \tag{4.5}$$

and

$$M \models \forall b \leq a \exists z \forall x > z \neg\theta(c_b, x, \vec{c}). \tag{4.6}$$

Since $\forall x > z \neg\theta(c_b, x, \vec{c}) \in \Pi_1$, by (4.6) and the fact that $M \models B\Pi_1$ we get that

$$M \models \exists t \forall b \leq a \exists z \leq t \forall x > z \neg\theta(c_b, x, \vec{c})$$

which implies that

$$M \models \exists z \forall b \leq a \forall x > z \neg\theta(c_b, x, \vec{c}).$$

Thus,

$$M \models \exists z \forall x > z \neg\exists y \leq a \theta(y, x, \vec{c}). \tag{4.7}$$

By (4.5) and (4.7) we have that

$$M \models \exists z \forall x > z (\exists y \leq a \theta(y, x, \vec{c}) \wedge \neg\exists y \leq a \theta(y, x, \vec{c}))$$

a contradiction.

The sentences in (4.3) and (4.4) are Π_2, T is a finite extension of $Tr_{\Pi_2}(M)$ and by Theorem 16 we get that (4.3) implies

$$M \models Con(T + \forall z \exists x > z \neg\exists y \leq c_a \theta(y, x, \vec{c}))$$

and that that (4.4) implies

$$M \models Con(T + \forall z \exists x > z \theta(c_b, x, \vec{c})), \quad \text{for some } b \leq a,\ b \in M.$$

□

By (4.3) we can assign a value to c greater than all values assigned to the constants in $T + \forall z \exists x > z \neg\exists y \leq c_a \theta(y, x, \vec{c})$ and get that

$$M \models Con(T + \neg\exists y \leq c_a \theta(y, c, \vec{c})).$$

Similarly, by (4.4) we get that

$$M \models Con(T + \theta(c_b, c, \vec{c})) \quad \text{for some } b \leq a,\ b \in M.$$

We can prove that:

Lemma 25. *There exists a set of sentences (in the sense of* M*):*

$$\Sigma = \bigcup_{i \in \mathbb{N}} \Sigma_i$$

such that:

1. Σ *contains* $I\Delta_0 + Tr_{\Pi_2}(M) + c{>}M$,
2. *for every sentence* $\exists x \varphi(x)$ *of* $LA^{*,H}$ *there exists a constant* d_φ *such that:*
$$\exists x \varphi(x) \to \varphi(d_\varphi) \in \Sigma,$$
3. *for every sentence* θ *of* $LA^{*,H}$,
$$\theta \in \Sigma \text{ or } \neg\theta \in \Sigma,$$
4. *for every formula* $\theta(y, c, \vec{c}) \in \Delta_1$ *of* $LA^{*,H}$*, either*
$$\neg \exists y \leq c_a \theta(y, c, \vec{c}) \in \Sigma \quad \text{or} \quad \theta(c_b, c, \vec{c}) \in \Sigma \text{ for some } b \leq a,\ b \in M.$$
5. *for all* $i \in \mathbb{N}$,
$$M \models \mathrm{Con}(\Sigma_i).$$

Proof. Let $\theta_1, \theta_2, \ldots$ be a recursive enumeration of the $\mathcal{LA}^{*,H}$ sentences.

For the base of the definition set by Lemma 23:

$$M \models \mathrm{Con}(I\Delta_0 + Tr_{\Pi_2}(M) + c > M)$$

which satisfies conditions 1.-5.

For the inductive step suppose Σ_i has been defined and satisfies condition 1.-5.

Using the standard argument for the completeness theorem, we can take care of condition 3. concerning θ_{i+1}, thus obtaining the set

$$\Sigma_i' = \Sigma_i \cup \{\psi_{i+1}\},$$

where

$$\psi_{i+1} \equiv \begin{cases} \theta_{i+1} & \text{if } M \models \mathrm{Con}(\Sigma_i + \theta_{i+1}) \\ \neg\theta_{i+1} & \text{otherwise} \end{cases}$$

If ψ_{i+1} is of the form $\exists x \phi(x)$ by Lemma 22 set:

$$\Sigma_i'' = \Sigma_i' \cup \{\phi(d_k)\}$$

where d_k is the first Henkin constant that does not occur in Σ_i'. Otherwise set:

$$\Sigma_i'' = \Sigma_i'.$$

Finally, if ψ_{i+1} is of the form $\theta(y, c, \vec{c}) \in \Delta_1$ then by Lemma 24 we add $\neg\exists y \leq c_a\theta(y, c, \vec{c})$ to Σ'',if

$$M \models \mathrm{Con}(T + \forall z \exists x > z \neg\exists y \leq c_a \theta(y, x, \vec{c}))$$

otherwise if

$$M \models \mathrm{Con}(T + \forall z \exists x > z\theta(c_b, x, \vec{c})), \quad \text{for some } b \leq a,\ b \in M.$$

we add $\theta(c_b, c, \vec{c})$ to Σ_i''. The derived set is the required Σ_{i+1}.
This completes the recursive definition of Σ_n so that conditions 1.-5. are satisfied. □

Set $\Sigma = \bigcup \Sigma_i$. Then

$$M \models \mathrm{Con}(\Sigma),$$

for, otherwise,

$$M \models \neg \mathrm{Con}(\Sigma),$$

and there exists $i \in \mathbb{N}$ such that:

$$M \models \neg \mathrm{Con}(\Sigma_i).$$

which contradicts the construction of the Σ_is.
Thus, Σ is a complete and consistent theory in $\mathcal{LA}^{*,H}$ extending $I\Delta_0$, containing the Π_2-truth of M and all sentences of the form $c > c_\alpha, \alpha \in M$.
Returning now to the proof of Theorem 18, let J^* be a model of (the standard part of) Σ. Then J^* is a model of Δ_0-induction and

$$J^* \models \mathrm{Tr}_{\Pi_2}(M). \tag{4.8}$$

Now let J be the reduct of J^* to $\mathcal{LA}$. It is easy to see that

$$J \models \mathrm{Tr}_{\Pi_2}(M). \tag{4.9}$$

and that M is isomorphic to a substructure of J. Let K be the substructure of J with universe the set of Σ_1-definable elements of J with parameters form the set

$$\{c_a^{J^*} \mid a \in M\} \cup \{c^{J^*}\},$$

i.e.

$$K = K^1\left(J, (\{c_a \mid a \in M\} \cup \{c\})^{J^*}\right).$$

Since $J \models I\Delta_0$, by Theorem 17 we get that

$$K \prec_1 J. \tag{4.10}$$

By (4.9) and (4.10) we get that

$$K \models Tr_{\Pi_2}(M). \tag{4.11}$$

Claim 1. M is a proper initial segment of K, i.e. $M \subset_e K$.

Indeed, suppose that $K \models b < a$, for some $a \in M$. Then $J \models b < a$ and so for some $\phi \in \Sigma_1$

$$J^* \models (\phi(c_b, c, \vec{c}) \wedge c_b < c_a) \wedge \forall x \phi(x, c, \vec{c}) \rightarrow x = c_b), \tag{4.12}$$

which means that

$$(\phi(c_b, c, \vec{c}) \wedge c_b < c_a) \wedge \forall x \phi(x, c, \vec{c}) \rightarrow x = c_b) \in \Sigma. \tag{4.13}$$

By the construction of Σ either

$$\neg \exists x < c_a \phi(x, c, \vec{c}) \in \Sigma \quad \text{or} \quad \phi(c_f, c, \vec{c}) \in \Sigma \text{ for a specific } f < a,\ f \in M.$$

By (4.13) and the fact that $J^* \models \Sigma$, it cannot be the case that $\neg \exists x < c_a \phi(x, c, \vec{c}) \in \Sigma$. Hence,

$$\phi(c_f, c, \vec{c}) \in \Sigma \text{ for a specific } f \leq a$$

which implies that $c_b = c_f \in \Sigma$. Thus b is (the image of) an element of M and so M is an initial segment of K

Now recall that c^{J^*} is an element of (the universe) of J and that $c_a^{J^*} < c^{J^*} \in \Sigma$ for all $\alpha \in M$, i.e.

$$J \models c_a^{J^*} < c^{J^*}, \quad \text{for all } a \in M.$$

It follows that M is (isomorphically embedded to) a proper initial segment of J. In addition,

$$K \models Tr_{\Pi_2}(M).$$

Therefore, by Proposition 2 for $n = 2$, we have that

$$M \prec_{2,e} K.$$

Since, clearly, $K \models I\Delta_0$, we see that K has all the required properties, i.e.

$$M \prec_{2,e} K \models I\Delta_0$$

which completes the proof. □

4.2 Σ_n-Collection

Let us finish this section by remarking that one can modify the proof of Theorem 18 to give an alternative proof of Theorem 15 for any $n \geq 2$. We will leave out the details which are the same as in the proof of Theorem 18 and give a sketch of the proof instead.

Theorem 19. *Every countable model M of $B\Sigma_n$, $n \geq 2$, has a proper Σ_n-elementary end extension K satisfying $I\Delta_0$, i.e.*

$$M \prec_{n,e} K \models I\Delta_0.$$

Proof. Let M be a model of Σ_n collection, where $n \geq 2$ and $\mathcal{LA}^*$ be the language obtained from $\mathcal{LA}$ by adding new constant symbols c and $\{c_\alpha : \alpha \in M\}$. We can now prove a Lemma similar to 23, namely

Lemma 26. $M \models \forall\varphi[\mathrm{Sat}_{\Pi_n}(\varphi) \to \mathrm{Con}(I\Delta_0 + \varphi + c{>}M)]$.

Proof of Lemma 26. By Theorem 1 we have that for all $n \geq 2$:

$$B\Sigma_n \Rightarrow I\Sigma_{n-1}.$$

Hence, by theorem 16

$$I\Sigma_{n-1} \vdash \mathrm{Con}(\mathsf{Tr}(\Pi_n)).$$

Since $I\Delta_0$ is Π_2-axiomatized and $c > M$ is a set of Δ_0-sentences we get that

$$M \models \mathrm{Con}(I\Delta_0 + \mathsf{Tr}_{\Pi_n}(M) + c > M).$$

□

Now let $LA^{*,H}$ be the extension of LA^* obtained by adding Henkin constants $\{d_a : a \in M\}$ and H the corresponding set of Henkin sentences.

Lemma 27. *There exists a set of sentences (in the sense of M):*

$$\Sigma = \bigcup_{i \in \mathbb{N}} \Sigma_i$$

such that:

1. *Σ contains $(I\Delta_0) + \mathsf{Tr}_{\Pi_n}(M) + c{>}M$,*
2. *for every sentence $\exists x\varphi(x)$ of $LA^{*,H}$ there exists a constant d_φ such that:*

$$\exists x\varphi(x) \to \varphi(d_\varphi) \in \Sigma,$$

3. *for every sentence θ of $LA^{*,H}$,*

$$\theta \in \Sigma \text{ or } \neg\theta \in \Sigma,$$

4. *for every formula* $\theta(y, c, \vec{c}) \in \Delta_{n-1}$ *of* $LA^{*,H}$*, either*

$$\neg\exists y \leq c_a \theta(y, c, \vec{c}) \in \Sigma \quad \textit{or} \quad \theta(c_b, c, \vec{c}) \in \Sigma \textit{ for some } b \leq a,\ b \in M.$$

5. *for all* $i \in \mathbb{N}$*,*

$$M \models \mathrm{Con}(\Sigma_i).$$

We now have a complete and consistent theory Σ for $\mathcal{LA}^{*,H}$ extending $I\Delta_0$, containing the Π_n-truth of M and all sentences of the form $c > c_\alpha, \alpha \in M$. Let J be a model of (the standard part of) Σ. Since $I\Delta_0$ is contained in Σ, J is a model of $I\Delta_0$.

$$J^* \models Tr_{\Pi_n}(M). \tag{4.14}$$

Now let J be the reduct of J^* to $\mathcal{LA}$. It is easy to see that

$$J \models Tr_{\Pi_n}(M). \tag{4.15}$$

and that M is isomorphic to a substructure of J. Let K be the substructure of J with universe the set of Σ_n-definable elements of J with parameters form the set

$$\{c_a^{J^*} \mid a \in M\} \cup \{c^{J^*}\},$$

i.e.

$$K = K^n\left(J, (\{c_a \mid a \in M\} \cup \{c\})^{J^*}\right).$$

Since $I\Sigma_{n-2}$ in Π_n axiomatizable, $J \models I\Sigma_{n-2}$. By Theorem 17 we get that

$$K \prec_{n-1} J. \tag{4.16}$$

By (4.15) and (4.16) we get that

$$K \models Tr_{\Pi_n}(M). \tag{4.17}$$

Claim 2. M is a proper initial segment of K, i.e. $M \subset_e K$.

Indeed, suppose that $K \models b < a$, for some $a \in M$. Then $J \models b < a$ and so for some $\phi \in \Sigma_n$

$$J^* \models (\phi(c_b, c, \vec{c}) \wedge c_b < c_a) \wedge \forall x \phi(x, c, \vec{c}) \rightarrow x = c_b), \tag{4.18}$$

which means that

$$(\phi(c_b, c, \vec{c}) \wedge c_b < c_a) \wedge \forall x \phi(x, c, \vec{c}) \rightarrow x = c_b) \in \Sigma. \tag{4.19}$$

By the construction of Σ either

$$\neg\exists x < c_a \phi(x, c, \vec{c}) \in \Sigma \quad \text{or} \quad \phi(c_f, c, \vec{c}) \in \Sigma \text{ for a specific } f < a,\ f \in M.$$

By (4.19) and the fact that $J^* \models \Sigma$, it cannot be the case that $\neg\exists x < c_a \phi(x, c, \vec{c}) \in \Sigma$. Hence,

$$\phi(c_f, c, \vec{c}) \in \Sigma \text{ for a specific } f \leq a$$

which implies that $c_b = c_f \in \Sigma$. Thus b is (the image of) an element of M and so M is an initial segment of K

Now recall that c^{J^*} is an element of (the universe) of J and that $c_a^{J^*} < c^{J^*} \in \Sigma$ for all $\alpha \in M$, i.e.

$$J \models c_a^{J^*} < c^{J^*}, \quad \text{for all } a \in M.$$

It follows that M is (isomorphically embedded to) a proper initial segment of J. In addition,

$$K \models \mathrm{Tr}_{\Pi_n}(M).$$

Therefore, by Proposition 2, we have that

$$M \prec_{n,e} K.$$

Since, clearly, $K \models I\Delta_0$, we see that K has all the required properties, i.e.

$$M \prec_{n,e} K \models I\Delta_0$$

which completes the proof. □

4.3 A note on cardinality

Given that Theorem 9 holds for models of any cardinality, it is natural to expect that Theorems 8 and 10 also hold for models of any cardinality. Indeed,

> *Such a possibility was first suggested by A. Wilkie,*

as mentioned in [3], in which P. Clote tried, using a formalization of a recursion theoretic argument, to show that Theorem 8 holds for every model of $B\Sigma_n$, $n \geq 2$. Unfortunately, Clote's approach fell short of his aim; however, it led to a proof of the following result (see [4]).

Theorem 20. *Every nonstandard model of* $I\Sigma_n$, $n \geq 2$, *has a proper* Σ_n*-elementary end extension satisfying* $I\Delta_0$.

Note that, since $B\Sigma_n \Rightarrow I\Sigma_{n-1}$ (as proved in [15]), for any $n \geq 1$, a straightforward consequence of Clote's result is

Theorem 21. *Every nonstandard model of* $B\Sigma_n$, $n \geq 3$, *has a proper* Σ_{n-1}*-elementary end extension satisfying* $I\Delta_0$.

The case for $n = 1$ remains open and as we have seen Theorem 12 is the best result we have so far in this direction.

We believe that a proof of Theorem 20 can be given, in the spirit of the proofs of the previous two sections, i.e. using a variant of the proof of the ACT.

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Index