# Model-theoretic investigations on ‘overwhelming majority’ default conditionals

**Christos Rantsoudis**
**Supervisor: Costas D. Koutras**

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**Abstract**

Defeasible conditionals of the form '*if A then normally B*' are usually interpreted with the aid of a '*normality*' ordering between possible states of affairs: $A \Rightarrow B$ is true if it happens that in the most '*normal*' (least *exceptional*) $A$-worlds, $B$ is also true. Another plausible interpretation of '*normality*' introduced in nonmonotonic reasoning dictates that $A \Rightarrow B$ is true iff $B$ is true in '*most*' $A$-worlds. A formal account of '*most*' in this *majority*-based approach to default reasoning has been given through the usage of weak filters and weak ultrafilters, capturing at least, a basic core of a size-oriented approach to defeasible reasoning. In this paper, we investigate *defeasible conditionals* constructed upon a notion of '*overwhelming majority*', defined as '*truth in a cofinite subset of* $\omega$', the first infinite ordinal. One approach employs the modal logic of the frame $(\omega, <)$, used in the temporal logic of discrete linear time. We introduce and investigate conditionals, defined modally over $(\omega, <)$; several modal definitions of the conditional connective are examined, with an emphasis on the nonmonotonic ones. An alternative interpretation of '*majority*' as sets *cofinal* (in $\omega$) rather than cofinite (subsets of $\omega$) is examined. For all these modal approaches over $(\omega, <)$, a decision procedure readily emerges, as the modal logic **KD4LZ** of this frame is well-known and a translation of the conditional sentences can be mechanically checked for validity. A second approach employs the conditional version of Scott-Montague semantics, in the form of $\omega$-many possible worlds, endowed with neighborhoods populated by its cofinite subsets. Again, different conditionals are introduced and examined. Although it is difficult to obtain a completeness theorem (since it is not easy to capture 'cofiniteness-in-$\omega$' syntactically) this research reveals the possible structure of '*overwhelming majority*' conditionals, whose relative strength is compared to (the conditional logic 'equivalent' of) KLM logics and other conditional logics in the literature.

# Contents

# Chapter 1

# Introduction

*Conditional Logic* is primarily concerned with the logical and semantic analysis of the rich class of *conditional statements*, identified with the sentences conforming with the '*if $X$ then $Y$*' structure. The topic has roots in antiquity and the medieval times but its contemporary development seems to start with F. Ramsey in the '30s and has blossomed after the late '60s [AC14]. There exist various conditionals of interest in Philosophy, Logic, Computer Science and Artificial Intelligence, including *counterfactual conditionals, causal conditionals, deontic conditionals, normality conditionals* (see [CdCH96] for a broad overview of applications); they represent different linguistic constructions with a common structural form ('*if ... then*') and the aim of the field is to provide a unifying formal logical account that accurately captures their essential meaning.

Artificial Intelligence has been interested in *conditional logics* for *default reasoning* already from the '80s (see the work of J. Delgrande [Del87, Del88]) *counterfactual reasoning* (M. Ginsberg, [Gin86]) and '*normality conditionals*' in nonmonotonic reasoning [Bel90, Bou94]. The reader is referred to the handbook article of J. Delgrande [Del98] for a broad overview of conditional logics for defeasible reasoning. The investigations on the intimate relation of conditional logics to nonmonotonic reasoning have been further triggered by the seminal work of S. Kraus, D. Lehmann and M. Magidor [KLM90, LM92], whose framework (KLM) has become the 'industry standard' for nonmonotonic consequence relations. There exist various possible-worlds semantics for conditional logics (see [Nut80, Del98]) and a connection to modal logic (known from D. Lewis' work [Lew73]) which has been further explored by the modal construction of '*normality conditionals*' [Lam91, Bou94].

A logic of '*normality conditionals*' for default reasoning, attempts to pin down the principles governing the statements of the form '*if $A$, then normally $B$ is the case*'. '*Normally*' is susceptible to a variety of interpretations. One is based on a '*normality*' ordering between possible worlds: $A \Rightarrow B$ is true if it happens that in the most '*normal*' (least *exceptional*) $A$-worlds, $B$ is also true [Lam91, Bou94]. Another, more recent one [Jau08] interprets '*normally*' as a '*majority*' quantifier: $A \Rightarrow B$ is true iff $B$ is true in '*most*' $A$-worlds. Questions of '*size*' in preferential nonmonotonic reasoning have been firstly introduced by K. Schlechta [Sch95, Sch97]; the notion of '*weak filter*' that emerged (as a '*core*' definition of '*large*' subsets) has been employed also in modal epistemic logics [KMZ14].

A majority-based account of default conditionals, depends heavily on what counts as a 'majority' of alternative situations, what is a 'large' set of possible worlds. It is difficult to state a good definition that would work for both the finite and the infinite case; the notions of *weak filters* and *weak ultrafilters* that have been used capture the minimum requirements of such a notion [Sch97, KMZ14]. In this paper, we experiment with a notion of '*overwhelming majority*', combined with the widely accepted intuition that $(A \Rightarrow B)$ could or should mean that $(A \wedge B)$ is more plausible than $(A \wedge \neg B)$. We define conditionals of this form to (essentially) mean that $A \wedge B$ is true in '*almost all*' ('*all, but finitely many*') points in the countable modal frame $(\omega, <)$ (the first infinite ordinal, strictly ordered under $<$), whose modal axiomatization (the normal modal logic **KD4LZ**) is known as the 'future' fragment of the temporal logic of discrete linear time [Gol92, Seg70]. This majority conditional is modally defined and this readily provides a decision procedure, as a modal translation of conditional formulas can be checked for validity in $(\omega, <)$ using any of the proof procedures known for **KD4LZ**. We examine the properties of this conditional, in particular with respect to the (conditional incarnation of the) '*conservative core*' of defeasible reasoning set by the KLM framework. The paradigm of '*overwhelming majority*' in our work is consistently represented with cofinite subsets of $\omega$. En route, we discuss variants: trying *cofinal* (rather than *cofinite*) subsets of $\omega$, and/or varying the modal definition of the conditional connective. Then, we discuss the possibility of defining conditional over cofinite subsets of $\omega$ in the neighborhood semantics for conditional logics; we prove that the conditionals defined can be very weak, even compared to the conditionals introduced in [Del06].

The paper is organized as follows: In Chapter 2 we provide the necessary background material, establishing notation and terminology. Chapter 3 contains the main results of this paper and is divided into two sections. In 3.1 we give the basic definitions of our approach and examine the conditional logic $\overset{\Rightarrow}{\mathbf{\Omega}}$ of '*majority default conditionals*' over $(\omega, <)$. Furthermore, we explore variants and their interaction with **Monotonicity**, as well as an alternative conditional logic $\overset{\Rightarrow}{\boldsymbol{\omega}}$. 3.2 comprises of an investigation of some additional logics defined using minimal (Scott-Montague) semantics and our results are summarized in a table at the end of the section. Finally, we conclude in Chapter 4 with some interesting questions for future research.

# Chapter 2

# Background

## 2.1 Modal Logic

We assume a language $\mathcal{L}$ of classical propositional logic, built upon the known *connectives* $\{\neg, \wedge, \vee, \rightarrow, \equiv\}$ over a countable set $\Phi = \{p_1, p_2, \ldots\}$ of *propositional variables.* The language $\mathcal{L}_\Box$ of propositional modal logic extends $\mathcal{L}$ with a modal necessity operator $\Box A$. We assume that the reader is acquainted with the basics of propositional Modal Logic; for details consult [BdRV01, Gol92]. A dual possibility operator is defined by $\Diamond A \equiv \neg\Box\neg A$.

**Modal Logics** are sets of formulas from $\mathcal{L}_\Box$, containing all propositional tautologies, and closed under the rules of **Modus Ponens** (**MP**) $\frac{A, A \rightarrow B}{B}$ and **Uniform Substitution** (**US**). A **normal modal logic** is a modal logic containing the axiom

$$\mathbf{K}.\ \ \Box A \wedge \Box(A \rightarrow B) \rightarrow \Box B$$

and closed under the **Rule of Necessitation**

$$\mathbf{RN}.\ \ \frac{A}{\Box A}$$

We denote by $\mathbf{KA_1A_2}\ldots\mathbf{A_n}$ the normal modal logic axiomatized by the axioms $\mathbf{A_1}\ldots\mathbf{A_n}$. A formula $B$ is a theorem of logic $\Lambda$ (denoted by $\vdash_\Lambda B$) iff $B \in \Lambda$.

Normal Modal Logics are interpreted over **relational possible-worlds models** or ***Kripke* models**. A *Kripke frame* $\mathfrak{F} = (W, R)$ consists of a set of possible worlds $W$ and a binary relation $R \subseteq W \times W$. We say that a world $w$ '*sees*' a world $v$ iff $wRv$. A valuation $V$ determines which propositional variables are true inside each possible world. Given a valuation $V$, a *Kripke model* $\mathfrak{M}$ is the triple $(W, R, V)$. Within a world $w$, the propositional connectives $\{\neg, \wedge, \vee, \rightarrow, \equiv\}$ are interpreted classically, while $\Box A$ is true at $w$ iff it is true in every world '*seen*' by $w$, i.e. $\mathfrak{M}, w \models \Box A$ iff $(\forall v \in W)(wRv \rightarrow \mathfrak{M}, v \models A)$. A formula $A$ is said to be *valid* in a model $\mathfrak{M}$ ($\mathfrak{M} \models A$) iff it is true at every world $w \in W$ of $\mathfrak{M}$. A formula $A$ is *valid* in a frame $\mathfrak{F}$ ($\mathfrak{F} \models A$) iff it is valid in every model $\mathfrak{M}$ of $\mathfrak{F}$. A formula $A$ is *valid* in a class of frames $C$ iff it is valid in every frame $\mathfrak{F}$ of $C$. We will denote by $\|A\|_\mathfrak{M}$ the set of worlds $w \in W$ in which $A$ is true at model $\mathfrak{M}$, i.e. $\|A\|_\mathfrak{M} = \{w \in W \mid \mathfrak{M}, w \models A\}$. If the model $\mathfrak{M}$ is obvious, the subscript is usually dropped and we denote it simply by $\|A\|$. **General Frames** are *possible-worlds frames* $\mathfrak{F} = (W, R, X)$, where $X$ is a subset of $2^W$ closed under union, intersection, complement and the operator $\Box Y = \{w \in W \mid (\forall u \in W)(wRu \to u \in Y)\}$. A *Kripke model* $\mathfrak{M} = (W, R, X, V)$ (an *admissible* model) based on the general frame $\mathfrak{F}$ requires that $V$ draws its valuations for propositional variables (and provably for every wff) from the set $X$ of admissible valuations (see [BdRV01]).

A normal modal logic $\Lambda$ is ***sound*** with respect to a class $C$ of frames iff $\vdash_\Lambda A$ implies $C \models A$. $\Lambda$ is ***complete*** with respect to $C$ iff $C \models A$ implies $\vdash_\Lambda A$. A logic $\Lambda$ is ***determined*** by a class $C$ of frames iff it is both *sound* and *complete* with respect to $C$.

We are going to make use of the following fact (see [Gol92, Seg70]). It is well known that the frame $(\omega, <)$ of the natural numbers with their natural strict ordering is axiomatized by the logic $\mathbf{\Omega}$, where $\mathbf{\Omega}$ is an abbreviation for the normal modal logic **K4DLZ**, axiomatized by:

**4**. $\Box A \to \Box\Box A$

**D**. $\Box A \to \Diamond A$

**L**. $\Box(A \wedge \Box A \to B) \vee \Box(B \wedge \Box B \to A)$

**Z**. $\Box(\Box A \to A) \to (\Diamond\Box A \to \Box A)$

The logic $\mathbf{\Omega}$ has been investigated in the context of axiomatizing the '*future*' fragment of discrete linear time. We will extensively exploit below that $\vdash_\Omega A$ iff $(\omega, <) \models A$.

## 2.2 Conditional Logics

The language $\mathcal{L}_{\Rightarrow}$ of propositional conditional logic extends $\mathcal{L}$ with a binary conditional connective $(A \Rightarrow B)$, interpreted for our purposes as '*$A$ normally implies $B$*' (also interpreted in the literature as a *counterfactual conditional, causal conditional, indicative conditional*, etc). The reader is referred to [Nut80] for more details; see also [Poz10].

The systems of conditional logic are defined as sets closed under certain rules and possessing certain axioms. Important rules include:

**RCEA**. $\dfrac{A \equiv B}{(A \Rightarrow C) \equiv (B \Rightarrow C)}$

**RCK**. $\dfrac{(A_1 \wedge ... \wedge A_n) \rightarrow B}{(C \Rightarrow A_1 \wedge ... \wedge C \Rightarrow A_n) \rightarrow (C \Rightarrow B)}$

**RCEC**. $\dfrac{A \equiv B}{(C \Rightarrow A) \equiv (C \Rightarrow B)}$

**RCE**. $\dfrac{A \rightarrow B}{A \Rightarrow B}$

Important axioms comprise:

**ID**. $A \Rightarrow A$

**CUT**. $(A \wedge B \Rightarrow C) \wedge (A \Rightarrow B) \rightarrow (A \Rightarrow C)$

**AC**. $(A \Rightarrow B) \wedge (A \Rightarrow C) \rightarrow (A \wedge B \Rightarrow C)$

**CC**. $(A \Rightarrow B) \wedge (A \Rightarrow C) \rightarrow (A \Rightarrow B \wedge C)$

**Loop**. $(A_0 \Rightarrow A_1 \wedge \ldots \wedge A_k \Rightarrow A_0) \rightarrow (A_0 \Rightarrow A_k)$

**OR**. $(A \Rightarrow C) \wedge (B \Rightarrow C) \rightarrow (A \vee B \Rightarrow C)$

**CV**. $(A \Rightarrow B) \wedge \neg(A \Rightarrow \neg C) \rightarrow (A \wedge C \Rightarrow B)$

**CSO**. $(A \Rightarrow B) \wedge (B \Rightarrow A) \rightarrow ((A \Rightarrow C) \equiv (B \Rightarrow C))$

**CM**. $(A \Rightarrow B \wedge C) \rightarrow (A \Rightarrow B) \wedge (A \Rightarrow C)$

**MP**. $(A \Rightarrow B) \rightarrow (A \rightarrow B)$

**MOD**. $(\neg A \Rightarrow A) \to (B \Rightarrow A)$

**CA**. $(A \Rightarrow B) \wedge (C \Rightarrow B) \to (A \wedge C \Rightarrow B)$

**CS**. $(A \wedge B) \to (A \Rightarrow B)$

**CEM**. $(A \Rightarrow B) \vee (A \Rightarrow \neg B)$

**SDA**. $(A \vee B \Rightarrow C) \to (A \Rightarrow C) \wedge (B \Rightarrow C)$

Principles of interest to the study of '*normality*' conditionals:

**Transitivity** $(A \Rightarrow B) \wedge (B \Rightarrow C) \to (A \Rightarrow C)$

**Weak Transitivity** $(A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$

**Monotonicity** $(A \Rightarrow B) \to (A \wedge C \Rightarrow B)$

**Weak Monotonicity** $(A \Rightarrow B) \Rightarrow (A \wedge C \Rightarrow B)$

**Modus Ponens** $A \wedge (A \Rightarrow B) \to B$

**Weak Modus Ponens** $A \wedge (A \Rightarrow B) \Rightarrow B$

Here are some important logics from the literature (see [Nut80]):

- **CU** = **RCEC** + **RCEA** + **RCK** + **ID** + **AC** + **CUT**
- **CE** = **RCEC** + **RCEA** + **RCK** + **ID** + **AC** + **CUT** + **CA**
- **V** = **RCEC** + **RCEA** + **RCK** + **ID** + **AC** + **CUT** + **CA** + **CV**

## 2.3 KLM Logics

The **KLM approach** [KLM90, LM92] to nonmonotonic reasoning (NMR) emerged in the early '90s from a study of **nonmonotonic consequence relations** and their connection to the *preferential semantics* for NMR. We only provide the basic facts in the interests of demonstrating their connection to the conditional logics defined; see the original articles for details.

In this section, we work with the language $\mathcal{L}$ of propositional logic and the new formation rule (in the metalanguage): $A \mid\!\sim B$ denotes a (KLM) default conditional or a nonmonotonic derivability relation ($A, B \in \mathcal{L}$). The following rules are sometimes called *Gabbay - Makinson conditions*:

**REF**. $A \mid\!\sim A$

**RW**. $\dfrac{\vdash A \rightarrow B \quad C \mid\!\sim A}{C \mid\!\sim B}$

**LLE**. $\dfrac{\vdash A \equiv B \quad A \mid\!\sim C}{B \mid\!\sim C}$

**CM**. $\dfrac{A \mid\!\sim B \quad A \mid\!\sim C}{A \wedge B \mid\!\sim C}$

**CUT**. $\dfrac{A \wedge B \mid\!\sim C \quad A \mid\!\sim B}{A \mid\!\sim C}$

**Loop**. $\dfrac{A_0 \mid\!\sim A_1 \ \dots \ A_k \mid\!\sim A_0}{A_0 \mid\!\sim A_k}$

**AND**. $\dfrac{A \mid\!\sim B \quad A \mid\!\sim C}{A \mid\!\sim B \wedge C}$

**RM**. $\dfrac{A \mid\!\sim C \quad A \wedge B \not\mid\!\sim C}{A \mid\!\sim \neg B}$

**OR**. $\dfrac{A \mid\!\sim C \quad B \mid\!\sim C}{A \vee B \mid\!\sim C}$

The KLM logics comprise the following systems, from the strongest to the weakest: **R** (*Rational*), **P** (*Preferential*), **CL** (*Loop-Cumulative*) and **C** (*Cumulative*).

- **R** consists of the propositional calculus plus **REF**, **LLE**, **RW**, **CM**, **AND**, **OR**, **RM**.
- **P** results from **R** by dropping **RM**.
- **CL** results from **P** by dropping **OR** and adding **Loop** and **CUT**.
- **C**, the weakest, results from **CL** by dropping **Loop**.

The following correspondences are known; [CL92, Poz10], see also Table 1, page 64:

- **R** corresponds to the '*flat*' fragment of conditional logic **V** (**CE** + **CV**).
- **P** corresponds to the '*flat*' fragment of conditional logic named **CE**.

- **C** corresponds to the '*flat*' fragment of **CU**.

A '*translation*' of the KLM conditions to conditional logic principle is known from [CL92]. It is readily checked in the correspondence shown in Table 1, at page 64.

## 2.4 Modal approaches to Default Conditionals

The most famous modal approach in order to define a conditional logic of normality is due to Boutilier and Lamarre using the modal logic **S4**, axiomatized by:

**K**. $\Box A \wedge \Box(A \to B) \to \Box B$

**T**. $\Box A \to A$

**4**. $\Box A \to \Box\Box A$

In [Lam91, Bou94] they define a *normality ordering* in sets of possible worlds, where '*world u sees the world v*' is translated as '*v is more normal (or less exceptional) than w*'. The conditional statement $A \Rightarrow B$ which is interpreted as '*if A then normally B*' is defined as follows:

$$(A \Rightarrow B) \quad \equiv_{def} \quad \Box(A \to \Diamond(A \wedge \Box(A \to B)))$$

Through this definition, the desired interpretation of '*B is true in the most normal worlds where A is also true*' becomes clear and the conditional logic of normality that is defined is shown to be equivalent with the modal logic **S4** mentioned above. This is achieved by defining the *necessity operator* $\Box A$ to be a shorthand for $(\neg A \Rightarrow A)$ and, accordingly, the *possibility operator* $\Diamond A$ to be a shorthand for $\neg(A \Rightarrow \neg A)$.

The conditional logics defined by this intuition are shown to contain many important axioms that are considered essential in the study of normality statements of this kind and to not contain others that are rather undesirable (a good example is the principles of interest to the study of '*normality*' conditionals mentioned above, against their weak counterparts). The cumulative results that have been shown can be again easily checked in Table 1, page 64, where they can also be compared with other known attempts in the literature that have been made in order to define such logics.

# Chapter 3

# '*Overwhelming Majority*' Conditionals

We wish to define (variants of) a default conditional of the form '*A normally implies B*'. The fundamental question is to provide a concrete interpretation of the statement '*normally*'. Earlier approaches resort to '*normality*' orderings ([Del88, Lam91, Bou94]: $A \Rightarrow B$ is true iff $B$ is true in the most normal $A$-worlds), and considerations of '*size*' ([Jau08]: $A \Rightarrow B$ is true iff $B$ is true in '*many*' ('*most*') $A$-worlds). Another fundamental intuition dictates that $(A \wedge B)$ is '*preferred*' over $(A \wedge \neg B)$ for $(A \Rightarrow B)$ to be true; another, similar view, is that $(A \rightarrow B)$ is more '*normal*' than $(A \rightarrow \neg B)$ [Boc01].

In this paper, we design '*majority default*' conditionals based on this intuition - note that we consistently work with the infinite set $\omega$ of possible worlds:

- $A \Rightarrow B$ is an '*overwhelming majority*' conditional, in the sense that $B$ is true in a very large set of $A$-worlds. We consider as '*large*' the cofinite subsets of $\omega$ (and '*small*' the finite ones). Obviously, this is an (extreme, but) intuitively acceptable form of '*overwhelming majority*'. We remind the reader that a set is cofinite iff its complement is finite (with respect to $\omega$ in this context).
- $A \Rightarrow B$ is true , either vacuously (if there are no '*many*' A-worlds) or essentially: iff $\|A \wedge B\|$ is much larger (it is a cofinite set) than $\|A \wedge \neg B\|$.

Throughout this chapter, we will be working with the set $\omega$ of countably many possible worlds, with the aim of providing different accounts of '*A normally implies B*' $(A \Rightarrow B)$ as '*B is true in all, but finitely many, A-worlds*'.

## 3.1 Conditionals modally defined over $(\omega, <)$

Our first approach is to define a '*majority*' conditional over the frame $(\omega, <)$ of natural numbers, strictly ordered under $<$. Conforming to the intuition(s) expressed above, we will define

$(A \Rightarrow B)$ as shorthand for:

$$(A \Rightarrow B) \quad \equiv_{def} \quad \Diamond\Box\neg A \ \vee \ \Diamond\Box(A \wedge B)$$

The import of such a modal definition over $(\omega, <)$ is that either there do not exist '*many*' $A$-worlds ($A$ settles down to be false, at some '*point*') or there exist '*overwhelmingly many*' '*points*' in which $A \wedge B$ is true ($A \wedge B$ is true in a cofinite subset of $\omega$). To state properly the conditional logic induced by this definition , we proceed to define the following translation of conditionals to the (mono)modal language $\mathcal{L}_\Box$:

**Definition 3.1.1** We recursively define the following translation $()^* : \mathcal{L}_\Rightarrow \to \mathcal{L}_\Box$

(i) $(p)^* = p$ , if $p \in \Phi$ ($p$ is a propositional variable)

(ii) $(A \circ B)^* = (A)^* \circ (B)^*$ for $\circ \in \{\wedge, \vee, \to, \equiv\}$

(iii) $(\neg A)^* = \neg(A)^*$

(iv) $(A \Rightarrow B)^* = \Diamond\Box\neg(A)^* \vee \Diamond\Box(A^* \wedge B^*)$

We proceed to define the logic $\overset{\Rightarrow}{\mathbf{\Omega}}$ of '*majority default conditionals*' over $(\omega, <)$:

**Definition 3.1.2 [Conditional Logic $\overset{\Rightarrow}{\mathbf{\Omega}}$].** The logic $\overset{\Rightarrow}{\mathbf{\Omega}}$ consists of all formulae $A \in \mathcal{L}_\Rightarrow$, such that:

$$A \in \overset{\Rightarrow}{\mathbf{\Omega}} \quad \text{iff} \quad (\omega, <) \models A^* \quad \text{iff} \quad \vdash_\Omega A^*$$

The second equivalence follows from the completeness of $\mathbf{\Omega} = \mathbf{K4DLZ}$ with respect to the frame $(\omega, <)$.

**Fact 3.1.3** Let $\mathfrak{M}$ be a model of $\mathfrak{F} = (\omega, <)$ and $n \in \omega$ an arbitrary world. We will say that $\mathfrak{M}, n \models (A \Rightarrow B)$ iff one of the following holds:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg A$

(ii) $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models A \wedge B$

Some comments on the definition of $\overrightarrow{\mathbf{\Omega}}$ are in order. This model-theoretic modal definition of the conditional has the advantage that it is a clear '*majority*' definition, easy to understand, with an intuitively acceptable '*largeness*' condition. It captures '*cofinite*' subsets of $\omega$ in an easy manner, in contrast to the difficulty of capturing this axiomatically. Further on, and perhaps more important, a decision procedure readily emerges from the definition:

To check whether a conditional $(A \Rightarrow B)$ is in $\overrightarrow{\mathbf{\Omega}}$, simply check whether $(A \Rightarrow B)^*$ has a tableaux proof in **K4DLZ**; such a proof procedure exists.

On the other hand, the ordering in $(\omega, <)$ has not any clear '*preference*' meaning here. It implicitly provides a '*temporal*' change flavour in the conditional defined, see below on the invalidity of **ID**.

Let us proceed to check the properties of $\overrightarrow{\mathbf{\Omega}}$. Throughout the proofs, $\mathfrak{F}$ refers to $(\omega, <)$ and all models $\mathfrak{M}$ are based on that frame.

**Theorem 3.1.4** The logic $\overrightarrow{\mathbf{\Omega}}$:

1. is closed under the rules **RCEA**, **RCK** and **RCEC**
2. contains the axioms **CUT**, **AC**, **CC**, **Loop**, **OR**, **CSO**, **CM**, **CA**, **Transitivity**, **Weak Transitivity** and **Weak Modus Ponens**

Proof. **RCEA**: Let $\mathfrak{M} \models (A \equiv B)$. We have to show that

$$\mathfrak{M} \models (A \Rightarrow C) \equiv (B \Rightarrow C)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \Rightarrow C)$. Then, either:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg A$, or

(ii) $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge C)$

Case (i): By $\mathfrak{M} \models (A \equiv B)$ we also have that $(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models (\neg A \equiv \neg B)$. This gives that $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg B$, and $\mathfrak{M}, n \models (B \Rightarrow C)$ follows.

Case (ii): Similarly, by $\mathfrak{M} \models (A \equiv B)$ we obtain

$$(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge C \equiv B \wedge C)$$

It follows that $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (B \wedge C)$, and thus $\mathfrak{M}, n \models (B \Rightarrow C)$.

So, if $\mathfrak{M}, n \models (A \Rightarrow C)$, then $\mathfrak{M}, n \models (B \Rightarrow C)$, which gives that

$$\mathfrak{M}, n \models (A \Rightarrow C) \rightarrow (B \Rightarrow C)$$

Similarly, $\mathfrak{M}, n \models (B \Rightarrow C) \rightarrow (A \Rightarrow C)$. Since the world $n$ was arbitrarily chosen, the proof is complete.

**RCK**: Let $\mathfrak{M} \models (A_1 \wedge \ldots \wedge A_n) \rightarrow B$. We have to show that

$$\mathfrak{M} \models (C \Rightarrow A_1 \wedge \ldots \wedge C \Rightarrow A_n) \rightarrow (C \Rightarrow B)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (C \Rightarrow A_1 \wedge \ldots \wedge C \Rightarrow A_n)$. Obviously, $\mathfrak{M}, n \models (C \Rightarrow A_1)$ and ... and $\mathfrak{M}, n \models (C \Rightarrow A_n)$. Then, either:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg C$, or

(ii) $(\exists n_3 > n)(\forall n_4 > n_3)(\mathfrak{M}, n_4 \models (C \wedge A_1)$ and ... and $\mathfrak{M}, n_4 \models (C \wedge A_n))$, which means that $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (C \wedge A_1 \wedge \ldots \wedge A_n)$

Case (i): By definition, $\mathfrak{M}, n \models (C \Rightarrow B)$ follows.

Case (ii): By $\mathfrak{M} \models (A_1 \wedge \ldots \wedge A_n) \rightarrow B$ we obtain

$$(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (C \wedge A_1 \wedge \ldots \wedge A_n) \rightarrow C \wedge B$$

It follows that $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (C \wedge B)$, and thus $\mathfrak{M}, n \models (C \Rightarrow B)$.

So, if $\mathfrak{M}, n \models (C \Rightarrow A_1 \wedge \ldots \wedge C \Rightarrow A_n)$, then $\mathfrak{M}, n \models (C \Rightarrow B)$, which gives that

$$\mathfrak{M}, n \models (C \Rightarrow A_1 \wedge \ldots \wedge C \Rightarrow A_n) \rightarrow (C \Rightarrow B)$$

Since the world $n$ was arbitrarily chosen, the proof is complete.

**RCEC**: Let $\mathfrak{M} \models (A \equiv B)$. We have to show that

$$\mathfrak{M} \models (C \Rightarrow A) \equiv (C \Rightarrow B)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (C \Rightarrow A)$. Then, either:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg C$, or

(ii) $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (C \wedge A)$

Case (i): By definition, $\mathfrak{M}, n \models (C \Rightarrow B)$ follows.

Case (ii): By $\mathfrak{M} \models (A \equiv B)$ we obtain

$$(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (C \wedge A \equiv C \wedge B)$$

It follows that $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (C \wedge B)$, and thus $\mathfrak{M}, n \models (C \Rightarrow B)$.

So, if $\mathfrak{M}, n \models (C \Rightarrow A)$, then $\mathfrak{M}, n \models (C \Rightarrow B)$, which gives that

$$\mathfrak{M}, n \models (C \Rightarrow A) \rightarrow (C \Rightarrow B)$$

Similarly, $\mathfrak{M}, n \models (C \Rightarrow B) \rightarrow (C \Rightarrow A)$. Since the world $n$ was arbitrarily chosen, the proof is complete.

**CUT**: We have to show that

$$\mathfrak{F} \models (A \wedge B \Rightarrow C) \wedge (A \Rightarrow B) \rightarrow (A \Rightarrow C)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \wedge B \Rightarrow C) \wedge (A \Rightarrow B)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$. Obviously, $\mathfrak{M}, n \models (A \wedge B \Rightarrow C)$ and $\mathfrak{M}, n \models (A \Rightarrow B)$. Then, either:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg A$, or

(ii) $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge B \wedge C)$

Case (i): By definition, $\mathfrak{M}, n \models (A \Rightarrow C)$ follows.

Case (ii): We also have that $(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge C)$, and thus $\mathfrak{M}, n \models (A \Rightarrow C)$.

So, if $\mathfrak{M}, n \models (A \wedge B \Rightarrow C) \wedge (A \Rightarrow B)$, then $\mathfrak{M}, n \models (A \Rightarrow C)$, which gives that

$$\mathfrak{M}, n \models (A \wedge B \Rightarrow C) \wedge (A \Rightarrow B) \rightarrow (A \Rightarrow C)$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete.

**AC**: We have to show that

$$\mathfrak{F} \models (A \Rightarrow B) \wedge (A \Rightarrow C) \rightarrow (A \wedge B \Rightarrow C)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (A \Rightarrow C)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$. Obviously, $\mathfrak{M}, n \models (A \Rightarrow B)$ and $\mathfrak{M}, n \models (A \Rightarrow C)$. Then, either:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg A$, or

(ii) $(\exists n_3 > n)(\forall n_4 > n_3)(\mathfrak{M}, n_4 \models (A \wedge B)$ and $\mathfrak{M}, n_4 \models (A \wedge C))$, which means that $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge B \wedge C)$

Case (i): We also have that $(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg(A \wedge B)$, and thus $\mathfrak{M}, n \models (A \wedge B \Rightarrow C)$.

Case (ii): By definition, $\mathfrak{M}, n \models (A \wedge B \Rightarrow C)$ follows.

So, if $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (A \Rightarrow C)$, then $\mathfrak{M}, n \models (A \wedge B \Rightarrow C)$, which gives that

$$\mathfrak{M}, n \models (A \Rightarrow B) \wedge (A \Rightarrow C) \rightarrow (A \wedge B \Rightarrow C)$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete.

**CC**: We have to show that

$$\mathfrak{F} \models (A \Rightarrow B) \wedge (A \Rightarrow C) \rightarrow (A \Rightarrow B \wedge C)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (A \Rightarrow C)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$. Obviously, $\mathfrak{M}, n \models (A \Rightarrow B)$ and $\mathfrak{M}, n \models (A \Rightarrow C)$. Then, either:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg A$, or

(ii) $(\exists n_3 > n)(\forall n_4 > n_3)(\mathfrak{M}, n_4 \models (A \wedge B)$ and $\mathfrak{M}, n_4 \models (A \wedge C))$, which means that $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge B \wedge C)$

Case (i): By definition, $\mathfrak{M}, n \models (A \Rightarrow B \wedge C)$ follows.

Case (ii): Similarly, by definition, $\mathfrak{M}, n \models (A \Rightarrow B \wedge C)$ follows.

So, if $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (A \Rightarrow C)$, then $\mathfrak{M}, n \models (A \Rightarrow B \wedge C)$, which gives that

$$\mathfrak{M}, n \models (A \Rightarrow B) \wedge (A \Rightarrow C) \rightarrow (A \Rightarrow B \wedge C)$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete.

**Loop**: We have to show that

$$\mathfrak{F} \models (A_0 \Rightarrow A_1 \wedge \ldots \wedge A_k \Rightarrow A_0) \rightarrow (A_0 \Rightarrow A_k)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A_0 \Rightarrow A_1 \wedge \ldots \wedge A_k \Rightarrow A_0)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$. Then, either:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg A_0$, or

(ii) $(\exists n_3 > n)(\forall n_4 > n_3)(\mathfrak{M}, n_4 \models (A_0 \wedge A_1)$ and ... and $\mathfrak{M}, n_4 \models (A_k \wedge A_0))$, which means that $(\exists n_3 > n)(\forall n_4 > n_3)\mathfrak{M}, n_4 \models (A_0 \wedge A_1 \wedge \ldots \wedge A_k)$

Case (i): By definition, $\mathfrak{M}, n \models (A_0 \Rightarrow A_k)$ follows.

Case (ii): We also have that $(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (A_0 \wedge A_k)$, and thus $\mathfrak{M}, n \models (A_0 \Rightarrow A_k)$.

So, if $\mathfrak{M}, n \models (A_0 \Rightarrow A_1 \wedge \ldots \wedge A_k \Rightarrow A_0)$, then $\mathfrak{M}, n \models (A_0 \Rightarrow A_k)$, which gives that

$$\mathfrak{M}, n \models (A_0 \Rightarrow A_1 \wedge \ldots \wedge A_k \Rightarrow A_0) \rightarrow (A_0 \Rightarrow A_k)$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete.

**OR**: We have to show that

$$\mathfrak{F} \models (A \Rightarrow C) \wedge (B \Rightarrow C) \rightarrow (A \vee B \Rightarrow C)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \Rightarrow C) \wedge (B \Rightarrow C)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$. Obviously, $\mathfrak{M}, n \models (A \Rightarrow C)$ and $\mathfrak{M}, n \models (B \Rightarrow C)$. Then one of the following must hold:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models (\neg A \wedge \neg B)$

(ii) $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (\neg A \wedge B \wedge C)$

(iii) $(\exists n_5 > n)(\forall n_6 > n_5)\ \mathfrak{M}, n_6 \models (\neg B \wedge A \wedge C)$

(iv) $(\exists n_7 > n)(\forall n_8 > n_7)\ \mathfrak{M}, n_8 \models (A \wedge B \wedge C)$

Case (i): Equivalently, we have that $(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg(A \vee B)$, and $\mathfrak{M}, n \models (A \vee B \Rightarrow C)$ follows.

Cases (ii) - (iv): We also have that $(\forall n_{4,6,8} > n_{3,5,7})\ \mathfrak{M}, n_{4,6,8} \models (A \vee B) \wedge C$, and thus $\mathfrak{M}, n \models (A \vee B \Rightarrow C)$.

So, if $\mathfrak{M}, n \models (A \Rightarrow C) \wedge (B \Rightarrow C)$, then $\mathfrak{M}, n \models (A \vee B \Rightarrow C)$, which gives that

$$\mathfrak{M}, n \models (A \Rightarrow C) \wedge (B \Rightarrow C) \rightarrow (A \vee B \Rightarrow C)$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete.

**CSO**: We have to show that

$$\mathfrak{F} \models (A \Rightarrow B) \wedge (B \Rightarrow A) \rightarrow ((A \Rightarrow C) \equiv (B \Rightarrow C))$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (B \Rightarrow A)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$. Obviously, $\mathfrak{M}, n \models (A \Rightarrow B)$ and $\mathfrak{M}, n \models (B \Rightarrow A)$. Then, either:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models (\neg A \wedge \neg B)$, or

(ii) $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge B)$

Case (i): By definition, $\mathfrak{M}, n \models (A \Rightarrow C)$ and $\mathfrak{M}, n \models (B \Rightarrow C)$, so $\mathfrak{M}, n \models (A \Rightarrow C) \equiv (B \Rightarrow C)$ follows.

Case (ii): Let $\mathfrak{M}, n \models (A \Rightarrow C)$. Then it follows that $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge C)$, because it cannot be the case that $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models \neg A$. By $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge B)$ we obtain

$$(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (B \wedge C)$$

It follows that $\mathfrak{M}, n \models (B \Rightarrow C)$, and thus $\mathfrak{M}, n \models (A \Rightarrow C) \rightarrow (B \Rightarrow C)$. Similarly, $\mathfrak{M}, n \models (B \Rightarrow C) \rightarrow (A \Rightarrow C)$, and thus $\mathfrak{M}, n \models (A \Rightarrow C) \equiv (B \Rightarrow C)$.

So, if $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (B \Rightarrow A)$, then $\mathfrak{M}, n \models (A \Rightarrow C) \equiv (B \Rightarrow C)$, which gives that

$$\mathfrak{M}, n \models (A \Rightarrow B) \wedge (B \Rightarrow A) \rightarrow ((A \Rightarrow C) \equiv (B \Rightarrow C))$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete.

**CM**: We have to show that

$$\mathfrak{F} \models (A \Rightarrow B \wedge C) \rightarrow (A \Rightarrow B) \wedge (A \Rightarrow C)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \Rightarrow B \wedge C)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$. Then, either:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg A$, or

(ii) $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge B \wedge C)$

Case (i): By definition, $\mathfrak{M}, n \models (A \Rightarrow B)$ and $\mathfrak{M}, n \models (A \Rightarrow C)$, so $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (A \Rightarrow C)$ follows.

Case (ii): We also have that $(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge B)$ and $(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge C)$, so $\mathfrak{M}, n \models (A \Rightarrow B)$ and $\mathfrak{M}, n \models (A \Rightarrow C)$, and thus $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (A \Rightarrow C)$.

So, if $\mathfrak{M}, n \models (A \Rightarrow B \wedge C)$, then $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (A \Rightarrow C)$, which gives that

$$\mathfrak{M}, n \models (A \Rightarrow B \wedge C) \rightarrow (A \Rightarrow B) \wedge (A \Rightarrow C)$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete.

**CA**: We have to show that

$$\mathfrak{F} \models (A \Rightarrow B) \wedge (C \Rightarrow B) \rightarrow (A \wedge C \Rightarrow B)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (C \Rightarrow B)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$. Obviously, $\mathfrak{M}, n \models (A \Rightarrow B)$ and $\mathfrak{M}, n \models (C \Rightarrow B)$. Then one of the following must hold:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models (\neg A \wedge \neg C)$

(ii) $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (\neg A \wedge C \wedge B)$

(iii) $(\exists n_5 > n)(\forall n_6 > n_5)\ \mathfrak{M}, n_6 \models (\neg C \wedge A \wedge B)$

(iv) $(\exists n_7 > n)(\forall n_8 > n_7)\ \mathfrak{M}, n_8 \models (A \wedge C \wedge B)$

Cases (i) - (iii): We also have that $(\forall n_{2,4,6} > n_{1,3,5})\ \mathfrak{M}, n_{2,4,6} \models \neg(A \wedge C)$, and thus $\mathfrak{M}, n \models (A \wedge C \Rightarrow B)$.

Case (iv): By definition, $\mathfrak{M}, n \models (A \wedge C \Rightarrow B)$ follows.

So, if $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (C \Rightarrow B)$, then $\mathfrak{M}, n \models (A \wedge C \Rightarrow B)$, which gives that

$$\mathfrak{M}, n \models (A \Rightarrow B) \wedge (C \Rightarrow B) \rightarrow (A \wedge C \Rightarrow B)$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete.

**Transitivity**: We have to show that

$$\mathfrak{F} \models (A \Rightarrow B) \wedge (B \Rightarrow C) \rightarrow (A \Rightarrow C)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (B \Rightarrow C)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$. Obviously, $\mathfrak{M}, n \models (A \Rightarrow B)$ and $\mathfrak{M}, n \models (B \Rightarrow C)$. Then one of the following must hold:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models (\neg A \wedge \neg B)$

(ii) $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (\neg A \wedge B \wedge C)$

(iii) $(\exists n_5 > n)(\forall n_6 > n_5)\ \mathfrak{M}, n_6 \models (A \wedge B \wedge C)$

Cases (i) - (ii): We also have that $(\forall n_{2,4} > n_{1,3})\ \mathfrak{M}, n_{2,4} \models \neg A$, and $\mathfrak{M}, n \models (A \Rightarrow C)$ follows.

Case (iii): Similarly, we also have that $(\forall n_6 > n_5)\ \mathfrak{M}, n_6 \models (A \wedge C)$, and thus $\mathfrak{M}, n \models (A \Rightarrow C)$.

So, if $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (B \Rightarrow C)$, then $\mathfrak{M}, n \models (A \Rightarrow C)$, which gives that

$$\mathfrak{M}, n \models (A \Rightarrow B) \wedge (B \Rightarrow C) \rightarrow (A \Rightarrow C)$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete.

**Weak Transitivity**: We have to show that

$$\mathfrak{F} \models (A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$$

Assume an arbitrary state $n \in \omega$ and $\mathfrak{M}$ a model of $\mathfrak{F}$. We have that $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$ iff one of the following holds:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg(A \Rightarrow B) \vee \neg(B \Rightarrow C)$, or

(ii) $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \Rightarrow B) \wedge (B \Rightarrow C) \wedge (A \Rightarrow C)$

Let (i) be false, that is let $(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models (A \Rightarrow B) \wedge (B \Rightarrow C)$ $(*)$. We will show that (ii) has to be true. By $(*)$ we have that $(\forall n_1 > n)(\exists n_2 > n_1)$ such that both the following disjunctions hold:

- $(\exists n_5 > n_2)(\forall n_6 > n_5)\ \mathfrak{M}, n_6 \models \neg A$ **or** $(\exists n_5 > n_2)(\forall n_6 > n_5)\ \mathfrak{M}, n_6 \models (A \wedge B)$
- $(\exists n_5 > n_2)(\forall n_6 > n_5)\ \mathfrak{M}, n_6 \models \neg B$ **or** $(\exists n_5 > n_2)(\forall n_6 > n_5)\ \mathfrak{M}, n_6 \models (B \wedge C)$

This means that one of the following must hold:

(a) $(\forall n_1 > n)(\exists n_5 > n_1)(\forall n_6 > n_5)\ \mathfrak{M}, n_6 \models (\neg A \wedge \neg B)$

(b) $(\forall n_1 > n)(\exists n_5 > n_1)(\forall n_6 > n_5)\ \mathfrak{M}, n_6 \models (\neg A \wedge B \wedge C)$

(c) $(\forall n_1 > n)(\exists n_5 > n_1)(\forall n_6 > n_5)\ \mathfrak{M}, n_6 \models (A \wedge B \wedge C)$

All of these cases give us that

$$(\forall n_1 > n)\ \mathfrak{M}, n_1 \models (A \Rightarrow B) \wedge (B \Rightarrow C) \wedge (A \Rightarrow C)$$

Consequently, we also have that

$$(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \Rightarrow B) \wedge (B \Rightarrow C) \wedge (A \Rightarrow C)$$

which is exactly (ii). So one of (i) or (ii) must hold, which means that

$$\mathfrak{M}, n \models (A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete.

**Weak Modus Ponens**: We have to show that

$$\mathfrak{F} \models A \wedge (A \Rightarrow B) \Rightarrow B$$

Assume an arbitrary state $n \in \omega$ and $\mathfrak{M}$ a model of $\mathfrak{F}$. We have that $\mathfrak{M}, n \models A \wedge (A \Rightarrow B) \Rightarrow B$ iff one of the following holds:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg A \vee \neg(A \Rightarrow B)$, or

(ii) $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models A \wedge (A \Rightarrow B) \wedge B$

Let (i) be false, that is let $(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models A \wedge (A \Rightarrow B)\ (*)$. We will show that (ii) has to be true. By $(*)$ we have that $(\forall n_1 > n)(\exists n_2 > n_1)$ such that $\mathfrak{M}, n_2 \models A$ and

$$(\exists n_5 > n_2)(\forall n_6 > n_5)\ \mathfrak{M}, n_6 \models \neg A \text{ or } (\exists n_5 > n_2)(\forall n_6 > n_5)\ \mathfrak{M}, n_6 \models (A \wedge B)$$

This means that

$$(\forall n_1 > n)(\exists n_5 > n_1)(\forall n_6 > n_5)\ \mathfrak{M}, n_6 \models (A \wedge B)$$

This gives us that

$$(\forall n_1 > n)\ \mathfrak{M}, n_1 \models (A \Rightarrow B)$$

From the last two, we also obtain

$$(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models A \wedge (A \Rightarrow B) \wedge B$$

which is exactly (ii). So one of (i) or (ii) must hold, which means that

$$\mathfrak{M}, n \models A \wedge (A \Rightarrow B) \Rightarrow B$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete. ∎

Having the intention to describe $\overset{\Rightarrow}{\boldsymbol{\Omega}}$ in full detail, we proceed now to identify the rules and axioms not present in this majority logic.

**Theorem 3.1.5** The logic $\overset{\Rightarrow}{\boldsymbol{\Omega}}$:

1. is not closed under the rule **RCE**
2. does not contain the axioms **ID**, **CV**, **MP**, **MOD**, **CS**, **CEM**, **SDA**, **Monotonicity**, **Weak Monotonicity** and **Modus Ponens**

Proof. **RCE**: We have to show that

$$\mathfrak{M} \models (A \to B) \text{ and } \mathfrak{M} \not\models (A \Rightarrow B)$$

for some model $\mathfrak{M}$ of $\mathfrak{F}$.

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \{n \mid n \text{ is even}\}$ and $V(B) = \omega$. Then $(\forall n \in \omega)\ \mathfrak{M}, n \models (A \to B)$, because $(\forall n \in \omega)\ \mathfrak{M}, n \models B$. Thus $\mathfrak{M} \models (A \to B)$.

But for an arbitrary state $n \in \omega$ we have that

$$(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models A \text{ and } (\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models \neg A$$

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B)$ and consequently $\mathfrak{M} \not\models (A \Rightarrow B)$.

**ID**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow A)$$

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \{n \,|\, n \text{ is even}\}$. Then for an arbitrary state $n \in \omega$ we have that

$$(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models A \text{ and } (\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models \neg A$$

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow A)$ and the proof is complete.

**CV**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \wedge \neg(A \Rightarrow \neg C) \rightarrow (A \wedge C \Rightarrow B)$$

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A) = V(B) = \omega$ and $V(C) = \{n \,|\, n \text{ is even}\}$. Then $(\forall n \in \omega)\ \mathfrak{M}, n \models (A \wedge B)$, so for an arbitrary state $n \in \omega$ we have that

$$(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models (A \wedge B)$$

and thus $\mathfrak{M}, n \models (A \Rightarrow B)$.

Additionally, for an arbitrary world $n \in \omega$ we have that

$$(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge C)$$

which gives $\mathfrak{M}, n \not\models (A \Rightarrow \neg C)$, or equivalently $\mathfrak{M}, n \models \neg(A \Rightarrow \neg C)$.

Consequently we have

$$\mathfrak{M}, n \models (A \Rightarrow B) \wedge \neg(A \Rightarrow \neg C)$$

But for $n \in \omega$, $\mathfrak{M}, n \not\models (A \wedge C \Rightarrow B)$, because we have that

$$(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge C) \text{ and } (\forall n_5 > n)(\exists n_6 > n_5)\ \mathfrak{M}, n_6 \models \neg C$$

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \wedge \neg(A \Rightarrow \neg C) \rightarrow (A \wedge C \Rightarrow B)$ and the proof is complete.

**MP**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \rightarrow (A \rightarrow B)$$

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega$ and $V(B) = \omega - \{n\}$. Then we have that

$$(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models (A \wedge B)$$

so by definition $\mathfrak{M}, n \models (A \Rightarrow B)$.

But $\mathfrak{M}, n \not\models (A \rightarrow B)$, because $\mathfrak{M}, n \models (A \wedge \neg B)$.

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \rightarrow (A \rightarrow B)$ and the proof is complete.

**MOD**: We have to show that

$$\mathfrak{F} \not\models (\neg A \Rightarrow A) \rightarrow (B \Rightarrow A)$$

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \{n \in \omega \,|\, n > n_1\}$ and $V(B) = \{n \,|\, n \text{ is even}\}$. Then we have that

$$(\exists n_2 > n_1)(\forall n_3 > n_2)\ \mathfrak{M}, n_3 \models A$$

so by definition $\mathfrak{M}, n_1 \models (\neg A \Rightarrow A)$.

But $\mathfrak{M}, n_1 \not\models (B \Rightarrow A)$, because

$$(\forall n_4 > n_1)(\exists n_5 > n_4)\mathfrak{M}, n_5 \models B \quad \text{and} \quad (\forall n_6 > n_1)(\exists n_7 > n_6)\mathfrak{M}, n_7 \models \neg B$$

It follows then that $\mathfrak{M}, n_1 \not\models (\neg A \Rightarrow A) \rightarrow (B \Rightarrow A)$ and the proof is complete.

**CS**: We have to show that

$$\mathfrak{F} \not\models (A \wedge B) \rightarrow (A \Rightarrow B)$$

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega$ and $V(B) = \{n \,|\, n \text{ is even}\}$. Then for an arbitrary even world $n$ we have that $\mathfrak{M}, n \models (A \wedge B)$. But $\mathfrak{M}, n \not\models (A \Rightarrow B)$, because

$$(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models A \quad \text{and} \quad (\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models \neg B$$

It follows then that $\mathfrak{M}, n \not\models (A \wedge B) \rightarrow (A \Rightarrow B)$ and the proof is complete.

**CEM**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \vee (A \Rightarrow \neg B)$$

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega$ and $V(B) = \{n \,|\, n \text{ is even}\}$. Then for an arbitrary state $n \in \omega$ we have that $\mathfrak{M}, n \not\models (A \Rightarrow B)$ and $\mathfrak{M}, n \not\models (A \Rightarrow \neg B)$, because

$$(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models A$$

$$(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models B$$

$$(\forall n_5 > n)(\exists n_6 > n_5)\ \mathfrak{M}, n_6 \models \neg B$$

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \vee (A \Rightarrow \neg B)$ and the proof is complete.

**SDA**: We have to show that

$$\mathfrak{F} \not\models (A \vee B \Rightarrow C) \rightarrow (A \Rightarrow C) \wedge (B \Rightarrow C)$$

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \{n \mid n \text{ is even}\}$ and $V(B) = V(C) = \omega$. Then for an arbitrary state $n \in \omega$ we have that $\mathfrak{M}, n \models (A \vee B \Rightarrow C)$, because

$$(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models (B \wedge C)$$

and consequently

$$(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models (A \vee B) \wedge C$$

But $\mathfrak{M}, n \not\models (A \Rightarrow C)$, because

$$(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models A \quad \text{and} \quad (\forall n_5 > n)(\exists n_6 > n_5)\ \mathfrak{M}, n_6 \models \neg A$$

and thus $\mathfrak{M}, n \not\models (A \Rightarrow C) \wedge (B \Rightarrow C)$.

It follows then that $\mathfrak{M}, n \not\models (A \vee B \Rightarrow C) \rightarrow (A \Rightarrow C) \wedge (B \Rightarrow C)$ and the proof is complete.

**Monotonicity**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \rightarrow (A \wedge C \Rightarrow B)$$

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A) = V(B) = \omega$ and $V(C) = \{n \mid n \text{ is even}\}$. Then $(\forall n \in \omega)\ \mathfrak{M}, n \models (A \wedge B)$, so for an arbitrary state $n \in \omega$ we have that

$$(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models (A \wedge B)$$

and thus $\mathfrak{M}, n \models (A \Rightarrow B)$.

But $\mathfrak{M}, n \not\models (A \wedge C \Rightarrow B)$, because

$$(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge C) \quad \text{and} \quad (\forall n_5 > n)(\exists n_6 > n_5)\ \mathfrak{M}, n_6 \models \neg C$$

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \rightarrow (A \wedge C \Rightarrow B)$ and the proof is complete.

**Weak Monotonicity**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \Rightarrow (A \wedge C \Rightarrow B)$$

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A) = V(B) = \omega$ and $V(C) = \{n \mid n \text{ is even}\}$. For an arbitrary state $n \in \omega$ we have that $\mathfrak{M}, n \models (A \Rightarrow B) \Rightarrow (A \wedge C \Rightarrow B)$ iff one of the following holds:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg(A \Rightarrow B)$

(ii) $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \Rightarrow B) \wedge (A \wedge C \Rightarrow B)$

Case (i): By construction we have that $(\forall n \in \omega)\ \mathfrak{M}, n \models (A \wedge B)$, so by definition

$$(\forall n \in \omega)\ \mathfrak{M}, n \models (A \Rightarrow B)$$

This gives us that $(\forall n \in \omega)\ \mathfrak{M}, n \not\models \neg(A \Rightarrow B)$ and the case (i) cannot hold.

Case (ii): By construction we have that

$$(\forall n_5 > n)(\exists n_6 > n_5)\ \mathfrak{M}, n_6 \models (A \wedge C) \quad \text{and} \quad (\forall n_7 > n)(\exists n_8 > n_7)\ \mathfrak{M}, n_8 \models \neg C$$

so by definition $(\forall n \in \omega)\ \mathfrak{M}, n \not\models (A \wedge C \Rightarrow B)$ and the case (ii) cannot hold.

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \Rightarrow (A \wedge C \Rightarrow B)$ and the proof is complete.

**Modus Ponens**: We have to show that

$$\mathfrak{F} \not\models A \wedge (A \Rightarrow B) \rightarrow B$$

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \{n\}$ and $V(B) = \emptyset$. Then $\mathfrak{M}, n \models A$ and by definition $\mathfrak{M}, n \models (A \Rightarrow B)$, which gives $\mathfrak{M}, n \models A \wedge (A \Rightarrow B)$.

But $\mathfrak{M}, n \not\models B$ by construction.

It follows then that $\mathfrak{M}, n \not\models A \wedge (A \Rightarrow B) \rightarrow B$ and the proof is complete. ∎

Observe that $\overset{\Rightarrow}{\mathbf{\Omega}}$ does not contain the **ID** axiom. This might appear strange; after all '*reflexivity seems to be satisfied universally by any kind of reasoning based on some notion of consequence*' [KLM90, p. 177] and defeasible conditionals are designed to incarnate some form of defeasible consequence. Yet, in the same sense as observed in [KLM90], conditionals that do not satisfy it '*probably express some notion of theory change*'. It seems that failure of **ID** is due to the unavoidable '*temporal*' flavour of $(\omega, <)$, whose ordering directly reminds the setting of discrete linear time. However, this seems appropriate for conditionals incorporating a notion of 'temporal' causation, in the form '*if $X$, then normally it should be the case that $Y$ holds* *in the future*' - "*normally, a strong earthquake implies a permanent change in future building codes*". Nevertheless, failure of **ID** is not a happy incident and it seems natural to consider alternative modal definitions of the conditional connective that would enforce the validity of **ID**. Below, we demonstrate that some plausible attempts to validate **ID** unfortunately result into a monotonic conditional logic.

As a first attempt, it seems natural to constrain the set of valuations, using the known recipe of general frames. For the rest of this section, to facilitate the exposition, we will use the symbol $(A \Rrightarrow B)$ for the monotonic conditional(s) we define below.

**Definition 3.1.6** Let $\mathfrak{F} = (\omega, <, X)$ be the general frame obtained by setting the set $X$ of admissible valuations to be the set of finite and cofinite subsets of $\omega$. We assume again that $A \Rrightarrow B$ as previously to be a shorthand for $\Diamond\Box\neg A \vee \Diamond\Box(A \wedge B)$. That is

$$A \Rrightarrow B \quad \text{iff} \quad \Diamond\Box\neg A \vee \Diamond\Box(A \wedge B)$$

Note that $X$ is well-defined, as it is closed under the boolean algebra operators and the modal operator (see [BdRV01, p. 30]).

**Proposition 3.1.7** The conditional defined in Definition 3.1.6 satisfies **Monotonicity**, i.e.

$$\mathfrak{F} \models (A \Rrightarrow B) \rightarrow (A \wedge C \Rrightarrow B)$$

PROOF. Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \Rrightarrow B)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$. Then, either:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg A$, or

(ii) $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge B)$

Case (i): We also have that $(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg(A \wedge C)$, and thus $\mathfrak{M}, n \models (A \wedge C \Rrightarrow B)$.

Case (ii): For a proposition $C$, since $\|C\| \in X$, one of the following must hold:

(a) $(\exists n_5 > n)(\forall n_6 > n_5)\ \mathfrak{M}, n_6 \models \neg C$

(b) $(\exists n_7 > n)(\forall n_8 > n_7)\ \mathfrak{M}, n_8 \models C$

In (a), we also have that $(\forall n_6 > n_5)\ \mathfrak{M}, n_6 \models \neg(A \wedge C)$, and thus $\mathfrak{M}, n \models (A \wedge C \Rrightarrow B)$.

In (b), by (ii) we obtain $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge B \wedge C)$, and $\mathfrak{M}, n \models (A \wedge C \Rrightarrow B)$ follows.

So, if $\mathfrak{M}, n \models (A \Rrightarrow B)$, then $\mathfrak{M}, n \models (A \wedge C \Rrightarrow B)$, which gives that

$$\mathfrak{M}, n \models (A \Rrightarrow B) \to (A \wedge C \Rrightarrow B)$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete. ∎

This phenomenon persists, even if we go back to $(\omega, <)$ and attempt to provide alternative modal definitions of the conditional. Not all of them conform to the '*true in all, but finitely many, worlds*' intuition, but they demonstrate the possibility of different variants.

**Theorem 3.1.8** Assume the following definitions:

(i) $A \Rrightarrow B \ \equiv_{def}\ \Box\Diamond\neg A \vee \Box\Diamond(A \wedge B)$

(ii) $A \Rrightarrow B \ \equiv_{def}\ \Box\Diamond\neg A \vee \Diamond\Box(A \wedge B)$

(iii) $A \Rrightarrow B \ \equiv_{def}\ \Diamond\Box(A \to B)$

(iv) $A \Rrightarrow B \ \equiv_{def}\ \Box\Diamond(A \to B)$

All $\Rrightarrow$ conditionals defined above satisfy **Monotonicity**.

Proof. In each case, we have to show that

$$\mathfrak{F} \models (A \Rrightarrow B) \to (A \wedge C \Rrightarrow B)$$

**(i)**: Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \Rrightarrow B)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$. Then, either:

(i) $(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg A$, or

(ii) $(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge B)$

Case (i): We also have that $(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg(A \wedge C)$, and thus $\mathfrak{M}, n \models (A \wedge C \Rrightarrow B)$.

Case (ii): For a proposition $C$ one of the following must hold:

(a) $(\forall n_5 > n)(\exists n_6 > n_5)\ \mathfrak{M}, n_6 \models \neg C$

(b) $(\exists n_7 > n)(\forall n_8 > n_7)\ \mathfrak{M}, n_8 \models C$

In (a), we also have that $(\exists n_6 > n_5)\ \mathfrak{M}, n_6 \models \neg(A \wedge C)$, and thus $\mathfrak{M}, n \models (A \wedge C \Rrightarrow B)$.

In (b), by (ii) we obtain $(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge B \wedge C)$, and $\mathfrak{M}, n \models (A \wedge C \Rrightarrow B)$ follows.

So, if $\mathfrak{M}, n \models (A \Rrightarrow B)$, then $\mathfrak{M}, n \models (A \wedge C \Rrightarrow B)$, which gives that

$$\mathfrak{M}, n \models (A \Rrightarrow B) \to (A \wedge C \Rrightarrow B)$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete.

**(ii)**: Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \Rrightarrow B)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$. Then, either:

(i) $(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg A$, or

(ii) $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge B)$

Case (i): We also have that $(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg(A \wedge C)$, and thus $\mathfrak{M}, n \models (A \wedge C \Rrightarrow B)$.

Case (ii): For a proposition $C$ one of the following must hold:

(a) $(\forall n_5 > n)(\exists n_6 > n_5)\ \mathfrak{M}, n_6 \models \neg C$

(b) $(\exists n_7 > n)(\forall n_8 > n_7)\ \mathfrak{M}, n_8 \models C$

In (a), we also have that $(\exists n_6 > n_5)\ \mathfrak{M}, n_6 \models \neg(A \wedge C)$, and thus $\mathfrak{M}, n \models (A \wedge C \Rrightarrow B)$.

In (b), by (ii) we obtain $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge B \wedge C)$, and $\mathfrak{M}, n \models (A \wedge C \Rrightarrow B)$ follows.

So, if $\mathfrak{M}, n \models (A \Rrightarrow B)$, then $\mathfrak{M}, n \models (A \wedge C \Rrightarrow B)$, which gives that

$$\mathfrak{M}, n \models (A \Rrightarrow B) \to (A \wedge C \Rrightarrow B)$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete.

**(iii)**: Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \Rrightarrow B)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$.

By definition, this means that

$$(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models (A \to B)$$

By propositional logic, we have that $(\forall n \in \omega)\ \mathfrak{M}, n \models (A \to B) \to (A \wedge C \to B)$.

From these, we obtain

$$(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models (A \wedge C \to B)$$

and $\mathfrak{M}, n \models (A \wedge C \Rrightarrow B)$ follows.

So, if $\mathfrak{M}, n \models (A \Rrightarrow B)$, then $\mathfrak{M}, n \models (A \wedge C \Rrightarrow B)$, which gives that

$$\mathfrak{M}, n \models (A \Rrightarrow B) \to (A \wedge C \Rrightarrow B)$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete.

**(iv)**: Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \Rrightarrow B)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$.

By definition, this means that

$$(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models (A \to B)$$

By propositional logic, we have that $(\forall n \in \omega)\ \mathfrak{M}, n \models (A \to B) \to (A \wedge C \to B)$.

From these, we obtain

$$(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models (A \wedge C \to B)$$

and $\mathfrak{M}, n \models (A \wedge C \Rrightarrow B)$ follows.

So, if $\mathfrak{M}, n \models (A \Rrightarrow B)$, then $\mathfrak{M}, n \models (A \wedge C \Rrightarrow B)$, which gives that

$$\mathfrak{M}, n \models (A \Rrightarrow B) \to (A \wedge C \Rrightarrow B)$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete. ∎

### 3.1.1 An alternative: cofinal vs cofinite in $(\omega, <)$

In this subsection, we discuss a possible alternative. Instead of working with the (obviously large) cofinite subsets of $\omega$, we will attempt to work with cofinal subsets: $S \subseteq \omega$ is cofinal in $\omega$ iff for every $n \in \omega$ there exists an $s \in S$, such that $n < s$.

We proceed to define the conditional $(A \Rightarrow B)$ as follows:

$$(A \Rightarrow B) \quad \equiv_{def} \quad \Diamond\Box\neg A \vee \Box\Diamond(A \wedge B)$$

**Definition 3.1.9 [Conditional Logic $\overset{\Rightarrow}{\boldsymbol{\omega}}$].** The logic $\overset{\Rightarrow}{\boldsymbol{\omega}}$ consists of all formulae $A \in \mathcal{L}_{\Rightarrow}$, such that:

$$A \in \overset{\Rightarrow}{\boldsymbol{\omega}} \quad \text{iff} \quad (\omega, <) \models A^* \quad \text{iff} \quad \vdash_{\mathbf{K4DLZ}} A^*$$

where $A^*$ is the obvious translation defined similarly to Definition 3.1.1.

**Fact 3.1.10** Let $\mathfrak{M}$ be a model of $\mathfrak{F} = (\omega, <)$ and $n \in \omega$ an arbitrary world. We will say that $\mathfrak{M}, n \models (A \Rightarrow B)$ iff one of the following holds:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg A$

(ii) $(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models A \wedge B$

The logic $\overrightarrow{\boldsymbol{\omega}}$ turns out to be quite interesting.

**Theorem 3.1.11** The logic $\overrightarrow{\boldsymbol{\omega}}$:

1. is closed under the rules **RCEA**, **RCEC** and **RCE**
2. contains the axioms **ID**, **CUT**, **Loop**, **OR**, **CV**, **CM**, **MOD**, **CEM** and **Weak Modus Ponens**

PROOF. **RCEA**: Let $\mathfrak{M} \models (A \equiv B)$. We have to show that

$$\mathfrak{M} \models (A \Rightarrow C) \equiv (B \Rightarrow C)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \Rightarrow C)$. Then, either:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg A$, or

(ii) $(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge C)$

Case (i): By $\mathfrak{M} \models (A \equiv B)$ we also have that $(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models (\neg A \equiv \neg B)$. This gives that $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg B$, and $\mathfrak{M}, n \models (B \Rightarrow C)$ follows.

Case (ii): Similarly, by $\mathfrak{M} \models (A \equiv B)$ we obtain

$$(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge C \equiv B \wedge C)$$

It follows that $(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (B \wedge C)$, and thus $\mathfrak{M}, n \models (B \Rightarrow C)$.

So, if $\mathfrak{M}, n \models (A \Rightarrow C)$, then $\mathfrak{M}, n \models (B \Rightarrow C)$, which gives that

$$\mathfrak{M}, n \models (A \Rightarrow C) \rightarrow (B \Rightarrow C)$$

Similarly, $\mathfrak{M}, n \models (B \Rightarrow C) \rightarrow (A \Rightarrow C)$. Since the world $n$ was arbitrarily chosen, the proof is complete.

**RCEC**: Let $\mathfrak{M} \models (A \equiv B)$. We have to show that

$$\mathfrak{M} \models (C \Rightarrow A) \equiv (C \Rightarrow B)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (C \Rightarrow A)$. Then, either:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg C$, or

(ii) $(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (C \wedge A)$

Case (i): By definition, $\mathfrak{M}, n \models (C \Rightarrow B)$ follows.

Case (ii): By $\mathfrak{M} \models (A \equiv B)$ we obtain

$$(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (C \wedge A \equiv C \wedge B)$$

It follows that $(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (C \wedge B)$, and thus $\mathfrak{M}, n \models (C \Rightarrow B)$.

So, if $\mathfrak{M}, n \models (C \Rightarrow A)$, then $\mathfrak{M}, n \models (C \Rightarrow B)$, which gives that

$$\mathfrak{M}, n \models (C \Rightarrow A) \rightarrow (C \Rightarrow B)$$

Similarly, $\mathfrak{M}, n \models (C \Rightarrow B) \rightarrow (C \Rightarrow A)$. Since the world $n$ was arbitrarily chosen, the proof is complete.

**RCE**: Let $\mathfrak{M} \models (A \rightarrow B)$. We have to show that

$$\mathfrak{M} \models (A \Rightarrow B)$$

Assume an arbitrary state $n \in \omega$. Then, either:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg A$, or

(ii) $(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models A$

Case (i): By definition, $\mathfrak{M}, n \models (A \Rightarrow B)$ follows.

Case (ii): By $\mathfrak{M} \models (A \rightarrow B)$ we obtain

$$(\forall n_3 > n)(\exists n_4 > n_3)(\mathfrak{M}, n_4 \models A \text{ and } \mathfrak{M}, n_4 \models A \rightarrow B)$$

It follows that $(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge B)$, and thus $\mathfrak{M}, n \models (A \Rightarrow B)$.

So in either case $\mathfrak{M}, n \models (A \Rightarrow B)$. Since the world $n$ was arbitrarily chosen, the proof is complete.

**ID**: We have to show that

$$\mathfrak{F} \models (A \Rightarrow A)$$

Assume an arbitrary state $n \in \omega$ and $\mathfrak{M}$ a model of $\mathfrak{F}$. Then, either:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg A$, or

(ii) $(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models A$

In both cases, by definition $\mathfrak{M}, n \models (A \Rightarrow A)$. Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete.

**CUT**: We have to show that

$$\mathfrak{F} \models (A \wedge B \Rightarrow C) \wedge (A \Rightarrow B) \rightarrow (A \Rightarrow C)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \wedge B \Rightarrow C) \wedge (A \Rightarrow B)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$. Obviously, $\mathfrak{M}, n \models (A \wedge B \Rightarrow C)$ and $\mathfrak{M}, n \models (A \Rightarrow B)$. Then, either:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg A$, or

(ii) $(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge B \wedge C)$

Case (i): By definition, $\mathfrak{M}, n \models (A \Rightarrow C)$ follows.

Case (ii): We also have that $(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge C)$, and thus $\mathfrak{M}, n \models (A \Rightarrow C)$.

So, if $\mathfrak{M}, n \models (A \wedge B \Rightarrow C) \wedge (A \Rightarrow B)$, then $\mathfrak{M}, n \models (A \Rightarrow C)$, which gives that

$$\mathfrak{M}, n \models (A \wedge B \Rightarrow C) \wedge (A \Rightarrow B) \rightarrow (A \Rightarrow C)$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete.

**Loop**: We have to show that

$$\mathfrak{F} \models (A_0 \Rightarrow A_1 \wedge \ldots \wedge A_k \Rightarrow A_0) \rightarrow (A_0 \Rightarrow A_k)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A_0 \Rightarrow A_1 \wedge \ldots \wedge A_k \Rightarrow A_0)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$. Then, either:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg A_0$, or

(ii) $(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (A_0 \wedge A_1)$ and ... and $(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (A_k \wedge A_0)$

Case (i): By definition, $\mathfrak{M}, n \models (A_0 \Rightarrow A_k)$ follows.

Case (ii): We need only that $(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (A_0 \wedge A_k)$, and thus $\mathfrak{M}, n \models (A_0 \Rightarrow A_k)$.

So, if $\mathfrak{M}, n \models (A_0 \Rightarrow A_1 \wedge \ldots \wedge A_k \Rightarrow A_0)$, then $\mathfrak{M}, n \models (A_0 \Rightarrow A_k)$, which gives that

$$\mathfrak{M}, n \models (A_0 \Rightarrow A_1 \wedge \ldots \wedge A_k \Rightarrow A_0) \rightarrow (A_0 \Rightarrow A_k)$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete.

**OR**: We have to show that

$$\mathfrak{F} \models (A \Rightarrow C) \wedge (B \Rightarrow C) \rightarrow (A \vee B \Rightarrow C)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \Rightarrow C) \wedge (B \Rightarrow C)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$. Obviously, $\mathfrak{M}, n \models (A \Rightarrow C)$ and $\mathfrak{M}, n \models (B \Rightarrow C)$. Then one of the following must hold:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models (\neg A \wedge \neg B)$

(ii) $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models \neg A$ and $(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (B \wedge C)$

(iii) $(\exists n_5 > n)(\forall n_6 > n_5)\ \mathfrak{M}, n_6 \models \neg B$ and $(\forall n_5 > n)(\exists n_6 > n_5)\ \mathfrak{M}, n_6 \models (A \wedge C)$

(iv) $(\forall n_7 > n)(\exists n_8 > n_7)\ \mathfrak{M}, n_8 \models (A \wedge C)$ and $(\forall n_7 > n)(\exists n_8 > n_7)\ \mathfrak{M}, n_8 \models (B \wedge C)$

Case (i): Equivalently, we have that $(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg(A \vee B)$, and $\mathfrak{M}, n \models (A \vee B \Rightarrow C)$ follows.

Cases (ii) - (iv): We also have that $(\exists n_{4,6,8} > n_{3,5,7})\ \mathfrak{M}, n_{4,6,8} \models (A \vee B) \wedge C$, and thus $\mathfrak{M}, n \models (A \vee B \Rightarrow C)$.

So, if $\mathfrak{M}, n \models (A \Rightarrow C) \wedge (B \Rightarrow C)$, then $\mathfrak{M}, n \models (A \vee B \Rightarrow C)$, which gives that

$$\mathfrak{M}, n \models (A \Rightarrow C) \wedge (B \Rightarrow C) \rightarrow (A \vee B \Rightarrow C)$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete.

**CV**: We have to show that

$$\mathfrak{F} \models (A \Rightarrow B) \wedge \neg(A \Rightarrow \neg C) \rightarrow (A \wedge C \Rightarrow B)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \Rightarrow B) \wedge \neg(A \Rightarrow \neg C)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$. Obviously, $\mathfrak{M}, n \models (A \Rightarrow B)$ and $\mathfrak{M}, n \not\models (A \Rightarrow \neg C)$. The only way this can be achieved is if

$$(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models (A \wedge B) \text{ and } (\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (\neg A \vee C)$$

But then we also have

$$(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models (A \wedge B \wedge C)$$

and, by definition, $\mathfrak{M}, n \models (A \wedge C \Rightarrow B)$ follows.

So, if $\mathfrak{M}, n \models (A \Rightarrow B) \wedge \neg(A \Rightarrow \neg C)$, then $\mathfrak{M}, n \models (A \wedge C \Rightarrow B)$, which gives that

$$\mathfrak{M}, n \models (A \Rightarrow B) \wedge \neg(A \Rightarrow \neg C) \rightarrow (A \wedge C \Rightarrow B)$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete.

**CM**: We have to show that

$$\mathfrak{F} \models (A \Rightarrow B \wedge C) \rightarrow (A \Rightarrow B) \wedge (A \Rightarrow C)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \Rightarrow B \wedge C)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$. Then, either:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg A$, or

(ii) $(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge B \wedge C)$

Case (i): By definition, $\mathfrak{M}, n \models (A \Rightarrow B)$ and $\mathfrak{M}, n \models (A \Rightarrow C)$, so $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (A \Rightarrow C)$ follows.

Case (ii): We also have that $(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge B)$ and $(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge C)$, so $\mathfrak{M}, n \models (A \Rightarrow B)$ and $\mathfrak{M}, n \models (A \Rightarrow C)$, and thus $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (A \Rightarrow C)$.

So, if $\mathfrak{M}, n \models (A \Rightarrow B \wedge C)$, then $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (A \Rightarrow C)$, which gives that

$$\mathfrak{M}, n \models (A \Rightarrow B \wedge C) \rightarrow (A \Rightarrow B) \wedge (A \Rightarrow C)$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete.

**MOD**: We have to show that

$$\mathfrak{F} \models (\neg A \Rightarrow A) \rightarrow (B \Rightarrow A)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (\neg A \Rightarrow A)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$. The only way this can be achieved is if

$$(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models A$$

For the proposition $B$ one of the following must hold:

(i) $(\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models \neg B$, or

(ii) $(\forall n_5 > n)(\exists n_6 > n_5)\ \mathfrak{M}, n_6 \models B$

Case (i): By definition, $\mathfrak{M}, n \models (B \Rightarrow A)$ follows.

Case (ii): By $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models A$ we also have that $(\forall n_5 > n)(\exists n_6 > n_5)\ \mathfrak{M}, n_6 \models (B \wedge A)$, and thus $\mathfrak{M}, n \models (B \Rightarrow A)$.

So, if $\mathfrak{M}, n \models (\neg A \Rightarrow A)$, then $\mathfrak{M}, n \models (B \Rightarrow A)$, which gives that

$$\mathfrak{M}, n \models (\neg A \Rightarrow A) \rightarrow (B \Rightarrow A)$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete.

**CEM**: We have to show that

$$\mathfrak{F} \models (A \Rightarrow B) \vee (A \Rightarrow \neg B)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \not\models (A \Rightarrow B)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$. The only way this can be achieved is if

$$(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models A \text{ and } (\exists n_3 > n)(\forall n_4 > n_3)\ \mathfrak{M}, n_4 \models (\neg A \vee \neg B)$$

But then we also have

$$(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models (A \wedge \neg B)$$

and, by definition, $\mathfrak{M}, n \models (A \Rightarrow \neg B)$ follows.

So, if $\mathfrak{M}, n \models \neg(A \Rightarrow B)$, then $\mathfrak{M}, n \models (A \Rightarrow \neg B)$, which gives that

$$\mathfrak{M}, n \models \neg(A \Rightarrow B) \rightarrow (A \Rightarrow \neg B)$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete.

**Weak Modus Ponens**: We have to show that

$$\mathfrak{F} \models A \wedge (A \Rightarrow B) \Rightarrow B$$

Assume an arbitrary state $n \in \omega$ and $\mathfrak{M}$ a model of $\mathfrak{F}$. We have that $\mathfrak{M}, n \models A \wedge (A \Rightarrow B) \Rightarrow B$ iff one of the following holds:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg A \vee \neg(A \Rightarrow B)$, or

(ii) $(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models A \wedge (A \Rightarrow B) \wedge B$

Let (i) be false, that is let $(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models A \wedge (A \Rightarrow B)$ $(*)$. We will show that (ii) has to be true. By $(*)$ we have that $(\forall n_1 > n)(\exists n_2 > n_1)$ such that $\mathfrak{M}, n_2 \models A$ and

$$(\exists n_5 > n_2)(\forall n_6 > n_5)\ \mathfrak{M}, n_6 \models \neg A \text{ or } (\forall n_5 > n_2)(\exists n_6 > n_5)\ \mathfrak{M}, n_6 \models (A \wedge B)$$

This means that

$$(\forall n_5 > n)(\exists n_6 > n_5)\ \mathfrak{M}, n_6 \models (A \wedge B)$$

This gives us that

$$(\forall n_1 > n)\ \mathfrak{M}, n_1 \models (A \Rightarrow B)$$

From the last two, we also obtain

$$(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models A \wedge (A \Rightarrow B) \wedge B$$

which is exactly (ii). So one of (i) or (ii) must hold, which means that

$$\mathfrak{M}, n \models A \wedge (A \Rightarrow B) \Rightarrow B$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete. ∎

**Theorem 3.1.12** The logic $\overset{\Rightarrow}{\omega}$:

1. is not closed under the rule **RCK**
2. does not contain the axioms **AC**, **CC**, **CSO**, **MP**, **CA**, **CS**, **SDA**, **Transitivity**, **Weak Transitivity**, **Monotonicity**, **Weak Monotonicity** and **Modus Ponens**

Proof. **RCK**: For $n = 2$: We have to show that

$$\mathfrak{M} \models (A_1 \wedge A_2 \to B) \text{ and } \mathfrak{M} \not\models (C \Rightarrow A_1) \wedge (C \Rightarrow A_2) \to (C \Rightarrow B)$$

for some model $\mathfrak{M}$ of $\mathfrak{F}$.

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A_1) = \{n \,|\, n \text{ is even}\}$, $V(A_2) = \{n \,|\, n \text{ is odd}\}$, $V(B) = \emptyset$ and $V(C) = \omega$. Then $(\forall n \in \omega)\ \mathfrak{M}, n \models (A_1 \wedge A_2 \to B)$, because $(\forall n \in \omega)\ \mathfrak{M}, n \models (\neg A_1 \vee \neg A_2)$. Thus $\mathfrak{M} \models (A_1 \wedge A_2 \to B)$.

For an arbitrary state $n \in \omega$ we have that

$$(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models (C \wedge A_1) \text{ and } (\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (C \wedge A_2)$$

This gives $\mathfrak{M}, n \models (C \Rightarrow A_1)$ and $\mathfrak{M}, n \models (C \Rightarrow A_2)$, which means that

$$\mathfrak{M}, n \models (C \Rightarrow A_1) \wedge (C \Rightarrow A_2)$$

But $\mathfrak{M}, n \not\models (C \Rightarrow B)$, because we have that

$$(\forall n_5 > n)(\exists n_6 > n_5)\ \mathfrak{M}, n_6 \models C \quad \text{and} \quad (\exists n_7 > n)(\forall n_8 > n_7)\ \mathfrak{M}, n_8 \models (\neg C \vee \neg B)$$

It follows then that $\mathfrak{M}, n \not\models (C \Rightarrow A_1) \wedge (C \Rightarrow A_2) \rightarrow (C \Rightarrow B)$ and consequently $\mathfrak{M} \not\models (C \Rightarrow A_1) \wedge (C \Rightarrow A_2) \rightarrow (C \Rightarrow B)$.

**AC**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \wedge (A \Rightarrow C) \rightarrow (A \wedge B \Rightarrow C)$$

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega$, $V(B) = \{n \,|\, n \text{ is even}\}$ and $V(C) = \{n \,|\, n \text{ is odd}\}$. Then for an arbitrary state $n \in \omega$ we have that

$$(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models (A \wedge B) \quad \text{and} \quad (\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge C)$$

This gives

$$\mathfrak{M}, n \models (A \Rightarrow B) \quad \text{and} \quad \mathfrak{M}, n \models (A \Rightarrow C)$$

and consequently

$$\mathfrak{M}, n \models (A \Rightarrow B) \wedge (A \Rightarrow C)$$

But $\mathfrak{M}, n \not\models (A \wedge B \Rightarrow C)$, because

$$(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models (A \wedge B) \quad \text{and} \quad (\exists n_5 > n)(\forall n_6 > n_5)\ \mathfrak{M}, n_6 \models \neg(A \wedge B \wedge C)$$

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \wedge (A \Rightarrow C) \rightarrow (A \wedge B \Rightarrow C)$ and the proof is complete.

**CC**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \wedge (A \Rightarrow C) \rightarrow (A \Rightarrow B \wedge C)$$

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega$, $V(B) = \{n \,|\, n \text{ is even}\}$ and $V(C) = \{n \,|\, n \text{ is odd}\}$. Then for an arbitrary state $n \in \omega$ we have that

$$(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models (A \wedge B) \quad \text{and} \quad (\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge C)$$

This gives

$$\mathfrak{M}, n \models (A \Rightarrow B) \quad \text{and} \quad \mathfrak{M}, n \models (A \Rightarrow C)$$

and consequently

$$\mathfrak{M}, n \models (A \Rightarrow B) \wedge (A \Rightarrow C)$$

But $\mathfrak{M}, n \not\models (A \Rightarrow B \wedge C)$, because

$$(\forall n_5 > n)(\exists n_6 > n_5)\ \mathfrak{M}, n_6 \models A \ \text{ and } \ (\exists n_7 > n)(\forall n_8 > n_7)\ \mathfrak{M}, n_8 \models \neg(A \wedge B \wedge C)$$

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \wedge (A \Rightarrow C) \rightarrow (A \Rightarrow B \wedge C)$ and the proof is complete.

**CSO**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \wedge (B \Rightarrow A) \rightarrow ((A \Rightarrow C) \equiv (B \Rightarrow C))$$

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega$, $V(B) = \{n \,|\, n \text{ is even}\}$ and $V(C) = \{n \,|\, n \text{ is odd}\}$. Then for an arbitrary state $n \in \omega$ we have that

$$(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models (A \wedge B) \ \text{ and } \ (\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge C)$$

This gives

$$\mathfrak{M}, n \models (A \Rightarrow B),\ \mathfrak{M}, n \models (B \Rightarrow A) \ \text{ and } \ \mathfrak{M}, n \models (A \Rightarrow C)$$

and consequently

$$\mathfrak{M}, n \models (A \Rightarrow B) \wedge (B \Rightarrow A) \ \text{ and } \ \mathfrak{M}, n \models A \Rightarrow C$$

But $\mathfrak{M}, n \not\models (B \Rightarrow C)$, because

$$(\forall n_5 > n)(\exists n_6 > n_5)\ \mathfrak{M}, n_6 \models B \ \text{ and } \ (\exists n_7 > n)(\forall n_8 > n_7)\mathfrak{M}, n_8 \models (\neg B \vee \neg C)$$

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \wedge (B \Rightarrow A) \rightarrow ((A \Rightarrow C) \rightarrow (B \Rightarrow C))$ and the proof is complete.

**MP**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \rightarrow (A \rightarrow B)$$

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega$ and $V(B) = \omega - \{n\}$. Then we have that

$$(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models (A \wedge B)$$

so by definition $\mathfrak{M}, n \models (A \Rightarrow B)$.

But $\mathfrak{M}, n \not\models (A \rightarrow B)$, because $\mathfrak{M}, n \models (A \wedge \neg B)$.

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \rightarrow (A \rightarrow B)$ and the proof is complete.

**CA**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \wedge (C \Rightarrow B) \rightarrow (A \wedge C \Rightarrow B)$$

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega - \{n = 3k \mid k \in \omega\}$, $V(B) = \omega - \{n = 3k - 1 \mid k \in \omega\}$ and $V(C) = \omega - \{n = 3k - 2 \mid k \in \omega\}$. Then for an arbitrary state $n \in \omega$ we have that

$$(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models (A \wedge B) \ \text{ and } \ (\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (C \wedge B)$$

This gives

$$\mathfrak{M}, n \models (A \Rightarrow B) \ \text{ and } \ \mathfrak{M}, n \models (C \Rightarrow B)$$

and consequently

$$\mathfrak{M}, n \models (A \Rightarrow B) \wedge (C \Rightarrow B)$$

But $\mathfrak{M}, n \not\models (A \wedge C \Rightarrow B)$, because

$$(\forall n_5 > n)(\exists n_6 > n_5)\ \mathfrak{M}, n_6 \models (A \wedge C) \ \text{ and } \ (\exists n_7 > n)(\forall n_8 > n_7)\ \mathfrak{M}, n_8 \models \neg(A \wedge B \wedge C)$$

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \wedge (C \Rightarrow B) \rightarrow (A \wedge C \Rightarrow B)$ and the proof is complete.

**CS**: We have to show that

$$\mathfrak{F} \not\models (A \wedge B) \rightarrow (A \Rightarrow B)$$

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \{n \mid n \text{ is even}\}$ and $V(B) = \{n\}$, where $n$ is an arbitrary even world. Then we have that $\mathfrak{M}, n \models (A \wedge B)$. But $\mathfrak{M}, n \not\models (A \Rightarrow B)$, because

$$(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models A \ \text{ and } \ (\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (\neg A \vee \neg B)$$

It follows then that $\mathfrak{M}, n \not\models (A \wedge B) \rightarrow (A \Rightarrow B)$ and the proof is complete.

**SDA**: We have to show that

$$\mathfrak{F} \not\models (A \vee B \Rightarrow C) \rightarrow (A \Rightarrow C) \wedge (B \Rightarrow C)$$

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \{n \mid n \text{ is even}\}$, $V(B) = \omega$ and $V(C) = \{n \mid n \text{ is odd}\}$. Then for an arbitrary state $n \in \omega$ we have that $\mathfrak{M}, n \models (A \vee B \Rightarrow C)$, because

$$(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models (B \wedge C)$$

and consequently

$$(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models (A \vee B) \wedge C$$

But $\mathfrak{M}, n \not\models (A \Rightarrow C)$, because

$$(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models A \ \text{ and } \ (\exists n_5 > n)(\forall n_6 > n_5)\ \mathfrak{M}, n_6 \models \neg A \vee \neg C$$

and thus $\mathfrak{M}, n \not\models (A \Rightarrow C) \wedge (B \Rightarrow C)$.

It follows then that $\mathfrak{M}, n \not\models (A \vee B \Rightarrow C) \rightarrow (A \Rightarrow C) \wedge (B \Rightarrow C)$ and the proof is complete.

**Transitivity**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \wedge (B \Rightarrow C) \rightarrow (A \Rightarrow C)$$

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \{n \,|\, n \text{ is even}\}$, $V(B) = \omega$ and $V(C) = \{n \,|\, n \text{ is odd}\}$. Then for an arbitrary state $n \in \omega$ we have that

$$(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models (A \wedge B) \ \text{ and } \ (\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (B \wedge C)$$

This gives

$$\mathfrak{M}, n \models (A \Rightarrow B) \ \text{ and } \ \mathfrak{M}, n \models (B \Rightarrow C)$$

and consequently

$$\mathfrak{M}, n \models (A \Rightarrow B) \wedge (B \Rightarrow C)$$

But $\mathfrak{M}, n \not\models (A \Rightarrow C)$, because

$$(\forall n_5 > n)(\exists n_6 > n_5)\ \mathfrak{M}, n_6 \models A \ \text{ and } \ (\exists n_7 > n)(\forall n_8 > n_7)\ \mathfrak{M}, n_8 \models \neg(A \wedge C)$$

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \wedge (B \Rightarrow C) \rightarrow (A \Rightarrow C)$ and the proof is complete.

**Weak Transitivity**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$$

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \{n \,|\, n \text{ is even}\}$, $V(B) = \omega$ and $V(C) = \{n \,|\, n \text{ is odd}\}$. For an arbitrary state $n \in \omega$ we have that $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$ iff one of the following holds:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg(A \Rightarrow B) \vee \neg(B \Rightarrow C)$

(ii) $(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \Rightarrow B) \wedge (B \Rightarrow C) \wedge (A \Rightarrow C)$

Case (i): By construction we have that

$$(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models (A \wedge B) \ \text{ and } \ (\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (B \wedge C)$$

so by definition

$$(\forall n \in \omega)\ \mathfrak{M}, n \models (A \Rightarrow B) \wedge (B \Rightarrow C)$$

This gives us that $(\forall n \in \omega)\ \mathfrak{M}, n \not\models \neg(A \Rightarrow B) \vee \neg(B \Rightarrow C)$ and the case (i) cannot hold.

Case (ii): By construction we have that

$$(\forall n_5 > n)(\exists n_6 > n_5)\ \mathfrak{M}, n_6 \models A \ \text{ and } \ (\exists n_7 > n)(\forall n_8 > n_7)\ \mathfrak{M}, n_8 \models \neg(A \wedge C)$$

so by definition $(\forall n \in \omega)\ \mathfrak{M}, n \not\models (A \Rightarrow C)$ and the case (ii) cannot hold.

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$ and the proof is complete.

**Monotonicity**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \rightarrow (A \wedge C \Rightarrow B)$$

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega$, $V(B) = \{n \,|\, n \text{ is even}\}$ and $V(C) = \{n \,|\, n \text{ is odd}\}$. Then for an arbitrary state $n \in \omega$ we have that

$$(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models (A \wedge B)$$

and thus $\mathfrak{M}, n \models (A \Rightarrow B)$.

But $\mathfrak{M}, n \not\models (A \wedge C \Rightarrow B)$, because

$$(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge C) \quad \text{and} \quad (\exists n_5 > n)(\forall n_6 > n_5)\ \mathfrak{M}, n_6 \models \neg(A \wedge B \wedge C)$$

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \rightarrow (A \wedge C \Rightarrow B)$ and the proof is complete.

**Weak Monotonicity**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \Rightarrow (A \wedge C \Rightarrow B)$$

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega$, $V(B) = \{n \,|\, n \text{ is even}\}$ and $V(C) = \{n \,|\, n \text{ is odd}\}$. For an arbitrary state $n \in \omega$ we have that $\mathfrak{M}, n \models (A \Rightarrow B) \Rightarrow (A \wedge C \Rightarrow B)$ iff one of the following holds:

(i) $(\exists n_1 > n)(\forall n_2 > n_1)\ \mathfrak{M}, n_2 \models \neg(A \Rightarrow B)$

(ii) $(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \Rightarrow B) \wedge (A \wedge C \Rightarrow B)$

Case (i): By construction we have that $(\forall n_1 > n)(\exists n_2 > n_1)\ \mathfrak{M}, n_2 \models (A \wedge B)$, so by definition

$$(\forall n \in \omega)\ \mathfrak{M}, n \models (A \Rightarrow B)$$

This gives us that $(\forall n \in \omega)\ \mathfrak{M}, n \not\models \neg(A \Rightarrow B)$ and the case (i) cannot hold.

Case (ii): By construction we have that

$$(\forall n_3 > n)(\exists n_4 > n_3)\ \mathfrak{M}, n_4 \models (A \wedge C) \quad \text{and} \quad (\exists n_5 > n)(\forall n_6 > n_5)\ \mathfrak{M}, n_6 \models \neg(A \wedge B \wedge C)$$

so by definition $(\forall n \in \omega)\ \mathfrak{M}, n \not\models (A \wedge C \Rightarrow B)$ and the case (ii) cannot hold.

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \Rightarrow (A \wedge C \Rightarrow B)$ and the proof is complete.

**Modus Ponens**: We have to show that

$$\mathfrak{F} \not\models A \wedge (A \Rightarrow B) \to B$$

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \{n\}$ and $V(B) = \emptyset$. Then $\mathfrak{M}, n \models A$ and by definition $\mathfrak{M}, n \models (A \Rightarrow B)$, which gives $\mathfrak{M}, n \models A \wedge (A \Rightarrow B)$.

But $\mathfrak{M}, n \not\models B$ by construction.

It follows then that $\mathfrak{M}, n \not\models A \wedge (A \Rightarrow B) \to B$ and the proof is complete. ■

## 3.2 Majority Conditionals over $\omega$ equipped with neighborhoods of cofinite subsets

In this section, we return to the original '*cofinite-as-large*' intuition and we take a more '*traditional*' approach. We resort to the minimal (Scott-Montague) semantics for conditionals introduced by Chellas [Che75], and we discuss variants of truth assignment to conditional statements in worlds whose neighborhoods contain cofinite (large) subsets of $\omega$. Models in this section are based on a frame $\mathfrak{F} = (\omega, f)$, where

$$f : \omega \times 2^{\omega} \to 2^{2^{\omega}}$$

maps worlds ($n \in \omega$) and propositions (sets of possible worlds, also subsets of $\omega$), to neighborhoods of cofinite subsets of $\omega$.

Comment: The conditional logics defined in the rest of this section are rather weak, compared to the previous ones and other known logics from the literature, since they only contain some basic rules and very few axioms. This can also be seen quite clearly in Table 1, at page 64.

**Definition 3.2.1 [Conditional Logic $\overset{\Rightarrow}{\mathfrak{m}}_{\mathbf{1}}$].** Let $\overset{\Rightarrow}{\mathfrak{m}}_{\mathbf{1}}$ be the logic consisting of all $A \in \mathcal{L}_{\Rightarrow}$ valid in $\mathfrak{F} = (\omega, f)$, where a conditional is evaluated as follows:
For a model $\mathfrak{M}$ over $\mathfrak{F}$, $\mathfrak{M}, n \models (A \Rightarrow B)$ iff either

(i) there exists $S \subseteq \|\neg A\|$ such that $S \in f(n, \|A\|)$, or

(ii) there exists $T \subseteq \|A \wedge B\|$ such that $T \in f(n, \|A\|)$

**Theorem 3.2.2** The logic $\overset{\Rightarrow}{\mathfrak{m}}_{\mathbf{1}}$:

1. is closed under the rules **RCEA** and **RCEC**
2. contains the axiom **CM**

PROOF. **RCEA**: Let $\mathfrak{M} \models (A \equiv B)$. This means that $\|A\| = \|B\|$, $\|\neg A\| = \|\neg B\|$ and $\|A \wedge C\| = \|B \wedge C\|$. We have to show that

$$\mathfrak{M} \models (A \Rightarrow C) \equiv (B \Rightarrow C)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \Rightarrow C)$. Then, either:

(i) there exists $S \subseteq \|\neg A\|$ such that $S \in f(n, \|A\|)$, or

(ii) there exists $T \subseteq \|A \wedge C\|$ such that $T \in f(n, \|A\|)$

Case (i): By $\|A\| = \|B\|$ and $\|\neg A\| = \|\neg B\|$ we also have that there exists $S \subseteq \|\neg B\|$ such that $S \in f(n, \|B\|)$ and by definition, $\mathfrak{M}, n \models (B \Rightarrow C)$ follows.

Case (ii): Similarly, by $\|A\| = \|B\|$ and $\|A \wedge C\| = \|B \wedge C\|$ we obtain that there exists $T \subseteq \|B \wedge C\|$ such that $T \in f(n, \|B\|)$ and thus $\mathfrak{M}, n \models (B \Rightarrow C)$.

So, if $\mathfrak{M}, n \models (A \Rightarrow C)$, then $\mathfrak{M}, n \models (B \Rightarrow C)$, which gives that

$$\mathfrak{M}, n \models (A \Rightarrow C) \rightarrow (B \Rightarrow C)$$

Similarly, $\mathfrak{M}, n \models (B \Rightarrow C) \rightarrow (A \Rightarrow C)$. Since the world $n$ was arbitrarily chosen, the proof is complete.

**RCEC**: Let $\mathfrak{M} \models (A \equiv B)$. This means that $\|C \wedge A\| = \|C \wedge B\|$. We have to show that

$$\mathfrak{M} \models (C \Rightarrow A) \equiv (C \Rightarrow B)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (C \Rightarrow A)$. Then, either:

(i) there exists $S \subseteq \|\neg C\|$ such that $S \in f(n, \|C\|)$, or

(ii) there exists $T \subseteq \|C \wedge A\|$ such that $T \in f(n, \|C\|)$

Case (i): By definition, we also have that $\mathfrak{M}, n \models (C \Rightarrow B)$.

Case (ii): By $\|C \wedge A\| = \|C \wedge B\|$ we obtain that there exists $T \subseteq \|C \wedge B\|$ such that $T \in f(n, \|C\|)$ and thus $\mathfrak{M}, n \models (C \Rightarrow B)$.

So, if $\mathfrak{M}, n \models (C \Rightarrow A)$, then $\mathfrak{M}, n \models (C \Rightarrow B)$, which gives that

$$\mathfrak{M}, n \models (C \Rightarrow A) \rightarrow (C \Rightarrow B)$$

Similarly, $\mathfrak{M}, n \models (C \Rightarrow B) \rightarrow (C \Rightarrow A)$. Since the world $n$ was arbitrarily chosen, the proof is complete.

**CM**: We have to show that

$$\mathfrak{F} \models (A \Rightarrow B \wedge C) \rightarrow (A \Rightarrow B) \wedge (A \Rightarrow C)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \Rightarrow B \wedge C)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$. Then, either:

(i) there exists $S \subseteq \|\neg A\|$ such that $S \in f(n, \|A\|)$, or

(ii) there exists $T \subseteq \|A \wedge B \wedge C\|$ such that $T \in f(n, \|A\|)$

Case (i): By definition, $\mathfrak{M}, n \models (A \Rightarrow B)$ and $\mathfrak{M}, n \models (A \Rightarrow C)$, so $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (A \Rightarrow C)$ follows.

Case (ii): We also have that there exists $T \subseteq \|A \wedge B \wedge C\| \subseteq \|A \wedge B\|$ such that $T \in f(n, \|A\|)$ and there exists $T \subseteq \|A \wedge B \wedge C\| \subseteq \|A \wedge C\|$ such that $T \in f(n, \|A\|)$. By definition then, $\mathfrak{M}, n \models (A \Rightarrow B)$ and $\mathfrak{M}, n \models (A \Rightarrow C)$, and thus $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (A \Rightarrow C)$.

So, if $\mathfrak{M}, n \models (A \Rightarrow B \wedge C)$, then $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (A \Rightarrow C)$, which gives that

$$\mathfrak{M}, n \models (A \Rightarrow B \wedge C) \rightarrow (A \Rightarrow B) \wedge (A \Rightarrow C)$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete. ∎

**Theorem 3.2.3** The logic $\overset{\Rightarrow}{\mathfrak{m}_1}$:

1. is not closed under the rules **RCK** and **RCE**
2. does not contain the axioms **ID**, **CUT**, **AC**, **CC**, **Loop**, **OR**, **CV**, **CSO**, **MP**, **MOD**, **CA**, **CS**, **CEM**, **SDA**, **Transitivity**, **Weak Transitivity**, **Monotonicity**, **Weak Monotonicity**, **Modus Ponens** and **Weak Modus Ponens**

Proof. **RCK**: For $n = 2$: We have to show that

$$\mathfrak{M} \models (A_1 \wedge A_2 \rightarrow B) \text{ and } \mathfrak{M} \not\models (C \Rightarrow A_1) \wedge (C \Rightarrow A_2) \rightarrow (C \Rightarrow B)$$

for some model $\mathfrak{M}$ of $\mathfrak{F}$.

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A_1) = \omega - \{n_1\}$, $V(A_2) = \omega - \{n_2\}$, $V(B) = \omega - \{n_1, n_2\}$, $V(C) = \omega$ and for an arbitrary world $n \in \omega$ let $f(n, \|C\|) = \{\omega - \{n_1\}, \omega - \{n_2\}\}$. Then $(\forall n \in \omega)\ \mathfrak{M}, n \models (A_1 \wedge A_2 \rightarrow B)$, because

$$\mathfrak{M}, n_1 \models \neg A_1,\ \mathfrak{M}, n_2 \models \neg A_2 \ \text{ and } \ (\forall n \neq n_1, n_2)\ \mathfrak{M}, n \models B$$

and thus $\mathfrak{M} \models (A_1 \wedge A_2 \rightarrow B)$.

By definition, we also have that $\mathfrak{M}, n \models (C \Rightarrow A_1)$ and $\mathfrak{M}, n \models (C \Rightarrow A_2)$, which means that

$$\mathfrak{M}, n \models (C \Rightarrow A_1) \wedge (C \Rightarrow A_2)$$

But $\mathfrak{M}, n \not\models (C \Rightarrow B)$, because we have that

$$\text{there is no } S \subseteq \|\neg C\| \text{ such that } S \in f(n, \|C\|) \text{ and}$$

$$\text{there is no } T \subseteq \|C \wedge B\| \text{ such that } T \in f(n, \|C\|)$$

It follows then that $\mathfrak{M}, n \not\models (C \Rightarrow A_1) \wedge (C \Rightarrow A_2) \rightarrow (C \Rightarrow B)$ and consequently $\mathfrak{M} \not\models (C \Rightarrow A_1) \wedge (C \Rightarrow A_2) \rightarrow (C \Rightarrow B)$.

**RCE**: We have to show that

$$\mathfrak{M} \models (A \rightarrow B) \ \text{ and } \ \mathfrak{M} \not\models (A \Rightarrow B)$$

for some model $\mathfrak{M}$ of $\mathfrak{F}$.

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \{n \mid n \text{ is even}\}$ and $V(B) = \omega$. Then $(\forall n \in \omega)\ \mathfrak{M}, n \models (A \rightarrow B)$, because $(\forall n \in \omega)\ \mathfrak{M}, n \models B$. Thus $\mathfrak{M} \models (A \rightarrow B)$.

But we have that both $\|\neg A\|$ and $\|A \wedge B\|$ are not co-finite, so for an arbitrary world $n \in \omega$

$$\text{there is no } S \subseteq \|\neg A\| \text{ such that } S \in f(n, \|A\|) \text{ and}$$

$$\text{there is no } T \subseteq \|A \wedge B\| \text{ such that } T \in f(n, \|A\|)$$

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B)$ and consequently $\mathfrak{M} \not\models (A \Rightarrow B)$.

**ID**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow A)$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \{n \mid n \text{ is even}\}$. Then we have that both $\|A\|$ and $\|\neg A\|$ are not co-finite, so for an arbitrary world $n \in \omega$

$$\text{there is no } S \subseteq \|\neg A\| \text{ such that } S \in f(n, \|A\|) \text{ and}$$

$$\text{there is no } T \subseteq \|A\| \text{ such that } T \in f(n, \|A\|)$$

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow A)$ and the proof is complete.

**CUT**: We have to show that

$$\mathfrak{F} \not\models (A \wedge B \Rightarrow C) \wedge (A \Rightarrow B) \rightarrow (A \Rightarrow C)$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega$, $V(B) = \omega - \{n_1\}$, $V(C) = \omega - \{n_2\}$ and for an arbitrary world $n \in \omega$ let $f(n, \|A \wedge B\|) = \{\omega - \{n_1, n_2\}\}$ and $f(n, \|A\|) = \{\omega - \{n_1\}\}$. By definition then, we have that

$$\mathfrak{M}, n \models (A \wedge B \Rightarrow C) \quad \text{and} \quad \mathfrak{M}, n \models (A \Rightarrow B)$$

and thus $\mathfrak{M}, n \models (A \wedge B \Rightarrow C) \wedge (A \Rightarrow B)$.

But $\mathfrak{M}, n \not\models (A \Rightarrow C)$, because

$$\text{there is no } S \subseteq \|\neg A\| \text{ such that } S \in f(n, \|A\|) \text{ and}$$

$$\text{there is no } T \subseteq \|A \wedge C\| \text{ such that } T \in f(n, \|A\|)$$

It follows then that $\mathfrak{M}, n \not\models (A \wedge B \Rightarrow C) \wedge (A \Rightarrow B) \rightarrow (A \Rightarrow C)$ and the proof is complete.

**CC**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \wedge (A \Rightarrow C) \rightarrow (A \Rightarrow B \wedge C)$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega$, $V(B) = \omega - \{n_1\}$, $V(C) = \omega - \{n_2\}$ and for an arbitrary world $n \in \omega$ let $f(n, \|A\|) = \{\omega - \{n_1\}, \omega - \{n_2\}\}$. By definition then, we have that

$$\mathfrak{M}, n \models (A \Rightarrow B) \quad \text{and} \quad \mathfrak{M}, n \models (A \Rightarrow C)$$

and thus $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (A \Rightarrow C)$.

But $\mathfrak{M}, n \not\models (A \Rightarrow B \wedge C)$, because

$$\text{there is no } S \subseteq \|\neg A\| \text{ such that } S \in f(n, \|A\|) \text{ and}$$

$$\text{there is no } T \subseteq \|A \wedge B \wedge C\| \text{ such that } T \in f(n, \|A\|)$$

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \wedge (A \Rightarrow C) \rightarrow (A \Rightarrow B \wedge C)$ and the proof is complete.

**Loop**: For $k = 2$: We have to show that

$$\mathfrak{F} \not\models (A_0 \Rightarrow A_1) \wedge (A_1 \Rightarrow A_2) \wedge (A_2 \Rightarrow A_0) \rightarrow (A_0 \Rightarrow A_2)$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A_0) = \omega$, $V(A_1) = \omega - \{n_1\}$, $V(A_2) = \omega - \{n_2\}$ and for an arbitrary world $n \in \omega$ let $f(n, \|A_0\|) = \{\omega - \{n_1\}\}$, $f(n, \|A_1\|) = \{\omega - \{n_1, n_2\}\}$ and $f(n, \|A_2\|) = \{\omega - \{n_2\}\}$. By definition then, we have that

$$\mathfrak{M}, n \models (A_0 \Rightarrow A_1),\ \mathfrak{M}, n \models (A_1 \Rightarrow A_2) \text{ and } \mathfrak{M}, n \models (A_2 \Rightarrow A_0)$$

and thus $\mathfrak{M}, n \models (A_0 \Rightarrow A_1) \wedge (A_1 \Rightarrow A_2) \wedge (A_2 \Rightarrow A_0)$.

But $\mathfrak{M}, n \not\models (A_0 \Rightarrow A_2)$, because

there is no $S \subseteq \|\neg A_0\|$ such that $S \in f(n, \|A_0\|)$ and

there is no $T \subseteq \|A_0 \wedge A_2\|$ such that $T \in f(n, \|A_0\|)$

It follows then that $\mathfrak{M}, n \not\models (A_0 \Rightarrow A_1) \wedge (A_1 \Rightarrow A_2) \wedge (A_2 \Rightarrow A_0) \rightarrow (A_0 \Rightarrow A_2)$ and the proof is complete.

**CSO**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \wedge (B \Rightarrow A) \rightarrow ((A \Rightarrow C) \equiv (B \Rightarrow C))$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega$, $V(B) = \omega - \{n_1\}$, $V(C) = \omega - \{n_2\}$ and for an arbitrary world $n \in \omega$ let $f(n, \|A\|) = \{\omega - \{n_1\}, \omega - \{n_2\}\}$ and $f(n, \|B\|) = \{\omega - \{n_1\}\}$. By definition then, we have that

$$\mathfrak{M}, n \models (A \Rightarrow B),\ \mathfrak{M}, n \models (B \Rightarrow A) \text{ and } \mathfrak{M}, n \models (A \Rightarrow C)$$

and thus $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (B \Rightarrow A)$ and $\mathfrak{M}, n \models (A \Rightarrow C)$.

But $\mathfrak{M}, n \not\models (B \Rightarrow C)$, because

there is no $S \subseteq \|\neg B\|$ such that $S \in f(n, \|B\|)$ and

there is no $T \subseteq \|B \wedge C\|$ such that $T \in f(n, \|B\|)$

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \wedge (B \Rightarrow A) \rightarrow ((A \Rightarrow C) \rightarrow (B \Rightarrow C))$ and the proof is complete.

**MP**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \rightarrow (A \rightarrow B)$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega$, $V(B) = \omega - \{n\}$ and for the world $n$ let $f(n, \|A\|) = \{\omega - \{n\}\}$. By definition then, we have that $\mathfrak{M}, n \models (A \Rightarrow B)$.

But $\mathfrak{M}, n \not\models (A \rightarrow B)$, because $\mathfrak{M}, n \models A \wedge \neg B$.

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \rightarrow (A \rightarrow B)$ and the proof is complete.

**CS**: We have to show that

$$\mathfrak{F} \not\models (A \wedge B) \rightarrow (A \Rightarrow B)$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = V(B) = \{n\}$ and for the world $n$ let $f(n, \|A\|) = \{\omega\}$. By definition then, we have that $\mathfrak{M}, n \models (A \wedge B)$.

But $\mathfrak{M}, n \not\models (A \Rightarrow B)$, because

$$\text{there is no } S \subseteq \|\neg A\| \text{ such that } S \in f(n, \|A\|) \text{ and}$$

$$\text{there is no } T \subseteq \|A \wedge B\| \text{ such that } T \in f(n, \|A\|)$$

It follows then that $\mathfrak{M}, n \not\models (A \wedge B) \rightarrow (A \Rightarrow B)$ and the proof is complete.

**CEM**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \vee (A \Rightarrow \neg B)$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = V(B) = \{n \,|\, n \text{ is even}\}$ and for an arbitrary state $n \in \omega$ let $f(n, \|A\|) = \{\omega\}$. Then we have that $\mathfrak{M}, n \not\models (A \Rightarrow B)$ and $\mathfrak{M}, n \not\models (A \Rightarrow \neg B)$, because

$$\text{there is no } S \subseteq \|\neg A\| \text{ such that } S \in f(n, \|A\|)$$

$$\text{there is no } T \subseteq \|A \wedge B\| \text{ such that } T \in f(n, \|A\|)$$

$$\text{there is no } R \subseteq \|A \wedge \neg B\| \text{ such that } R \in f(n, \|A\|)$$

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \vee (A \Rightarrow \neg B)$ and the proof is complete.

**Transitivity**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \wedge (B \Rightarrow C) \rightarrow (A \Rightarrow C)$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega$, $V(B) = \omega - \{n_1\}$, $V(C) = \omega - \{n_2\}$ and for an arbitrary world $n \in \omega$ let $f(n, \|A\|) = \{\omega - \{n_1\}\}$ and $f(n, \|B\|) = \{\omega - \{n_1, n_2\}\}$. By definition then, we have that

$$\mathfrak{M}, n \models (A \Rightarrow B) \text{ and } \mathfrak{M}, n \models (B \Rightarrow C)$$

and thus $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (B \Rightarrow C)$.

But $\mathfrak{M}, n \not\models (A \Rightarrow C)$, because

$$\text{there is no } S \subseteq \|\neg A\| \text{ such that } S \in f(n, \|A\|) \text{ and}$$

$$\text{there is no } T \subseteq \|A \wedge C\| \text{ such that } T \in f(n, \|A\|)$$

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \wedge (B \Rightarrow C) \rightarrow (A \Rightarrow C)$ and the proof is complete.

**Weak Transitivity**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega - \{n_1\}$, $V(B) = \omega - \{n_2\}$, $V(C) = \omega$ and for $n^* \in \omega$ let the following hold:

(i) $(\forall n \in \omega - \{n^*\})\ f(n, \|A\|) = \{\omega - \{n_1\}, \omega - \{n_1, n_2\}\}$, while $f(n^*, \|A\|) = \{\omega\}$

(ii) $(\forall n \in \omega - \{n^*\})\ f(n, \|B\|) = \{\omega - \{n_2\}\}$, while $f(n^*, \|B\|) = \{\omega\}$

(iii) $(\forall n \in \omega)\ f(n, \|(A \Rightarrow B) \wedge (B \Rightarrow C)\|) = \{\omega\}$

By definition then, we have that

$$\|A \Rightarrow B\| = \|B \Rightarrow C\| = \|A \Rightarrow C\| = \omega - \{n^*\}$$

This means that

$$\|(A \Rightarrow B) \wedge (B \Rightarrow C) \wedge (A \Rightarrow C)\| = \omega - \{n^*\} \ (1) \text{ and } \|\neg(A \Rightarrow B) \vee \neg(B \Rightarrow C)\| = \{n^*\} \ (2)$$

For an arbitrary state $n \in \omega$ we have that $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$ iff one of the following holds:

(a) there exists $S \subseteq \|\neg(A \Rightarrow B) \vee \neg(B \Rightarrow C)\|$ such that $S \in f(n, \|(A \Rightarrow B) \wedge (B \Rightarrow C)\|)$

(b) there exists $T \subseteq \|(A \Rightarrow B) \wedge (B \Rightarrow C) \wedge (A \Rightarrow C)\|$ such that $T \in f(n, \|(A \Rightarrow B) \wedge (B \Rightarrow C)\|)$

Case (a): By (2) and (iii) there is no $S \subseteq \|\neg(A \Rightarrow B) \vee \neg(B \Rightarrow C)\|$ such that $S \in f(n, \|(A \Rightarrow B) \wedge (B \Rightarrow C)\|)$.

Case (b): By (1) and (iii) there is no $T \subseteq \|(A \Rightarrow B) \wedge (B \Rightarrow C) \wedge (A \Rightarrow C)\|$ such that $T \in f(n, \|(A \Rightarrow B) \wedge (B \Rightarrow C)\|)$.

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$ and the proof is complete.

**Weak Monotonicity**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \Rightarrow (A \wedge C \Rightarrow B)$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega - \{n_1\}$, $V(B) = V(C) = \omega$ and for $n^* \in \omega$ let the following hold:

(i) $(\forall n \in \omega - \{n^*\})\ f(n, \|A\|) = \{\omega - \{n_1\}\}$, while $f(n^*, \|A\|) = \{\omega\}$

(ii) $(\forall n \in \omega - \{n^*\})\ f(n, \|A \wedge C\|) = \{\omega - \{n_1\}\}$, while $f(n^*, \|A \wedge C\|) = \{\omega\}$

(iii) $(\forall n \in \omega)\ f(n, \|(A \Rightarrow B)\|) = \{\omega\}$

By definition then, we have that

$$\|A \Rightarrow B\| = \|A \wedge C \Rightarrow B\| = \omega - \{n^*\}$$

This means that

$$\|(A \Rightarrow B) \wedge (A \wedge C \Rightarrow B)\| = \omega - \{n^*\}\ (1) \quad \text{and} \quad \|\neg(A \Rightarrow B)\| = \{n^*\}\ (2)$$

For an arbitrary state $n \in \omega$ we have that $\mathfrak{M}, n \models (A \Rightarrow B) \Rightarrow (A \wedge C \Rightarrow B)$ iff one of the following holds:

(a) there exists $S \subseteq \|\neg(A \Rightarrow B)\|$ such that $S \in f(n, \|A \Rightarrow B\|)$

(b) there exists $T \subseteq \|(A \Rightarrow B) \wedge (A \wedge C \Rightarrow B)\|$ such that $T \in f(n, \|A \Rightarrow B\|)$

Case (a): By (2) and (iii) there is no $S \subseteq \|\neg(A \Rightarrow B)\|$ such that $S \in f(n, \|A \Rightarrow B\|)$.

Case (b): By (1) and (iii) there is no $T \subseteq \|(A \Rightarrow B) \wedge (A \wedge C \Rightarrow B)\|$ such that $T \in f(n, \|A \Rightarrow B\|)$.

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \Rightarrow (A \wedge C \Rightarrow B)$ and the proof is complete.

**Modus Ponens**: We have to show that

$$\mathfrak{F} \not\models A \wedge (A \Rightarrow B) \rightarrow B$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega$, $V(B) = \omega - \{n\}$ and for the world $n$ let $f(n, \|A\|) = \{\omega - \{n\}\}$. Then $\mathfrak{M}, n \models A$ and by definition $\mathfrak{M}, n \models (A \Rightarrow B)$, which gives $\mathfrak{M}, n \models A \wedge (A \Rightarrow B)$.

But $\mathfrak{M}, n \not\models B$ by construction.

It follows then that $\mathfrak{M}, n \not\models A \wedge (A \Rightarrow B) \rightarrow B$ and the proof is complete.

**Weak Modus Ponens**: We have to show that

$$\mathfrak{F} \not\models A \wedge (A \Rightarrow B) \Rightarrow B$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega - \{n_1\}$ and $V(B) = \omega$ and for $n^* \in \omega$ let the following hold:

(i) $(\forall n \in \omega - \{n^*\})\ f(n, \|A\|) = \{\omega - \{n_1\}\}$, while $f(n^*, \|A\|) = \{\omega\}$

(ii) $(\forall n \in \omega)\ f(n, \|A \wedge (A \Rightarrow B)\|) = \{\omega\}$

By definition then, we have that

$$\|A \Rightarrow B\| = \omega - \{n^*\}$$

This means that

$$\|A \wedge (A \Rightarrow B) \wedge B\| = \omega - \{n_1, n^*\}\ (1) \quad \text{and} \quad \|\neg A \vee \neg(A \Rightarrow B)\| = \{n_1, n^*\}\ (2)$$

For an arbitrary state $n \in \omega$ we have that $\mathfrak{M}, n \models A \wedge (A \Rightarrow B) \Rightarrow B$ iff one of the following holds:

(a) there exists $S \subseteq \|\neg A \vee \neg(A \Rightarrow B)\|$ such that $S \in f(n, \|A \wedge (A \Rightarrow B)\|)$

(b) there exists $T \subseteq \|A \wedge (A \Rightarrow B) \wedge B\|$ such that $T \in f(n, \|A \wedge (A \Rightarrow B)\|)$

Case (a): By (2) and (ii) there is no $S \subseteq \|\neg A \vee \neg(A \Rightarrow B)\|$ such that $S \in f(n, \|A \wedge (A \Rightarrow B)\|)$.

Case (b): By (1) and (ii) there is no $T \subseteq \|A \wedge (A \Rightarrow B) \wedge B\|$ such that $T \in f(n, \|A \wedge (A \Rightarrow B)\|)$.

It follows then that $\mathfrak{M}, n \not\models A \wedge (A \Rightarrow B) \Rightarrow B$ and the proof is complete.

**AC**, **OR**, **CV**, **MOD**, **CA**, **SDA**, **Monotonicity**: For all these axioms the counterexample is obvious. For an arbitrary world $n \in \omega$ just let $f(n, \|A\|)$, $f(n, \|\neg A\|)$, $f(n, \|B\|)$, $f(n, \|C\|)$, $f(n, \|A \wedge B\|)$, $f(n, \|A \vee B\|)$ and $f(n, \|A \wedge C\|)$ be different from one another, according to the respective axiom, and all are invalid in $n$. ∎

Now for another variant. While the previous logic took subsets of truth sets to be included in a neighborhood, here we take only the truth sets themselves.

**Definition 3.2.4 [Conditional Logic $\overset{\Rightarrow}{\mathfrak{m}_2}$].** Let $\overset{\Rightarrow}{\mathfrak{m}_2}$ be the logic consisting of all $A \in \mathcal{L}_{\Rightarrow}$ valid in $\mathfrak{F} = (\omega, f)$, where a conditional is evaluated as follows:
For a model $\mathfrak{M}$ over $\mathfrak{F}$, $\mathfrak{M}, n \models (A \Rightarrow B)$ iff either

(i) $\|\neg A\| \in f(n, \|A\|)$, or

(ii) $\|A \wedge B\| \in f(n, \|A\|)$

**Theorem 3.2.5** The logic $\overset{\Rightarrow}{\mathfrak{m}_2}$ is closed under the rules **RCEA** and **RCEC**

Proof. **RCEA**: Let $\mathfrak{M} \models (A \equiv B)$. This means that $\|A\| = \|B\|$, $\|\neg A\| = \|\neg B\|$ and $\|A \wedge C\| = \|B \wedge C\|$. We have to show that

$$\mathfrak{M} \models (A \Rightarrow C) \equiv (B \Rightarrow C)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \Rightarrow C)$. Then, either:

(i) $\|\neg A\| \in f(n, \|A\|)$, or

(ii) $\|A \wedge C\| \in f(n, \|A\|)$

Case (i): By $\|A\| = \|B\|$ and $\|\neg A\| = \|\neg B\|$ we also have that $\|\neg B\| \in f(n, \|B\|)$ and by definition, $\mathfrak{M}, n \models (B \Rightarrow C)$ follows.

Case (ii): Similarly, by $\|A\| = \|B\|$ and $\|A \wedge C\| = \|B \wedge C\|$ we obtain $\|B \wedge C\| \in f(n, \|B\|)$ and thus $\mathfrak{M}, n \models (B \Rightarrow C)$.

So, if $\mathfrak{M}, n \models (A \Rightarrow C)$, then $\mathfrak{M}, n \models (B \Rightarrow C)$, which gives that

$$\mathfrak{M}, n \models (A \Rightarrow C) \rightarrow (B \Rightarrow C)$$

Similarly, $\mathfrak{M}, n \models (B \Rightarrow C) \to (A \Rightarrow C)$. Since the world $n$ was arbitrarily chosen, the proof is complete.

**RCEC**: Let $\mathfrak{M} \models (A \equiv B)$. This means that $\|C \wedge A\| = \|C \wedge B\|$. We have to show that

$$\mathfrak{M} \models (C \Rightarrow A) \equiv (C \Rightarrow B)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (C \Rightarrow A)$. Then, either:

(i) $\|\neg C\| \in f(n, \|C\|)$, or

(ii) $\|C \wedge A\| \in f(n, \|C\|)$

Case (i): By definition, we also have that $\mathfrak{M}, n \models (C \Rightarrow B)$.

Case (ii): By $\|C \wedge A\| = \|C \wedge B\|$ we obtain $\|C \wedge B\| \in f(n, \|C\|)$ and thus $\mathfrak{M}, n \models (C \Rightarrow B)$.

So, if $\mathfrak{M}, n \models (C \Rightarrow A)$, then $\mathfrak{M}, n \models (C \Rightarrow B)$, which gives that

$$\mathfrak{M}, n \models (C \Rightarrow A) \to (C \Rightarrow B)$$

Similarly, $\mathfrak{M}, n \models (C \Rightarrow B) \to (C \Rightarrow A)$. Since the world $n$ was arbitrarily chosen, the proof is complete. ∎

**Theorem 3.2.6** The logic $\overrightarrow{\mathfrak{m}_2}$:

1. is not closed under the rules **RCK** and **RCE**
2. does not contain any of the axioms **ID**, **CUT**, **AC**, **CC**, **Loop**, **OR**, **CV**, **CSO**, **CM**, **MP**, **MOD**, **CA**, **CS**, **CEM**, **SDA**, **Transitivity**, **Weak Transitivity**, **Monotonicity**, **Weak Monotonicity**, **Modus Ponens** and **Weak Modus Ponens**

Proof. **RCK**, **RCE**, **ID**, **CUT**, **AC**, **CC**, **Loop**, **OR**, **CV**, **CSO**, **MP**, **MOD**, **CA**, **CS**, **CEM**, **SDA**, **Transitivity**, **Weak Transitivity**, **Monotonicity**, **Weak Monotonicity**, **Modus Ponens**, **Weak Modus Ponens**: The counterexamples of all these rules and axioms are identical to the respective cases of the logic $\overrightarrow{\mathfrak{m}_1}$. The counterexample of the axiom **CM** follows.

**CM**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B \wedge C) \to (A \Rightarrow B) \wedge (A \Rightarrow C)$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega$, $V(B) = \omega - \{n_1\}$, $V(C) = \omega - \{n_2\}$ and for an arbitrary world $n \in \omega$ let $f(n, \|A\|) = \{\omega - \{n_1, n_2\}\}$. By definition then, we have that

$$\mathfrak{M}, n \models (A \Rightarrow B \wedge C)$$

But $\mathfrak{M}, n \not\models (A \Rightarrow B)$, because

$$\|\neg A\| \notin f(n, \|A\|) \text{ and } \|A \wedge B\| \notin f(n, \|A\|)$$

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B \wedge C) \to (A \Rightarrow B) \wedge (A \Rightarrow C)$ and the proof is complete. ∎

Before proceeding to the last definition, let $\mathfrak{F} = (\omega, f)$, where

$$f : \omega \times 2^{\omega} \to 2^{2^{\omega}}$$

maps worlds and propositions, to neighborhoods of cofinite subsets of $\omega$ and the following two hold:

(i) If $S \in f(n, X)$ and $S \subseteq T$ then $T \in f(n, X)$ (upwards closed)

(ii) If $S, T \in f(n, X)$ then $S \cap T \in f(n, X)$ (closed under intersections)

The function $f$ is well defined, because if $S$ is cofinite and $S \subseteq T$ then $T$ is also cofinite and if $S, T$ are cofinite then $S \cap T$ is also cofinite.

**Definition 3.2.7 [Conditional Logic $\overset{\Rightarrow}{\mathfrak{m}}_3$].** Let $\overset{\Rightarrow}{\mathfrak{m}}_3$ be the logic consisting of all $A \in \mathcal{L}_{\Rightarrow}$ valid in $\mathfrak{F} = (\omega, f)$, where a conditional is evaluated as follows:
For a model $\mathfrak{M}$ over $\mathfrak{F}$, $\mathfrak{M}, n \models (A \Rightarrow B)$ iff either

(i) $\|\neg A\| \in f(n, \|A\|)$, or

(ii) $\|A \wedge B\| \in f(n, \|A\|)$

**Fact 3.2.8** If we replaced (i) and (ii) in the above definition with:

(i) there exists $S \subseteq \|\neg A\|$ such that $S \in f(n, \|A\|)$

(ii) there exists $T \subseteq \|A \wedge B\|$ such that $T \in f(n, \|A\|)$

then the definition of $(A \Rightarrow B)$ would be equivalent, because of the condition (i) in the definition of $f$.

**Theorem 3.2.9** The logic $\vec{\mathfrak{m}}_3^{\Rightarrow}$:

1. is closed under the rules **RCEA**, **RCK** and **RCEC**
2. contains the axioms **CC** and **CM**

PROOF. **RCEA**: Let $\mathfrak{M} \models (A \equiv B)$. This means that $\|A\| = \|B\|$, $\|\neg A\| = \|\neg B\|$ and $\|A \wedge C\| = \|B \wedge C\|$. We have to show that

$$\mathfrak{M} \models (A \Rightarrow C) \equiv (B \Rightarrow C)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \Rightarrow C)$. Then, either:

(i) $\|\neg A\| \in f(n, \|A\|)$, or

(ii) $\|A \wedge C\| \in f(n, \|A\|)$

Case (i): By $\|A\| = \|B\|$ and $\|\neg A\| = \|\neg B\|$ we also have that $\|\neg B\| \in f(n, \|B\|)$ and by definition, $\mathfrak{M}, n \models (B \Rightarrow C)$ follows.

Case (ii): Similarly, by $\|A\| = \|B\|$ and $\|A \wedge C\| = \|B \wedge C\|$ we obtain $\|B \wedge C\| \in f(n, \|B\|)$ and thus $\mathfrak{M}, n \models (B \Rightarrow C)$.

So, if $\mathfrak{M}, n \models (A \Rightarrow C)$, then $\mathfrak{M}, n \models (B \Rightarrow C)$, which gives that

$$\mathfrak{M}, n \models (A \Rightarrow C) \rightarrow (B \Rightarrow C)$$

Similarly, $\mathfrak{M}, n \models (B \Rightarrow C) \rightarrow (A \Rightarrow C)$. Since the world $n$ was arbitrarily chosen, the proof is complete.

**RCK**: Let $\mathfrak{M} \models (A_1 \wedge \ldots \wedge A_n) \rightarrow B$. This means that $\|C \wedge A_1 \wedge \ldots \wedge A_n\| \subseteq \|C \wedge B\|$. We have to show that

$$\mathfrak{M} \models (C \Rightarrow A_1 \wedge \ldots \wedge C \Rightarrow A_n) \rightarrow (C \Rightarrow B)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (C \Rightarrow A_1 \wedge \ldots \wedge C \Rightarrow A_n)$. Obviously, $\mathfrak{M}, n \models (C \Rightarrow A_1)$ and ... and $\mathfrak{M}, n \models (C \Rightarrow A_n)$. Then, either:

(i) $\|\neg C\| \in f(n, \|C\|)$, or

(ii) $\|C \wedge A_1\| \in f(n, \|C\|)$ and ... and $\|C \wedge A_n\| \in f(n, \|C\|)$. By the condition (ii) in the definition of $f$, this gives $\|C \wedge A_1 \wedge \ldots \wedge A_n\| \in f(n, \|C\|)$

Case (i): By definition, we also have that $\mathfrak{M}, n \models (C \Rightarrow B)$.

Case (ii): By $\|C \wedge A_1 \wedge \ldots \wedge A_n\| \subseteq \|C \wedge B\|$ and the condition (i) in the definition of $f$, we obtain $\|C \wedge B\| \in f(n, \|C\|)$ and thus $\mathfrak{M}, n \models (C \Rightarrow B)$.

So, if $\mathfrak{M}, n \models (C \Rightarrow A_1 \wedge \ldots \wedge C \Rightarrow A_n)$, then $\mathfrak{M}, n \models (C \Rightarrow B)$, which gives that

$$\mathfrak{M}, n \models (C \Rightarrow A_1 \wedge \ldots \wedge C \Rightarrow A_n) \rightarrow (C \Rightarrow B)$$

Since the world $n$ was arbitrarily chosen, the proof is complete.

**RCEC**: Let $\mathfrak{M} \models (A \equiv B)$. This means that $\|C \wedge A\| = \|C \wedge B\|$. We have to show that

$$\mathfrak{M} \models (C \Rightarrow A) \equiv (C \Rightarrow B)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (C \Rightarrow A)$. Then, either:

(i) $\|\neg C\| \in f(n, \|C\|)$, or

(ii) $\|C \wedge A\| \in f(n, \|C\|)$

Case (i): By definition, we also have that $\mathfrak{M}, n \models (C \Rightarrow B)$.

Case (ii): By $\|C \wedge A\| = \|C \wedge B\|$ we obtain $\|C \wedge B\| \in f(n, \|C\|)$ and thus $\mathfrak{M}, n \models (C \Rightarrow B)$.

So, if $\mathfrak{M}, n \models (C \Rightarrow A)$, then $\mathfrak{M}, n \models (C \Rightarrow B)$, which gives that

$$\mathfrak{M}, n \models (C \Rightarrow A) \rightarrow (C \Rightarrow B)$$

Similarly, $\mathfrak{M}, n \models (C \Rightarrow B) \rightarrow (C \Rightarrow A)$. Since the world $n$ was arbitrarily chosen, the proof is complete.

**CC**: We have to show that

$$\mathfrak{F} \models (A \Rightarrow B) \wedge (A \Rightarrow C) \rightarrow (A \Rightarrow B \wedge C)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (A \Rightarrow C)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$. Obviously, $\mathfrak{M}, n \models (A \Rightarrow B)$ and $\mathfrak{M}, n \models (A \Rightarrow C)$. Then, either:

(i) $\|\neg A\| \in f(n, \|A\|)$, or

(ii) $\|A \wedge B\| \in f(n, \|A\|)$ and $\|A \wedge C\| \in f(n, \|A\|)$

Case (i): By definition, $\mathfrak{M}, n \models (A \Rightarrow B \wedge C)$ follows.

Case (ii): By the condition (ii) in the definition of $f$, we also have that $\|A \wedge B \wedge\ C\| \in f(n, \|A\|)$, and thus $\mathfrak{M}, n \models (A \Rightarrow B \wedge C)$.

So, if $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (A \Rightarrow C)$, then $\mathfrak{M}, n \models (A \Rightarrow B \wedge C)$, which gives that

$$\mathfrak{M}, n \models (A \Rightarrow B) \wedge (A \Rightarrow C) \to (A \Rightarrow B \wedge C)$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete.

**CM**: We have to show that

$$\mathfrak{F} \models (A \Rightarrow B \wedge C) \to (A \Rightarrow B) \wedge (A \Rightarrow C)$$

Assume an arbitrary state $n \in \omega$, such that $\mathfrak{M}, n \models (A \Rightarrow B \wedge C)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$. Then, either:

(i) $\|\neg A\| \in f(n, \|A\|)$, or

(ii) $\|A \wedge B \wedge C\| \in f(n, \|A\|)$

Case (i): By definition, $\mathfrak{M}, n \models (A \Rightarrow B)$ and $\mathfrak{M}, n \models (A \Rightarrow C)$, so $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (A \Rightarrow C)$ follows.

Case (ii): We also have that $\|A \wedge B \wedge C\| \subseteq \|A \wedge B\|$ and $\|A \wedge B \wedge C\| \subseteq \|A \wedge C\|$. By condition (i) in the definition of $f$, this gives

$$\|A \wedge B\| \in f(n, \|A\|) \text{ and } \|A \wedge C\| \in f(n, \|A\|)$$

By definition then, $\mathfrak{M}, n \models (A \Rightarrow B)$ and $\mathfrak{M}, n \models (A \Rightarrow C)$, and thus $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (A \Rightarrow C)$.

So, if $\mathfrak{M}, n \models (A \Rightarrow B \wedge C)$, then $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (A \Rightarrow C)$, which gives that

$$\mathfrak{M}, n \models (A \Rightarrow B \wedge C) \to (A \Rightarrow B) \wedge (A \Rightarrow C)$$

Since the world $n$ and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete. ∎

**Theorem 3.2.10** The logic $\overset{\Rightarrow}{\mathfrak{m}}_3$:

1. is not closed under the rule **RCE**
2. does not contain the axioms **ID**, **CUT**, **AC**, **Loop**, **OR**, **CV**, **CSO**, **MP**, **MOD**, **CA**, **CS**, **CEM**, **SDA**, **Transitivity**, **Weak Transitivity**, **Monotonicity**, **Weak Monotonicity**, **Modus Ponens** and **Weak Modus Ponens**

PROOF. **RCE**: We have to show that

$$\mathfrak{M} \models (A \rightarrow B) \text{ and } \mathfrak{M} \not\models (A \Rightarrow B)$$

for some model $\mathfrak{M}$ of $\mathfrak{F}$.

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \{n \mid n \text{ is even}\}$ and $V(B) = \omega$. Then $(\forall n \in \omega)\ \mathfrak{M}, n \models (A \rightarrow B)$, because $(\forall n \in \omega)\ \mathfrak{M}, n \models B$. Thus $\mathfrak{M} \models (A \rightarrow B)$.

But we have that both $\|\neg A\|$ and $\|A \wedge B\|$ are not co-finite, so for an arbitrary world $n \in \omega$

$$\|\neg A\| \notin f(n, \|A\|) \text{ and } \|A \wedge B\| \notin f(n, \|A\|)$$

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B)$ and consequently $\mathfrak{M} \not\models (A \Rightarrow B)$.

**ID**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow A)$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \{n \mid n \text{ is even}\}$. Then we have that both $\|A\|$ and $\|\neg A\|$ are not co-finite, so for an arbitrary world $n \in \omega$

$$\|\neg A\| \notin f(n, \|A\|) \text{ and } \|A\| \notin f(n, \|A\|)$$

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow A)$ and the proof is complete.

**CUT**: We have to show that

$$\mathfrak{F} \not\models (A \wedge B \Rightarrow C) \wedge (A \Rightarrow B) \rightarrow (A \Rightarrow C)$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = V(B) = \omega$, $V(C) = \omega - \{n_1\}$ and for an arbitrary world $n \in \omega$ let $f(n, \|A\|) = \{\omega\}$ and $f(n, \|A \wedge B\|) = \{\omega, \omega - \{n_1\}\}$. By definition then, we have that

$$\mathfrak{M}, n \models (A \wedge B \Rightarrow C) \text{ and } \mathfrak{M}, n \models (A \Rightarrow B)$$

and thus $\mathfrak{M}, n \models (A \wedge B \Rightarrow C) \wedge (A \Rightarrow B)$.

But $\mathfrak{M}, n \not\models (A \Rightarrow C)$, because

$$\|\neg A\| \notin f(n, \|A\|) \text{ and } \|A \wedge C\| \notin f(n, \|A\|)$$

It follows then that $\mathfrak{M}, n \not\models (A \wedge B \Rightarrow C) \wedge (A \Rightarrow B) \rightarrow (A \Rightarrow C)$ and the proof is complete.

**Loop**: For $k = 2$: We have to show that

$$\mathfrak{F} \not\models (A_0 \Rightarrow A_1) \wedge (A_1 \Rightarrow A_2) \wedge (A_2 \Rightarrow A_0) \rightarrow (A_0 \Rightarrow A_2)$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A_0) = \omega$, $V(A_1) = \omega - \{n_1\}$, $V(A_2) = \omega - \{n_2\}$ and for an arbitrary world $n \in \omega$ let $f(n, \|A_0\|) = \{\omega, \omega - \{n_1\}\}$, $f(n, \|A_1\|) = \{\omega, \omega - \{n_1\}, \omega - \{n_2\}, \omega - \{n_1, n_2\}\}$ and $f(n, \|A_2\|) = \{\omega, \omega - \{n_2\}\}$. By definition then, we have that

$$\mathfrak{M}, n \models (A_0 \Rightarrow A_1),\ \ \mathfrak{M}, n \models (A_1 \Rightarrow A_2)\ \ \text{and}\ \ \mathfrak{M}, n \models (A_2 \Rightarrow A_0)$$

and thus $\mathfrak{M}, n \models (A_0 \Rightarrow A_1) \wedge (A_1 \Rightarrow A_2) \wedge (A_2 \Rightarrow A_0)$.

But $\mathfrak{M}, n \not\models (A_0 \Rightarrow A_2)$, because

$$\|\neg A_0\| \notin f(n, \|A_0\|) \text{ and } \|A_0 \wedge A_2\| \notin f(n, \|A_0\|)$$

It follows then that $\mathfrak{M}, n \not\models (A_0 \Rightarrow A_1) \wedge (A_1 \Rightarrow A_2) \wedge (A_2 \Rightarrow A_0) \rightarrow (A_0 \Rightarrow A_2)$ and the proof is complete.

**CSO**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \wedge (B \Rightarrow A) \rightarrow ((A \Rightarrow C) \equiv (B \Rightarrow C))$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega$, $V(B) = \omega - \{n_1\}$, $V(C) = \omega - \{n_2\}$ and for an arbitrary world $n \in \omega$ let $f(n, \|A\|) = \{\omega, \omega - \{n_1\}, \omega - \{n_2\}, \omega - \{n_1, n_2\}\}$ and $f(n, \|B\|) = \{\omega, \omega - \{n_1\}\}$. By definition then, we have that

$$\mathfrak{M}, n \models (A \Rightarrow B),\ \ \mathfrak{M}, n \models (B \Rightarrow A)\ \ \text{and}\ \ \mathfrak{M}, n \models (A \Rightarrow C)$$

and thus $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (B \Rightarrow A)$ and $\mathfrak{M}, n \models (A \Rightarrow C)$.

But $\mathfrak{M}, n \not\models (B \Rightarrow C)$, because

$$\|\neg B\| \notin f(n, \|B\|) \text{ and } \|B \wedge C\| \notin f(n, \|B\|)$$

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \wedge (B \Rightarrow A) \rightarrow ((A \Rightarrow C) \rightarrow (B \Rightarrow C))$ and the proof is complete.

**MP**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \rightarrow (A \rightarrow B)$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega$, $V(B) = \omega - \{n\}$ and for the world $n$ let $f(n, \|A\|) = \{\omega, \omega - \{n\}\}$. By definition then, we have that $\mathfrak{M}, n \models (A \Rightarrow B)$.

But $\mathfrak{M}, n \not\models (A \rightarrow B)$, because $\mathfrak{M}, n \models A \wedge \neg B$.

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \rightarrow (A \rightarrow B)$ and the proof is complete.

**CS**: We have to show that

$$\mathfrak{F} \not\models (A \wedge B) \rightarrow (A \Rightarrow B)$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = V(B) = \{n\}$ and for the world $n$ let $f(n, \|A\|) = \{\omega\}$. By definition then, we have that $\mathfrak{M}, n \models (A \wedge B)$.

But $\mathfrak{M}, n \not\models (A \Rightarrow B)$, because

$$\|\neg A\| \notin f(n, \|A\|) \text{ and } \|A \wedge B\| \notin f(n, \|A\|)$$

It follows then that $\mathfrak{M}, n \not\models (A \wedge B) \rightarrow (A \Rightarrow B)$ and the proof is complete.

**CEM**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \vee (A \Rightarrow \neg B)$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = V(B) = \{n \,|\, n \text{ is even}\}$ and for an arbitrary state $n \in \omega$ let $f(n, \|A\|) = \{\omega\}$. Then we have that $\mathfrak{M}, n \not\models (A \Rightarrow B)$ and $\mathfrak{M}, n \not\models (A \Rightarrow \neg B)$, because

$$\|\neg A\| \notin f(n, \|A\|),\ \|A \wedge B\| \notin f(n, \|A\|) \text{ and } \|A \wedge \neg B\| \notin f(n, \|A\|)$$

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \vee (A \Rightarrow \neg B)$ and the proof is complete.

**Transitivity**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \wedge (B \Rightarrow C) \rightarrow (A \Rightarrow C)$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega$, $V(B) = \omega - \{n_1\}$, $V(C) = \omega - \{n_2\}$ and for an arbitrary world $n \in \omega$ let $f(n, \|A\|) = \{\omega, \omega - \{n_1\}\}$ and $f(n, \|B\|) = \{\omega, \omega - \{n_1\}, \omega - \{n_2\}, \omega - \{n_1, n_2\}\}$. By definition then, we have that

$$\mathfrak{M}, n \models (A \Rightarrow B) \text{ and } \mathfrak{M}, n \models (B \Rightarrow C)$$

and thus $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (B \Rightarrow C)$.

But $\mathfrak{M}, n \not\models (A \Rightarrow C)$, because

$$\|\neg A\| \notin f(n, \|A\|) \text{ and } \|A \wedge C\| \notin f(n, \|A\|)$$

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \wedge (B \Rightarrow C) \rightarrow (A \Rightarrow C)$ and the proof is complete.

**Weak Transitivity**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega - \{n_1\}$, $V(B) = \omega - \{n_2\}$, $V(C) = \omega$ and for $n^* \in \omega$ let the following hold:

(i) $(\forall n \in \omega - \{n^*\})\ f(n, \|A\|) = \{\omega, \omega - \{n_1\}, \omega - \{n_2\}, \omega - \{n_1, n_2\}\}$, while $f(n^*, \|A\|) = \{\omega\}$

(ii) $(\forall n \in \omega - \{n^*\})\ f(n, \|B\|) = \{\omega, \omega - \{n_2\}\}$, while $f(n^*, \|B\|) = \{\omega\}$

(iii) $(\forall n \in \omega)\ f(n, \|(A \Rightarrow B) \wedge (B \Rightarrow C)\|) = \{\omega\}$

By definition then, we have that

$$\|A \Rightarrow B\| = \|B \Rightarrow C\| = \|A \Rightarrow C\| = \omega - \{n^*\}$$

This means that

$\|(A \Rightarrow B) \wedge (B \Rightarrow C) \wedge (A \Rightarrow C)\| = \omega - \{n^*\}$ (1) and $\|\neg(A \Rightarrow B) \vee \neg(B \Rightarrow C)\| = \{n^*\}$ (2)

For an arbitrary state $n \in \omega$ we have that $\mathfrak{M}, n \models (A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$ iff one of the following holds:

(a) $\|\neg(A \Rightarrow B) \vee \neg(B \Rightarrow C)\| \in f(n, \|(A \Rightarrow B) \wedge (B \Rightarrow C)\|)$

(b) $\|(A \Rightarrow B) \wedge (B \Rightarrow C) \wedge (A \Rightarrow C)\| \in f(n, \|(A \Rightarrow B) \wedge (B \Rightarrow C)\|)$

Case (a): By (2) and (iii) we have $\|\neg(A \Rightarrow B) \vee \neg(B \Rightarrow C)\| \notin f(n, \|(A \Rightarrow B) \wedge (B \Rightarrow C)\|)$.

Case (b): By (1) and (iii) we have $\|(A \Rightarrow B) \wedge (B \Rightarrow C) \wedge (A \Rightarrow C)\| \notin f(n, \|(A \Rightarrow B) \wedge (B \Rightarrow C)\|)$.

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$ and the proof is complete.

**Weak Monotonicity**: We have to show that

$$\mathfrak{F} \not\models (A \Rightarrow B) \Rightarrow (A \wedge C \Rightarrow B)$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega - \{n_1\}$, $V(B) = V(C) = \omega$ and for $n^* \in \omega$ let the following hold:

(i) $(\forall n \in \omega - \{n^*\})\ f(n, \|A\|) = \{\omega, \omega - \{n_1\}\}$, while $f(n^*, \|A\|) = \{\omega\}$

(ii) $(\forall n \in \omega - \{n^*\})\ f(n, \|A \wedge C\|) = \{\omega, \omega - \{n_1\}\}$, while $f(n^*, \|A \wedge C\|) = \{\omega\}$

(iii) $(\forall n \in \omega)\ f(n, \|A \Rightarrow B\|) = \{\omega\}$

By definition then, we have that

$$\|A \Rightarrow B\| = \|A \wedge C \Rightarrow B\| = \omega - \{n^*\}$$

This means that

$$\|(A \Rightarrow B) \wedge (A \wedge C \Rightarrow B)\| = \omega - \{n^*\}\ (1) \quad \text{and} \quad \|\neg(A \Rightarrow B)\| = \{n^*\}\ (2)$$

For an arbitrary state $n \in \omega$ we have that $\mathfrak{M}, n \models (A \Rightarrow B) \Rightarrow (A \wedge C \Rightarrow B)$ iff one of the following holds:

(a) $\|\neg(A \Rightarrow B)\| \in f(n, \|A \Rightarrow B\|)$

(b) $\|(A \Rightarrow B) \wedge (A \wedge C \Rightarrow B)\| \in f(n, \|A \Rightarrow B\|)$

Case (a): By (2) and (iii) we have $\|\neg(A \Rightarrow B)\| \notin f(n, \|A \Rightarrow B\|)$.

Case (b): By (1) and (iii) we have $\|(A \Rightarrow B) \wedge (A \wedge C \Rightarrow B)\| \notin f(n, \|A \Rightarrow B\|)$.

It follows then that $\mathfrak{M}, n \not\models (A \Rightarrow B) \Rightarrow (A \wedge C \Rightarrow B)$ and the proof is complete.

**Modus Ponens**: We have to show that

$$\mathfrak{F} \not\models A \wedge (A \Rightarrow B) \rightarrow B$$

Let $\mathfrak{M} = (\omega, <, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega$ and $V(B) = \omega - \{n\}$ and for the world $n$ let $f(n, \|A\|) = \{\omega, \omega - \{n\}\}$. Then $\mathfrak{M}, n \models A$ and by definition $\mathfrak{M}, n \models (A \Rightarrow B)$, which gives $\mathfrak{M}, n \models A \wedge (A \Rightarrow B)$.

But $\mathfrak{M}, n \not\models B$ by construction.

It follows then that $\mathfrak{M}, n \not\models A \wedge (A \Rightarrow B) \rightarrow B$ and the proof is complete.

**Weak Modus Ponens**: We have to show that

$$\mathfrak{F} \not\models A \wedge (A \Rightarrow B) \Rightarrow B$$

Let $\mathfrak{M} = (\omega, f, V)$ be a model of $\mathfrak{F}$ such that $V(A) = \omega - \{n_1\}$ and $V(B) = \omega$ and for $n^* \in \omega$ let the following hold:

(i) $(\forall n \in \omega - \{n^*\})\ f(n, \|A\|) = \{\omega, \omega - \{n_1\}\}$, while $f(n^*, \|A\|) = \{\omega\}$

(ii) $(\forall n \in \omega)\ f(n, \|A \wedge (A \Rightarrow B)\|) = \{\omega\}$

By definition then, we have that

$$\|A \Rightarrow B\| = \omega - \{n^*\}$$

This means that

$$\|A \wedge (A \Rightarrow B) \wedge B\| = \omega - \{n_1, n^*\}\ (1) \quad \text{and} \quad \|\neg A \vee \neg(A \Rightarrow B)\| = \{n_1, n^*\}\ (2)$$

For an arbitrary state $n \in \omega$ we have that $\mathfrak{M}, n \models A \wedge (A \Rightarrow B) \Rightarrow B$ iff one of the following holds:

(a) $\|\neg A \vee \neg(A \Rightarrow B)\| \in f(n, \|A \wedge (A \Rightarrow B)\|)$

(b) $\|A \wedge (A \Rightarrow B) \wedge B\| \in f(n, \|A \wedge (A \Rightarrow B)\|)$

Case (a): By (2) and (ii) we have $\|\neg A \vee \neg(A \Rightarrow B)\| \notin f(n, \|A \wedge (A \Rightarrow B)\|)$.

Case (b): By (1) and (ii) we have $\|A \wedge (A \Rightarrow B) \wedge B\| \notin f(n, \|A \wedge (A \Rightarrow B)\|)$.

It follows then that $\mathfrak{M}, n \not\models A \wedge (A \Rightarrow B) \Rightarrow B$ and the proof is complete.

**AC**, **OR**, **CV**, **MOD**, **CA**, **SDA**, **Monotonicity**: For all these axioms the counterexample is obvious. For an arbitrary world $n \in \omega$ just let $f(n, \|A\|)$, $f(n, \|\neg A\|)$, $f(n, \|B\|)$, $f(n, \|C\|)$, $f(n, \|A \wedge B\|)$, $f(n, \|A \vee B\|)$ and $f(n, \|A \wedge C\|)$ be different from one another, according to the respective axiom, and all are invalid in $n$. ∎

The following table summarizes our results and shows clearly the relation of our model-theoretically defined logics to some of the well-known conditional logics in the literature.

The position of the conditional logics defined in this paper can be easily identified in the last column. Note that on the left of the table we have the KLM systems, with the the first column comprising of the axioms and rules and the second column the *Cumulative, Loop-Cumulative, Preferential* and *Rational* systems respectively. The rest of the table consists of conditional logics (known and new) with the most important axioms and rules listed in the third column. Furthermore, the correspondence between the axioms and rules of the KLM systems and the first 9 of the conditional logics can be readily seen. The same holds for the KLM systems themselves with respect to the first 3 conditional logics (with the exception of the KLM system **CL**) as was already noted at 2.3, on page 8. The logics **CT4** and **C4** were also already discussed in 2.4 as the conditional logics of normality defined by Boutilier and Lamarre respectively. The logic **NP** was introduced by James P. Delgrande in 1987 [Del87] and was one of the first attempts of defining a normality conditional. Finally, the quite weak logics **Λ** and **C** are due to Victor Jauregui [Jau08] and James P. Delgrande [Del03] respectively.

In the table we have made it easily recognizable which axioms are valid in each logic or system and which are not. We have only indicated the ones that are specifically proven to be valid or invalid and have left blank the cases where it is not suggested anywhere in the literature. A tick means that a logic possesses the axiom (or rule) and a shaded box, that it does not.

| KLM | | | | | |
|---|---|---|---|---|---|
| Axioms and Rules | | Systems | | | |
| | | C | CL | P | R |
| **REF** | $A \mid\!\sim A$ | ✓ | ✓ | ✓ | ✓ |
| **LLE** | $\dfrac{\vdash A\equiv B \quad A \mid\!\sim C}{B \mid\!\sim C}$ | ✓ | ✓ | ✓ | ✓ |
| **RW** | $\dfrac{\vdash A\rightarrow B \quad C \mid\!\sim A}{C \mid\!\sim B}$ | ✓ | ✓ | ✓ | ✓ |
| **CUT** | $\dfrac{A\wedge B \mid\!\sim C \quad A \mid\!\sim B}{A \mid\!\sim C}$ | ✓ | ✓ | ✓ | ✓ |
| **CM** | $\dfrac{A \mid\!\sim B \quad A \mid\!\sim C}{A\wedge B \mid\!\sim C}$ | ✓ | ✓ | ✓ | ✓ |
| **AND** | $\dfrac{A \mid\!\sim B \quad A \mid\!\sim C}{A \mid\!\sim B\wedge C}$ | ✓ | ✓ | ✓ | ✓ |
| **Loop** | $\dfrac{A_0 \mid\!\sim A_1 \; \ldots \; A_k \mid\!\sim A_0}{A_0 \mid\!\sim A_k}$ | | ✓ | ✓ | ✓ |
| **OR** | $\dfrac{A \mid\!\sim C \quad B \mid\!\sim C}{A\vee B \mid\!\sim C}$ | | | ✓ | ✓ |
| **RM** | $\dfrac{A \mid\!\sim C \quad A\wedge B \not\mid\!\sim C}{A \mid\!\sim \neg B}$ | | | | ✓ |

| CONDITIONAL LOGICS | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Axioms and Rules | | Systems | | | | | | | | | | | | |
| | | CU | CE | V | CT4 | C4 | A | C | NP | $\overset{\Rightarrow}{\Omega}$ | $\overset{\Rightarrow}{\omega}$ | $\overset{\Rightarrow}{\mathfrak{m}}_1$ | $\overset{\Rightarrow}{\mathfrak{m}}_2$ | $\overset{\Rightarrow}{\mathfrak{m}}_3$ |
| **ID** | $A \Rightarrow A$ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | | ✓ | | | |
| **RCEA** | $\dfrac{A \equiv B}{(A\Rightarrow C) \equiv (B\Rightarrow C)}$ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| **RCK** | $\dfrac{(A_1 \wedge \ldots \wedge A_n) \rightarrow B}{(C\Rightarrow A_1 \wedge \ldots \wedge C\Rightarrow A_n) \rightarrow (C\Rightarrow B)}$ | ✓ | ✓ | ✓ | | ✓ | | | ✓ | ✓ | | | | ✓ |
| **CUT** | $(A\wedge B \Rightarrow C) \wedge (A\Rightarrow B) \rightarrow (A\Rightarrow C)$ | ✓ | ✓ | ✓ | ✓ | ✓ | | | ✓ | ✓ | ✓ | | | |
| **AC** | $(A\Rightarrow B) \wedge (A\Rightarrow C) \rightarrow (A\wedge B \Rightarrow C)$ | ✓ | ✓ | ✓ | ✓ | ✓ | | | | ✓ | | | | |
| **CC** | $(A\Rightarrow B) \wedge (A\Rightarrow C) \rightarrow (A \Rightarrow B\wedge C)$ | ✓ | ✓ | ✓ | ✓ | ✓ | | | ✓ | ✓ | | | | ✓ |
| **Loop** | $(A_0\Rightarrow A_1 \wedge \ldots \wedge A_k\Rightarrow A_0) \rightarrow (A_0\Rightarrow A_k)$ | | ✓ | ✓ | | | | | | ✓ | ✓ | | | |
| **OR** | $(A\Rightarrow C) \wedge (B\Rightarrow C) \rightarrow (A\vee B \Rightarrow C)$ | | ✓ | ✓ | ✓ | ✓ | | | ✓ | ✓ | ✓ | | | |
| **CV** | $(A\Rightarrow B) \wedge \neg(A\Rightarrow\neg C) \rightarrow (A\wedge C \Rightarrow B)$ | | | ✓ | | | | | ✓ | | ✓ | | | |
| **RCEC** | $\dfrac{A \equiv B}{(C\Rightarrow A) \equiv (C\Rightarrow B)}$ | ✓ | ✓ | ✓ | | ✓ | | | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| **RCE** | $\dfrac{A \rightarrow B}{A \Rightarrow B}$ | ✓ | ✓ | ✓ | | | | | | | ✓ | | | |
| **CSO** | $(A\Rightarrow B) \wedge (B\Rightarrow A) \rightarrow ((A\Rightarrow C) \equiv (B\Rightarrow C))$ | ✓ | ✓ | ✓ | ✓ | ✓ | | | | ✓ | | | | |
| **CM** | $(A \Rightarrow B\wedge C) \rightarrow (A\Rightarrow B) \wedge (A\Rightarrow C)$ | ✓ | ✓ | ✓ | | | ✓ | | | ✓ | ✓ | ✓ | | ✓ |
| **MP** | $(A\Rightarrow B) \rightarrow (A\rightarrow B)$ | | | | | | | | | | | | | |
| **MOD** | $(\neg A\Rightarrow A) \rightarrow (B\Rightarrow A)$ | | ✓ | ✓ | | | | | | | ✓ | | | |
| **CA** | $(A\Rightarrow B) \wedge (C\Rightarrow B) \rightarrow (A\wedge C \Rightarrow B)$ | | ✓ | ✓ | | | | | | ✓ | | | | |
| **CS** | $(A \wedge B) \rightarrow (A\Rightarrow B)$ | | | | | | | | | | | | | |
| **CEM** | $(A\Rightarrow B) \vee (A\Rightarrow\neg B)$ | | | | | | | | | | ✓ | | | |
| **SDA** | $(A\vee B \Rightarrow C) \rightarrow (A\Rightarrow C) \wedge (B\Rightarrow C)$ | | | | | | | | | | | | | |
| **Transitivity** | $(A\Rightarrow B) \wedge (B\Rightarrow C) \rightarrow (A\Rightarrow C)$ | | | | | | | | | ✓ | | | | |
| **Weak Transitivity** | $(A\Rightarrow B) \wedge (B\Rightarrow C) \Rightarrow (A\Rightarrow C)$ | | | | ✓ | | | | | ✓ | | | | |
| **Monotonicity** | $(A\Rightarrow B) \rightarrow (A\wedge C \Rightarrow B)$ | | | | | | | | | | | | | |
| **Weak Monotonicity** | $(A\Rightarrow B) \Rightarrow (A\wedge C \Rightarrow B)$ | | | | ✓ | | | | | | | | | |
| **Weak Modus Ponens** | $A \wedge (A\Rightarrow B) \Rightarrow B$ | | | | ✓ | | | | | ✓ | ✓ | | | |

Table 1: Logics $\overset{\Rightarrow}{\Omega}$, $\overset{\Rightarrow}{\omega}$, $\overset{\Rightarrow}{\mathfrak{m}}_i$ vs KLM systems and known Conditional Logics.

# Chapter 4

# Conclusions

In this paper, we have worked on a majority-based account of normality conditionals, based on the intuition that a cofinite subset of $\omega$ is obviously much larger than its complement. The attempt of defining conditionals modally over the frame $(\omega, <)$ has the obvious advantage that its modal axiomatization directly leads to a (for instance, tableaux-based) decision procedure, through an obvious translation. The other direction of employing Scott-Montague type semantics with neighborhoods of cofinite subsets, demonstrates the flexibility of the approach, as even weak logics can be defined by tuning the truth definitions.

The expected difficulty of obtaining a complete axiomatization, is partly due to the fact that conditional logic lacks the sophisticated model-theoretic machinery of modal logics that allows to prove the completeness result for the logic of $(\omega, <)$ (p-morphisms, bulldozing, cluster analysis of transitive frames, etc.). The experimentation with cofinite sets as the guiding principle behind '*overwhelming majority*' is however very instructive, as it allows to delineate the core rules of such an approach.

It is interesting to check, as a question that readily emerges form this work, the nonmonotonic consequence relations that emerge from these conditionals and also try to place them exactly in the universe of conditional logics (e.g [Nut80]).

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