The Admissible Rules of Intermediate Logics

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Πρόλογος

Ένα από τα πιο ενδιαφέροντα χαρακτηριστικά των κατασκευαστικών φορμαλισμών που τους διαχωρίζει από τους αντίστοιχους κλασικούς είναι η αποδοχή μη–παραγόμενων κανόνων. Το γεγονός αυτό είχε παρατηρηθεί ήδη από την δεκαετία του 50, όμως μόλις το 2001 βρέθηκε μια πλήρης απαρίθμηση αυτών των κανόνων για την περίπτωση της ιντουισιονιστικής προτασιακής λογικής. Το σχετικό θεώρημα οφείλεται στη Rosalie Iemhoff και βασίστηκε σε αποτελέσματα των Dick de Jongh, Albert Visser και Silvio Ghilardi. Τα μαθηματικά εργαλεία που χρησιμοποιήθηκαν στη διάρκεια της απόδειξης αυτού του θεωρήματος οδηγούν σε χαρακτηρισμό της ιντουισιονιστικής προτασιακής λογικής τόσο σε συντακτικό όσο και σε σημασιολογικό επίπεδο.

Στη συνέχεια το ενδιαφέρον στρέφεται στις συνεπείς επεκτάσεις της ιντουισιονιστικής προτασιακής λογικής, στις λεγόμενες ενδιάμεσες προτασιακές λογικές. Εκμεταλλευόμενοι τον κυρίαρχο ρόλο της ιντουισιονιστικής προτασιακής λογικής στο πλέγμα των ενδιάμεσων προτασιακών λογικών και γενικεύοντας κατάλληλα τις μεθόδους που έχουν ήδη αναπτυχθεί, καταλήγουμε σε αποτελέσματα περιγραφής των αποδεκτών κανόνων για μερικές από τις πιο γνωστές, ιστορικές και σημαντικές λογικές. Η πλήρης περιγραφή των αποδεκτών κανόνων οποιασδήποτε ενδιάμεσης προτασιακής λογικής είναι η επόμενη πρόκληση.

Ευχαριστίες

Θα ήθελα να εκφράσω την ευγνωμοσύνη μου στην Joan Rand Moschovakis για το χρόνο και την ενέργεια που αφιέρωσε κατά τη διάρκεια συγγραφής αυτής της εργασίας. Την ευχαριστώ επίσης για τις αναρίθμητες διορθώσεις που αφορούσαν τόσο στα μαθηματικά όσο και στην αγγλική γλώσσα. Κυρίως όμως την ευχαριστώ για το ότι έκανε πολύ περισσότερα από ότι ο ρόλος της απαιτούσε. Η ηθική υποστήριξη και ο ζήλος που επέδειξε για τη συνέχιση των σπουδών μου μετά την αποφοίτηση μου από το ΜΠΛΑ είναι ανεκτίμητα.

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Contents

1 Definitions and basic results

1.1 Syntax

The *alphabet* of the propositional language $\mathcal{L}$ consists of:

- the *propositional variables* $p_0, p_1, \ldots$
- the *propositional constant* $\bot$
- the *propositional connectives*: $\wedge$, $\vee$, $\rightarrow$
- the *punctuation marks*: (and)

The *formulas* of $\mathcal{L}$ are inductively defined as

- the constant $\bot$ and all propositional variables are $\mathcal{L}$–formulas
- if φ, ψ are $\mathcal{L}$–formulas then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$ are also $\mathcal{L}$–formulas

The set of variables and the set of formulas of $\mathcal{L}$ are denoted by Var$\mathcal{L}$ and For$\mathcal{L}$ respectively.

We prefer to limit the alphabet, in order to have shorter inductive definitions and proofs. As a result, the propositional constant $\top$ and the propositional connectives $\neg$ and $\leftrightarrow$ are not included in our language, but are introduced as abbreviations in the usual way:

$$\begin{aligned} \top &= (\bot \rightarrow \bot) \\ (\neg\varphi) &= (\varphi \rightarrow \bot) \\ (\varphi \leftrightarrow \psi) &= ((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)) \end{aligned}$$

We will also use the notation $\bigwedge_{i=1}^{n} \varphi_i = \varphi_1 \wedge \varphi_1 \ldots \varphi_n$ and $\bigwedge\Gamma$ to denote the conjunction of the formulas of the finite set Γ. The corresponding notation for disjunction is defined analogously. Finally, we define

$$\bigwedge\emptyset = \top \qquad \text{and} \qquad \bigvee\emptyset = \bot$$

A restricted finite language is presented in § 2.7.1.

1.2 Intuitionistic propositional calculus

As the main objective of this text is to investigate metamathematical properties, we will present a Hilbert–style formal system for intuitionistic propositional logic. However, in the few cases where an actual formal proof will be needed, we will deploy the corresponding natural deduction system.

Intuitionistic Propositional Calculus (IPC) consists of the following axiom schemes

$\to$	$A \to (B \to A)$
	$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
$\wedge$	$A \to (B \to A \wedge B)$
	$A \wedge B \to A$
	$A \wedge B \to B$
$\vee$	$A \to A \vee B$
	$B \to A \vee B$
	$(A \to C) \to ((B \to C) \to (A \vee B \to C))$
$\bot$	$\bot \to A$

and its unique inference rule is modus ponens, namely

From A and $A \to B$ conclude B

Classical propositional calculus CPC is obtained by replacing the $\bot$–axiom with the law of double negation $\neg\neg A \to A$ or equivalently with the law of the excluded middle $A \vee \neg A$.

Definition 1.1. A *derivation* in IPC of a formula φ from assumptions Γ is a finite sequence of formulas $\theta_1, \ldots, \theta_n = \varphi$, each of which either is an axiom of IPC or belongs to Γ or is obtained by applying the modus ponens rule to two formulas occurring earlier in the sequence. If such a derivation exists, we write $\Gamma \vdash \varphi$.

Theorem 1.2 (Deduction Theorem)**.** *If* $\Gamma \cup \{\varphi\} \vdash \psi$ *then* $\Gamma \vdash \varphi \to \psi$*.*

Proof. By induction on the length of any given derivation $\theta_1, \ldots, \theta_n \equiv \psi$, we prove that $\Gamma \vdash \varphi \to \theta_i$. □

1.3 Intermediate logics

Definition 1.3. A *substitution* is an extension to For$\mathcal{L}$ of a function from Var$\mathcal{L}$ to For$\mathcal{L}$, that commutes with the connectives. Notice that by this requirement $\bot$ and $\top$ are fixed points of every substitution.

Definition 1.4. A *logic* in the language $\mathcal{L}$ is any set $L \subseteq$ For$\mathcal{L}$ which satisfies the following conditions:

1. L is closed under modus ponens, i.e L is a theory
2. L is closed under substitution, i.e. if $\varphi \in L$ then $\sigma(\varphi) \in L$ for every substitution σ

According to the above definition IPC and CPC are logics. The set For$\mathcal{L}$ is also a logic; the *inconsistent* logic.

Lemma 1.5. *For every logic L and every formula φ,*

$$\vdash_L \varphi \iff \text{for every substitution } \sigma,\ \vdash_L \sigma(\varphi)$$

Proof. The left–to–right holds since L is closed under substitution and the right–to–left is shown using the identity substitution. □

Observe that in classical propositional logic we only have to be concerned about variable free substitutions.

Definition 1.6. An *intermediate* logic in the language $\mathcal{L}$ is any consistent logic extending IPC.

The term "intermediate" is justified by the following theorem.

Theorem 1.7.

1. *For every variable free formula φ, $\varphi \in \mathrm{IPC}$ or $\neg\varphi \in \mathrm{IPC}$*
2. *All intermediate logics contain the same variable free formulas*
3. *Every intermediate logic is a subset of classical logic*

Proof.

1. By induction on the construction of φ
2. By the first item and the consistency of every intermediate logic
3. $$\begin{aligned} \varphi \in L &\Rightarrow \text{ for every substitution } \sigma : \sigma(\varphi) \in L \\ &\Rightarrow \text{ for every variable free substitution } \sigma : \sigma(\varphi) \in L \\ &\Rightarrow \text{ for every variable free substitution } \sigma : \sigma(\varphi) \in \mathrm{CPC} \quad \text{[by the second item]} \\ &\Rightarrow \varphi \in \mathrm{CPC} \quad \text{[by the nature of classical logic]} \end{aligned}$$

□

Definition 1.8. *Derivations in an intermediate logic* L are defined similarly to IPC, but now in addition to the axioms of IPC we can use the extra axioms of L. In case we are not aware of an axiomatisation of L, then this merely means that we may use every L–theorem.

Observe that

$$\begin{aligned} \Gamma \vdash_L \varphi &\iff \text{there is a finite set } \Delta \text{ of formulas in } L \text{ such that } \Gamma \cup \Delta \vdash \varphi \\ &\iff \Gamma \cup L \vdash \varphi \end{aligned}$$

therefore the deduction theorem holds for every intermediate logic.

The *set of consequences* of a set of formulas Γ in an intermediate logic L is denoted by $Cn^L(\Gamma)$, i.e.

$$Cn^L(\Gamma) = \{\varphi \in \mathrm{For}\mathcal{L} \mid \Gamma \vdash_L \varphi\}$$

1.4 The slash method

Kleene introduced in [Kle62] the notion of slash to investigate disjunction and existence properties under implication for intuitionistic arithmetic. We present Aczel's alternative version which has the additional property of being closed under deduction.

Definition 1.9. (Aczel slash for IPC) Let Γ be a set of formulas and φ a formula. $\Gamma \mid \varphi$ is defined by induction on the construction of φ

$\Gamma \mid \varphi \iff \Gamma \vdash \varphi$, if φ is a propositional variable or $\bot$

$\Gamma \mid \varphi \wedge \psi \iff \Gamma \mid \varphi$ and $\Gamma \mid \psi$

$\Gamma \mid \varphi \vee \psi \iff \Gamma \mid \varphi$ or $\Gamma \mid \psi$

$\Gamma \mid \varphi \to \psi \iff \Gamma \vdash \varphi \to \psi$ and $(\Gamma \mid \varphi \Rightarrow \Gamma \mid \psi)$

Theorem 1.10. *Let Γ be a (possibly empty) set of formulas.*

1. $\Gamma \mid \varphi \Rightarrow \Gamma \vdash \varphi$
2. $(\forall \psi \in \Gamma : \Gamma \mid \psi) \Rightarrow (\Gamma \vdash \varphi \Rightarrow \Gamma \mid \varphi)$

Proof.

1. By induction on formula φ
2. By induction on the derivation $\Gamma \vdash \varphi$

□

If we denote Kleene's slash with $|^K$ then a straightforward induction on the construction of formula φ establishes that

$$\Gamma |^K \varphi \text{ and } \Gamma \vdash \varphi \iff \Gamma \mid \varphi$$

Therefore, result 1.10.2 holds for both versions.

The next theorem, as well as theorem 1.26, indicate the power of the slash–method in obtaining results related to the disjunction property.

Definition 1.11. A set of formulas X has the *disjunction property* if

$$\varphi \vee \psi \in X \Rightarrow \varphi \in X \text{ or } \psi \in X$$

Theorem 1.12. IPC *has the disjunction property*

Proof. Let $\vdash \varphi \vee \psi$. Then $\mid \varphi \vee \psi$, by theorem 1.10.2 for empty Γ, hence $\mid \varphi$ or $\mid \psi$, therefore $\vdash \varphi$ or $\vdash \psi$ by theorem 1.10.1. □

In 1968 de Jongh confirmed Kleene's conjecture that IPC is characterised in terms of the slash relation. In order to state this result we need some generalisations.

Definition 1.13. A formula φ has the L–disjunction property if for all formulas ψ, θ

$$\varphi \vdash_L \psi \vee \theta \Rightarrow \varphi \vdash_L \psi \text{ or } \varphi \vdash_L \theta$$

The slash relation can be defined for every intermediate logic L, by replacing $\vdash$ with $\vdash_L$.

Theorem 1.14 (De Jongh)**.** IPC *is the only intermediate logic with the property that for every formula* φ

$$\varphi \,|_L\, \varphi \iff \varphi \text{ has the } L\text{–disjunction property}$$

Proof. In [dJ70] □

1.5 Propositional rules

Definition 1.15. A *propositional rule* is an expression of the form $\dfrac{\varphi_1, \ldots, \varphi_n}{\psi}$, where $\varphi_1, \ldots, \varphi_n, \psi$ are propositional formulas.

In the current framework we may as well assume that every rule, except modus ponens, has a single premise.

Definition 1.16. The *derivations in an intermediate logic with additional rules R* are defined similarly to the derivations in intermediate logics, but now in addition to modus ponens we can use the rules of R. In other words,

$\Gamma \vdash_L^R \varphi \iff$ there is finite sequence of formulas $\theta_1, \ldots, \theta_n = \varphi$, each of which is
either an axiom of L
or it belongs to Γ
or there are $i_1, \ldots, i_k < i$ such that $\dfrac{\theta_{i_1}, \ldots, \theta_{i_k}}{\theta_i}$ is an instance of
a rule of R or of modus ponens

Note that adding a rule to a formal system is weaker than adding the corresponding axiom scheme. For example, it is not generally true that $\varphi \to \psi$ is derivable in IPC plus the inference rule $\dfrac{\varphi}{\psi}$.

Definition 1.17. A set of formulas X is *closed under a rule* if whenever there are formulas $\varphi_1, \ldots, \varphi_n$ in X such that $\varphi_1, \ldots, \varphi_n/\psi$ is an instance of the rule, then ψ is also in X. X is *closed under a set of rules* R if it is closed under every rule of R.

Definition 1.18. A rule $\dfrac{\varphi, \ldots, \varphi_n}{\psi}$ is *derivable* in an intermediate logic L if the conclusion is derivable in L from the premises, i.e. if $\bigwedge_{i=1}^{n} \varphi_i \vdash_L \psi$.

In view of lemma 1.5, instead of demanding "for every substitution σ, $\sigma(\bigwedge_{i=1}^{n} \varphi_i) \vdash_L \sigma(\psi)$" we chose the more intuitive "$\bigwedge_{i=1}^{n} \varphi_i \vdash_L \psi$", which by the deduction theorem is also equivalent to "$\vdash_L \bigwedge_{i=1}^{n} \varphi_i \to \psi$".

Modus ponens is derivable in IPC. It is also not difficult to establish that

$$\frac{\neg\neg\neg\varphi}{\neg\varphi} \qquad \text{and} \qquad \frac{\varphi}{\psi \to \varphi}$$

are both derivable in IPC. Since derivability is stable under extensions, these rules are also derivable in every intermediate logic.

The addition of derivable rules to a formal system may result in shorter formal proofs, therefore it is a method used to obtain proof–theoretic results. However, it is gratuitous in terms of theory, since it does not enlarge the set of provable formulas. On the other hand, the addition of non–derivable rules invalidates the corresponding deduction theorem, thus rendering $\vdash_L^R$ impractical.

1.6 Admissibility

The admissible rules of a theory are the rules under which the theory is closed. In our context this is formed as

Definition 1.19. The rule φ/ψ is *admissible in the intermediate logic* L if the conclusion is derivable in L whenever the premise is derivable in L, i.e. if for every substitution σ

$$\vdash_L \sigma(\varphi) \Rightarrow \vdash_L \sigma(\psi)$$

In such a case we write $\varphi \mathrel{|\!\!\sim_L} \psi$.

Observe that every derivable rule is also admissible.

Theorem 1.20. *For every intermediate logic L and every formula φ*

1. $\mathrel{|\!\!\sim_L} \varphi \iff \top/\varphi$ *is admissible in* $L \iff \vdash_L \varphi$

2. *If every admissible rule of L is also admissible in an intermediate logic L' then $L \subseteq L'$*

Proof.

1. $$\begin{aligned} \mathrel{|\!\!\sim_L} \varphi &\iff \top \mathrel{|\!\!\sim_L} \varphi \\ &\iff \forall\sigma\,(\vdash_L \sigma(\top) \Rightarrow \vdash_L \sigma\varphi) \\ &\iff \forall\sigma\,(\vdash_L \top \Rightarrow \vdash_L \sigma\varphi) && [\text{since } \sigma(\top) \equiv \top] \\ &\iff \forall\sigma \vdash_L \sigma\varphi && [\text{since} \vdash_L \top] \\ &\iff \vdash_L \varphi && [\text{by lemma 1.5}] \end{aligned}$$

2. $\varphi \in L \Rightarrow \top/\varphi$ is admissible in $L \Rightarrow \top/\varphi$ is admissible in $L' \Rightarrow \varphi \in L'$

□

The converse of the second part of the above theorem is not valid, since unlike derivability, admissibility is not stable under extensions.

Theorem 1.21. *Consider an intermediate logic L and formulas φ, ψ.*

1. *If $\varphi \mathrel{|\!\!\sim_L} \psi$ then $\vdash_{\mathrm{CPC}} \varphi \to \psi$, therefore $L + (\varphi \to \psi)$ is consistent*
2. *Every admissible rule of* CPC *is derivable*

Proof.

1. Let σ be variable free substitution. By theorem 1.7, $\sigma(\varphi) \in \mathrm{IPC}$ or $\neg\sigma(\varphi) \in \mathrm{IPC}$. In the first case we have that

$$\vdash \sigma(\varphi) \Rightarrow \vdash_L \sigma(\varphi) \Rightarrow \vdash_L \sigma(\psi) \Rightarrow \vdash_L \sigma(\varphi) \to \sigma(\psi) \Rightarrow \vdash_L \sigma(\varphi \to \psi) \Rightarrow \vdash_{\mathrm{CPC}} \sigma(\varphi \to \psi)$$

and in the second that

$$\vdash \neg\sigma(\varphi) \Rightarrow \vdash \sigma(\varphi \to \psi) \Rightarrow \vdash_{\mathrm{CPC}} \sigma(\varphi \to \psi)$$

Therefore in any case $\sigma(\varphi \to \psi)$ is in CPC, so $\varphi \to \psi$ is in CPC, by theorem 1.7.

The extended logic is consistent since

$$L + (\varphi \to \psi) \subseteq \mathrm{CPC} + (\varphi \to \psi) = \mathrm{CPC}$$

2. By the first item for $L = \mathrm{CPC}$

□

Intuitionistically, the situation is as usual more complicated. Probably Harrop first observed that the Kreisel–Putnam rule

$$\frac{\neg A \to B \vee C}{(\neg A \to B) \vee (\neg A \to C)}$$

is admissible though not derivable in IPC. Later, Mints observed that

$$\frac{(A \to B) \to A \vee C}{((A \to B) \to A) \vee ((A \to B) \to C)}$$

is a rule of the same kind. Abstracting more, de Jongh and Visser discovered an infinite collection of non–derivable, admissible rules and conjectured that they form a basis for intuitionistic propositional logic.

Definition 1.22. A set or rules R is a *basis for the admissible rules* of an intermediate logic L if for all formulas φ, ψ,

$$\varphi \mathrel{|\!\!\sim_L} \psi \iff \varphi \vdash_L^R \psi$$

Iemhoff's theorem which confirmed in [Iem01b] this conjecture is one of the main theorems presented in this thesis. Meanwhile, Rybakov showed that the problem of whether a rule is admissible in IPC or not is decidable and that there is no finite basis for IPC.

We end this section with some remarks about the bases.

Lemma 1.23. *Let L be an intermediate logic and let R be a set of rules that are derivable in L. Then,*

1. *for all formulas φ, ψ*

$$\varphi \vdash_L^R \psi \iff \varphi \vdash_L \psi$$

2. *if R is a basis for admissibility in L, then L does not have non–derivable admissible rules*

Proof.

1. By induction on the length of given derivation
2. By the previous item

□

It will also become clear in § 4.4, that even if two logics have the same basis for admissibility, they do not have the same set of admissible rules, unless they are equal.

1.7 The Visser rules

Definition 1.24. For $n \geq 1$, V_n is the rule

$$\frac{(A \to B \vee C) \vee D}{\bigvee_{i=1}^{n} (A \to E_i) \vee (A \to B) \vee (A \to C) \vee D}$$

where $A \equiv \bigwedge_{i=1}^{n} (E_i \to F_i)$. The collection of all Visser rules is denoted by V. The restricted V_n rule is defined by omitting the disjunct D from both premise and conclusion.

Observe, that both Kreisel–Putnam and Mints rule are instances of the restricted V_1 rule.

Theorem 1.25. *Let L be an intermediate logic.*

1. *If V_n is admissible in L then so is the restricted V_n. If L has the disjunction property then the converse also holds*
2. *The V_n and the restricted V_n rule are equiderivable in L*
3. *If V_n is admissible (derivable) in L, then for every $m \leq n$, V_m is also admissible (derivable) in L*

4. *If V_1 is not admissible in L then none of the Visser rules is admissible in L*

5. *[First observed by Roziere in [Roz93]] If V_1 is derivable in L, then every Visser rule is derivable in L. Therefore, either all or none of the Visser rules are derivable in L*

Proof.

1. and 2. By the fact that the formulas $\varphi \leftrightarrow \varphi \vee \bot$ and $(\varphi \to \psi) \to (\varphi \vee \theta \to \psi \vee \theta)$ are both derivable in IPC

3. Just observe that $A \equiv \bigwedge_{i=1}^{m} (E_i \to F_i)$ is equivalent to $A \wedge \bigwedge_{i=1}^{n-m} (E_1 \to F_1)$

4. By the previous item

5. Let L be an intermediate logic in which V_1 is derivable. For clarity we will only show that V_2 is also derivable in L. The general case is similar. So,

$$(E_1 \to F_1) \wedge (E_2 \to F_2) \to B \vee C$$

is equivalent to

$$(E_1 \to F_1) \to ((E_2 \to F_2) \to B \vee C)$$

and using the assumption to the antecedent and the transitivity of implication we obtain

$$(E_1 \to F_1) \to ((E_2 \to F_2) \to E_2) \vee ((E_2 \to F_2) \to B) \vee ((E_2 \to F_2) \to C)$$

Now by two consecutive uses of V_1 (as a scheme) or equivalently by using the generalised form 1.27 of the Visser rules we get

$$\begin{aligned}((E_1 \to F_1) \to E_1) \vee ((E_1 \to F_1) \to ((E_2 \to F_2) \to E_2))\\ \vee ((E_1 \to F_1) \to ((E_2 \to F_2) \to B))\\ \vee ((E_1 \to F_1) \to ((E_2 \to F_2) \to C))\end{aligned}$$

which obviously implies

$$(A \to E_1) \vee (A \to E_2) \vee (A \to B) \vee (A \to C)$$

where $A \equiv (E_1 \to F_1) \wedge (E_2 \to F_2)$.

□

Theorem 1.26. *The Visser rules are admissible in* IPC

Proof. By the fact that IPC has the disjunction property, theorem 1.12, and the previous theorem it suffices to show that the restricted Visser rules are admissible in IPC. So assume that $A \vdash B \vee C$, where $A \equiv \bigwedge_{i=1}^{n} (E_i \to F_i)$. If $A \mid A$ then $A \mid B \vee C$ by theorem 1.10.2, hence $A \mid B$ or $A \mid C$, therefore $A \vdash B$ or $A \vdash C$ by theorem 1.10.1. If $A \nmid A$ then there exists an $i \leq n$ such that $A \nmid E_i \to F_i$, hence $A \mid E_i$, since $A \vdash E_i \to F_i$, therefore $A \vdash E_i$ by theorem 1.10.1. □

The following generalised form of the Visser Rules is particularly convenient.

Definition 1.27. For each m, the V_{nm} rule is of the form

$$\frac{(A \to \bigvee_{i=1}^{m} B_i) \vee C}{\bigvee_{i=1}^{n} (A \to E_i) \vee \bigvee_{i=1}^{m} (A \to B_i) \vee C}$$

where $A \equiv \bigwedge_{i=1}^{n} (E_i \to F_i)$.

Observe that $V_{n2} = V_n$ and that

$$V_{n0} = \frac{\neg A \vee C}{\bigvee_{i=1}^{n} (A \to E_i) \vee C} \qquad \text{and} \qquad V_{n1} = \frac{(A \to B) \vee C}{\bigvee_{i=1}^{n} (A \to E_i) \vee (A \to B) \vee C}$$

are both derivable in IPC. Therefore, rarely will we refer to them.

Theorem 1.28. *Let X be a set of propositional formulas which is closed under deduction in* IPC. *If it is closed under the V_n rule, then it is closed under every V_{nm} rule.*

Proof. Trying to simplify a little bit the notation, we define the formulas

$$A = \bigwedge_{i=1}^{n} (E_i \to F_i) \qquad e = \bigvee_{i=1}^{n} (A \to E_i) \qquad b^k = \bigvee_{i=1}^{k} (A \to B_i)$$

It is already mentioned that for $m = 0, 1$ the corresponding implications are derivable in IPC, therefore they are in X. For $m \geq 2$, the proof proceeds by induction on m. The basis case is treated by assumption. For the inductive step assume that

$$(A \to \bigvee_{i=1}^{m+1} B_i) \vee C \in X$$

Considering $\bigvee_{i=1}^{m+1} B_i$ as $\bigvee_{i=1}^{m} B_i \vee B_{m+1}$, the fact that X is closed under the V_n rule implies that

$$e \vee (A \to \bigvee_{i=1}^{m} B_i) \vee (A \to B_{m+1}) \vee C \in X$$

Reading the above formula as $(A \to \bigvee_{i=1}^{m} B_i) \vee \big(e \vee (A \to B_{m+1}) \vee C\big)$ and applying the induction hypothesis we get that

$$e \vee b^m \vee \big(e \vee (A \to B_{m+1}) \vee C\big) \in X$$

which is equivalent to

$$e \vee b^{m+1} \vee C \in X$$

□

1.8 A proof system for admissibility

The fact that we cannot rely on the well–known properties of $\vdash_L$ in order to manipulate $\vdash^R_L$ makes this notion rather cumbersome. Furthermore, there is a lurking danger of using accidentally properties that do not hold in general, for example the deduction theorem, in the middle of a long, involved proof. So, instead of manipulating $\vdash^R_L$–relation directly, we will develop an equivalent, but easier to handle proof system.

1.8.1 A class of proof systems

Although we intend to use a single proof system, we define a whole class of them, exploiting the fact that the theorem below holds in such a broad context.

Definition 1.29. Fix an intermediate logic L and a set of rules R. The $PS_{L,R}$–proof system is specified by the following axioms and rules:

Axioms L If $\varphi \vdash_L \psi$ then $\varphi \rhd \psi$

R If $\dfrac{\varphi_1, \dots, \varphi_n}{\psi}$ is an instance of a rule of R then $\bigwedge_{i=1}^{n} \varphi_i \rhd \psi$

Rules $$\frac{\theta \rhd \varphi \qquad \theta \rhd \psi}{\theta \rhd \varphi \wedge \psi}\ \text{Conj} \qquad \frac{\varphi \rhd \theta \qquad \theta \rhd \psi}{\varphi \rhd \psi}\ \text{Cut}$$

Theorem 1.30. *For every intermediate logic L, every set of rules R and all formulas φ, ψ*

$$\varphi \vdash^R_L \psi \iff PS_{L,R} \vdash \varphi \rhd \psi$$

Proof. $\Rightarrow$) Let $\xi_1, \dots, \xi_n$ be a derivation of $\varphi \vdash^R_L \psi$. We will inductively prove that $PS_{L,R} \vdash \varphi \rhd \xi_i$, for every $i \leq n$. Formula ξ_1 is either derivable in L or it is φ. In either case $\varphi \vdash_L \xi_1$, hence $PS_{L,R} \vdash \varphi \rhd \xi_1$ by the L–axiom.

Now consider ξ_{i+1}. If it is L–derivable or φ, then it is treated as above. If it is derived from an application of modus ponens, then there are formulas $\xi_j, \xi_j \to \xi_{i+1}$ which occur earlier in the sequence and so

$$\dfrac{\dfrac{\overset{\text{by I.H.}\ \vdots}{\varphi \rhd \xi_j} \qquad \overset{\vdots\ \text{by I.H.}}{\varphi \rhd \xi_j \to \xi_{i+1}}}{\varphi \rhd \xi_j \wedge (\xi_j \to \xi_{i+1})}\ \text{Conj} \qquad \overset{\text{by the L–axiom}}{\overset{\Downarrow}{\xi_j \wedge (\xi_j \to \xi_i) \rhd \xi_{i+1}}}}{\varphi \rhd \xi_{i+1}}\ \text{Cut}$$

If ξ_{i+1} is derived from an application of a rule of R, then there are formulas $\xi_{i_1}, \dots, \xi_{i_m}$ which occur earlier in the sequence such that $\dfrac{\xi_{i_1}, \dots, \xi_{i_m}}{\xi_{i+1}}$ is an instance of a rule of R. Therefore,

$$\dfrac{\dfrac{\begin{matrix}\dfrac{\begin{matrix}\text{by I.H.}\ \vdots \\ \varphi \rhd \xi_{i_1}\end{matrix} \qquad \begin{matrix}\vdots\ \text{by I.H.} \\ \varphi \rhd \xi_{i_2}\end{matrix}}{\varphi \rhd \xi_{i_1} \wedge \xi_{i_2}}\ \text{Conj} \\ \vdots\end{matrix} \qquad \begin{matrix}\vdots\ \text{by I.H.} \\ \varphi \rhd \xi_{i_n}\end{matrix}}{\varphi \rhd \bigwedge_{j=1}^{n} \xi_{i_j}}\ \text{Conj} \qquad \begin{matrix}\text{the R–axiom} \\ \Downarrow \\ \bigwedge_{j=1}^{n} \xi_{i_j} \rhd \xi_{i+1}\end{matrix}}{\varphi \rhd \xi_{i+1}}\ \text{Cut}$$

$\Leftarrow$) By induction on the depth of any given $PS_{L,R}$–proof of $\varphi \rhd \psi$. A single line proof of $\varphi \rhd \psi$ is due to the L or the R–axiom. In either case, $\varphi \vdash^{R}_{L} \psi$.

Now assume that there is an $(n+1)$–deep $PS_{L,R}$–proof of $\varphi \rhd \psi$. If the last rule applied is that of conjunction, then $\psi \equiv \psi_1 \wedge \psi_2$ and there are $PS_{L,R}$–proofs of $\varphi \rhd \psi_1$ and $\varphi \rhd \psi_2$ of depth $\leq n$. The induction hypothesis implies that $\varphi \vdash^{R}_{L} \psi_1$ and $\varphi \vdash^{R}_{L} \psi_2$, therefore $\varphi \vdash^{R}_{L} \psi_1 \wedge \psi_2$ by concatenating the derivations and appending the obvious three lines. The case of the cut rule is similar. $\square$

1.8.2 The AR–proof system

As our main aim is to study the connection of the Visser rules with the admissible rules of intuitionistic propositional logic, it is logical to focus on $PS_{\mathrm{IPC},V}$. This proof system was defined by Rosalie Iemhoff in [Iem01b] and it will be denoted as AR, standing for Admissible Rules, and its axioms as I and V respectively.

Theorem 1.31 (Iemhoff)**.** *Let φ, ψ be formulas*

1. *If $AR \vdash \varphi \rhd \psi$ then $AR \vdash \varphi \vee \theta \rhd \psi \vee \theta$*
2. *If $AR \vdash \varphi \rhd \psi$ then $AR \vdash \theta \vee \varphi \rhd \theta \vee \psi$*
3. *If $AR \vdash \varphi \rhd \theta$ and $AR \vdash \psi \rhd \theta$ then $AR \vdash (\varphi \vee \psi) \rhd \theta$*
4. *If $AR \vdash \varphi \rhd \theta$ then $AR \vdash \varphi \wedge \psi \rhd \theta$*

Proof.

1. The proof proceeds by induction on the depth of any given proof. For axioms, just observe that $\varphi \to \psi \vdash \varphi \vee \theta \to \psi \vee \theta$ and that if φ/ψ is an instance of a Visser rule then $\varphi \vee \theta/\psi \vee \theta$ is also an instance of the same Visser rule.

Now assume that there is an $(n+1)$–deep AR–proof of $\varphi \rhd \psi$. If the last rule applied is that of conjunction, then $\psi \equiv \psi_1 \wedge \psi_2$ and there are AR–proofs of $\varphi \rhd \psi_1$ and $\varphi \rhd \psi_2$ of depth $\leq n$. Therefore,

$$\dfrac{\dfrac{\overset{\text{by I.H.}\ \vdots}{\varphi\lor\theta\rhd\psi_1\lor\theta} \qquad \overset{\vdots\ \text{by I.H.}}{\varphi\lor\theta\rhd\psi_2\lor\theta}}{\varphi\lor\theta\rhd(\psi_1\lor\theta)\land(\psi_2\lor\theta)}\,\text{Conj} \qquad \overset{\text{by the I–axiom}}{\overset{\Downarrow}{(\psi_1\lor\theta)\land(\psi_2\lor\theta)\rhd(\psi_1\land\psi_2)\lor\theta}}}{\varphi\lor\theta\rhd(\psi_1\land\psi_2)\lor\theta}\,\text{Cut}$$

The case of the cut rule is similar.

2. Assuming $AR\vdash\varphi\rhd\psi$, we get that

$$\dfrac{\dfrac{\overset{\text{by the } I \text{ axiom}}{\overset{\Downarrow}{\theta\lor\varphi\rhd\varphi\lor\theta}} \qquad \overset{\vdots\ \text{by the previous item}}{\varphi\lor\theta\rhd\psi\lor\theta}}{\theta\lor\varphi\rhd\psi\lor\theta}\,\text{Cut} \qquad \overset{\text{by the I–axiom}}{\overset{\Downarrow}{\psi\lor\theta\rhd\theta\lor\psi}}}{\theta\lor\varphi\rhd\theta\lor\psi}\,\text{Cut}$$

3. Assuming $AR\vdash\varphi\rhd\theta$ and $AR\vdash\psi\rhd\theta$, we get that

$$\dfrac{\overset{\text{by item 1}\ \vdots}{\varphi\lor\psi\rhd\theta\lor\psi} \qquad \dfrac{\overset{\vdots\ \text{by item 2}}{\theta\lor\psi\rhd\theta\lor\theta} \qquad \overset{\text{by the I–axiom}}{\overset{\Downarrow}{\theta\lor\theta\rhd\theta}}}{\theta\lor\psi\rhd\theta}\,\text{Cut}}{\varphi\lor\psi\rhd\theta}\,\text{Cut}$$

4. Apply the cut rule to assumption and to $\varphi\land\psi\rhd\varphi$, which is valid by the I–axiom. □

1.9 Saturation

Definition 1.32. Let L be an intermediate logic and let X, Y be sets of formulas.

1. X is *L–saturated* if $X\vdash_L\varphi\lor\psi \Rightarrow \varphi\in X$ or $\psi\in X$
2. X is *strongly L–saturated in Y* if for every $n\in\omega$ and for all $\varphi_1,\dots,\varphi_n$

$$\text{If } X\vdash_L\bigvee_{i=1}^{n}\varphi_i \text{ then } \varphi_i\in Y \text{ for some } i\leq n$$

Note that the following items could have been added in the previous definition.

1. X is *strongly L–saturated* if for every $n\in\omega$ and for all $\varphi_1,\dots,\varphi_n$

$$\text{If } X\vdash_L\bigvee_{i=1}^{n}\varphi_i \text{ then } \varphi_i\in X \text{ for some } i\leq n$$

2. X is *L–saturated in Y* if $X\vdash_L\varphi\lor\psi \Rightarrow \varphi\in Y$ or $\psi\in Y$

However, they both seem to lack mathematical interest. The first because it coincides with the notion "X is L–saturated" and the second because it is weaker than what we need.

We start by listing some basic properties of the saturated sets.

Theorem 1.33 (Saturation properties)**.** *Let L be a logic and X, Y be sets of formulas.*

1. *X is L–saturated $\iff$ X is closed under deduction in L and has the disjunction property*
2. *X is L–saturated $\iff$ X is strongly L–saturated in X*
3. *If X is L–saturated, then every subset of X is strongly L–saturated in X*
4. *If X is strongly L–saturated in Y, then $X \subseteq Y$*
5. *If X is strongly L–saturated in Y and $Z \subseteq X$ then Z is also strongly L–saturated in Y*
6. *If X is strongly L–saturated in Y and $X \vdash_L \varphi$ then $X \cup \{\varphi\}$ is also strongly L–saturated in Y*

Proof. The first is proved using the fact that $\vdash \varphi \leftrightarrow \varphi \vee \varphi$. The rest are completely trivial. □

Theorem 1.34. *Consider an intermediate logic L and a set of formulas Γ. Then for all formulas ψ, φ the following are equivalent:*

- $\Gamma \vdash_L \psi \to \varphi$
- *for every L–saturated set $Y \supseteq \Gamma$, $\psi \in Y \Rightarrow \varphi \in Y$*

Proof. The left-to-right direction is obvious; the converse will be established by proving the contrapositive. Assume that $\Gamma \nvdash_L \psi \to \varphi$ and let $\xi_0, \xi_1, \dots$ be an enumeration of all formulas in which every formula appears infinitely often. We inductively define a sequence $Y_0 \subseteq Y_1 \subseteq \dots$ of sets of formulas satisfying the invariant property $Y_i \nvdash_L \varphi$, as follows:

$$Y_0 = \Gamma \cup \{\psi\}, \qquad \xi_i \in Y_{i+1} \iff Y_i \cup \{\xi_i\} \nvdash_L \varphi$$

Clearly, $Y = \bigcup_i Y_i$ contains ψ, but not φ. We will show that Y is also L–saturated, so assume that $Y \vdash_L \eta \vee \theta$, hence there is an i such that $Y_i \vdash_L \eta \vee \theta$. Pick indices j, k such that $k \geq j > i$, $\xi_j \equiv \eta$ and $\xi_k \equiv \theta$. If $Y_j \cup \{\eta\} \vdash_L \varphi$ and $Y_k \cup \{\theta\} \vdash_L \varphi$, then $Y_k \cup \{\eta \vee \theta\} \vdash_L \varphi$, hence $Y_k \vdash_L \varphi$, a contradiction. Therefore, $\eta \in Y_{j+1}$ or $\theta \in Y_{k+1}$, and so η or θ is in Y. □

Corollary 1.35. *Consider an intermediate logic L, a set of formulas Γ and a formula φ. Then,*

1. $\Gamma \vdash_L \varphi \iff$ *φ is contained in every L–saturated superset of Γ*
2. $\Gamma \vdash_L \neg\varphi \iff$ *there is no consistent L–saturated superset of Γ containing φ*

Proof. Apply theorem 1.34 to $\psi \equiv \top$ and $\varphi \equiv \bot$ respectively. □

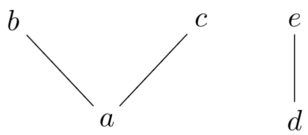
2 Kripke model constructions

2.1 Kripke models

We start by setting up the notation for the semantics.

Definition 2.1. A *Kripke frame* is a partially ordered set, i.e. a pair $\langle W, \leq \rangle$ where W is a non–empty set and $\leq$ is a partial order on W. The elements of W are called nodes. If W has a $\leq$–minimum element then the frame is *rooted*.

According to the preceding definition, a Kripke frame is not necessarily rooted or finite or even connected if it is considered as a graph! For example, the following structure is a well-defined Kripke frame.



Definition 2.2. A *Kripke model* over the language L is a tuple $\langle W, \leq, V \rangle$, where $\langle W, \leq \rangle$ is a Kripke frame and V is a function from W to $\mathcal{P}(\mathrm{Var}\mathcal{L})$ that satisfies the following monotonicity condition:

$$u \leq v \Rightarrow V(u) \subseteq V(v), \text{ for all } u, v \in W$$

Definition 2.3. Let $K = \langle W, \leq, V \rangle$ be a Kripke model and let u be a node of W. By induction on the construction of a formula φ we define the notion of being *true in* K *at* u (or φ is *forced* at u) as follows

- $K, u \Vdash p \iff p \in V(u)$
- $K, u \nVdash \bot$
- $K, u \Vdash \varphi \wedge \psi \iff K, u \Vdash \varphi$ and $K, u \Vdash \psi$
- $K, u \Vdash \varphi \vee \psi \iff K, u \Vdash \varphi$ or $K, u \Vdash \psi$
- $K, u \Vdash \varphi \rightarrow \psi \iff \forall v \geq u,\ K, v \Vdash \varphi \Rightarrow K, v \Vdash \psi$

It follows from the definition that the abbreviated connectives behave properly

- $K, u \Vdash \top$
- $K, u \Vdash \neg\varphi \iff \forall v \geq u\ K, v \nVdash \varphi$
- $K, u \Vdash \varphi \leftrightarrow \psi \iff \forall v \geq u\ (K, v \Vdash \varphi \iff K, v \Vdash \psi)$

Although we demand the monotonicity condition only for propositional variables, it turns out that it holds for every formula.

Lemma 2.4. *For every formula φ, for every Kripke model K and for all nodes u, v of K*

$$\text{If } K, u \Vdash \varphi \text{ and } u \leq v \text{ then } K, v \Vdash \varphi$$

Proof. By induction on the construction of φ. □

Definition 2.5.

- *A Kripke model K satisfies* a formula φ (notation $K \models \varphi$) if φ is true at every node of K
- *A class of Kripke models $\mathscr{K}$ satisfies* a formula φ if φ is satisfied in every model of $\mathscr{K}$
- *A Kripke frame F satisfies* a formula φ if φ is satisfied in every model based on F
- *A class of Kripke frames $\mathscr{F}$ satisfies* a formula φ if φ is satisfied in every frame of $\mathscr{F}$
- The *theory of a Kripke model K*, denoted by $Th(K)$, is the set of formulas satisfied by K. The term "theory" is justified by the fact it is closed under modus ponens. The notion is extended to classes of models and frames in the obvious way.
- Two Kripke models K, M are *equivalent* if they have the same theory, i.e. if for all formulas φ:
$$K \models \varphi \iff M \models \varphi$$

The theory of every *rooted* Kripke model has obviously the disjunction property. Therefore, if a rooted Kripke model is a model of an intermediate logic L, then its theory is by theorem 1.33 an L–saturated set. This result does not in general hold for non–rooted models, as the following two–node example establishes

Figure 1: A model the theory of which does not have the disjunction property

2.1.1 Generated submodels

Only the case of implication in the inductive definition 2.3 differentiates the Kripke model truth from the classical one, since all the other connectives are also treated locally. However even in this case, the nodes below a given node u cannot affect what formulas are true at u. This observation suggests the following definition.

Definition 2.6.

- A frame $\langle W, \leq \rangle$ is a *subframe* of a frame $\langle Z, \preceq \rangle$ if $W \subseteq Z$ and $\leq$ is the restriction of $\preceq$ in W

- A subframe is *generated* if it is upwards closed
- The subframe of a frame F *generated by a node* u consists of all nodes of F greater or equal to u and is denoted by F_u. Such frames are also called generated rooted subframes
- A model $\langle W, \leq, V\rangle$ is a (generated) *submodel* of $\langle Z, \preceq, U\rangle$ if $\langle W, \leq\rangle$ is a (generated) subframe of $\langle Z, \preceq\rangle$ and $V(w) = U(w)$ for every node w of W

Theorem 2.7. *Let u be a node of a Kripke model K. Then, for every formula φ*

$$K_u \models \varphi \iff K, u \Vdash \varphi$$

Proof. By induction on the construction of φ. □

2.1.2 Soundness

Theorem 2.8 (Soundness)**.** *If $\Gamma \vdash \varphi$ then φ is satisfied in every Kripke model the theory of which is a superset of Γ.*

Proof. This is a quite easy and rather tedious proof, so we will only provide a sketch of it. First, we show that each axiom of IPC is satisfied by every Kripke model. The closure under modus ponens is treated by the fact that a theory of a model is obviously a theory. Then we prove by induction on the length of the given derivation $\xi_1, \ldots, \xi_n$ of $\Gamma \vdash \varphi$ that each ξ_i is satisfied in every Kripke model the theory of which is a superset of Γ. □

The completeness theorem of IPC with respect to Kripke frames is deferred until § 2.3.

2.1.3 Kripke frames and intermediate logics

Definition 2.9. Given a substitution σ and a Kripke model K, we construct the Kripke model $\sigma^*(K)$ based on the frame of K and with assignment V_σ defined as:

$$p \in V_\sigma(u) \iff K_u \models \sigma(p)$$

for every propositional variable p and every node u of K. The monotonicity condition K satisfies implies that $\sigma^*(K)$ is a well–defined Kripke model.

Lemma 2.10. *For every Kripke model K, every substitution σ and every formula φ*

$$\sigma^*(K) \models \varphi \iff K \models \sigma(\varphi)$$

Proof. By a straightforward induction on the construction of formula φ □

Theorem 2.11. *The set of formulas satisfied in the arbitrary class of Kripke frames is an intermediate logic.*

Proof. Let L be the set of formulas satisfied in a class of Kripke frames $\mathscr{F}$. By the soundness theorem 2.8, IPC $\subseteq L$. Consider a formula φ in L, a substitution σ and a Kripke model $K = \langle W, \leq, V \rangle$ based on some frame F in $\mathscr{F}$. The model $\sigma^*(K)$ is also based on F, thus it satisfies φ, therefore $K \models \sigma(\varphi)$ by lemma 2.10. □

The converse of the above theorem does not hold, see for example [CZ97], where a Kripke incomplete intermediate logic is constructed. This result demonstrates the inadequacy of the Kripke semantics in the general framework of intermediate logics and turns our attention to algebraic semantics (Heyting algebras) and relational semantics (general frames).

2.1.4 Extension property

We end this section by defining a property which, as we will show in § 3.3, characterises intuitionistic propositional logic. The variations related to intermediate logics and the Visser rules are defined in § 4.3.

Definition 2.12. Let $K_1, \ldots, K_n$ be rooted Kripke models and let X be a set of formulas. The structure $(\sum_{i=1}^{n} K_i)^X$ is constructed by taking an isomorphic copy of each K_i so that their frames are disjoint and then adding below the roots a new node r at which a propositional variable is true if and only if it belongs to X. Therefore $(\sum_{i=1}^{n} K_i)^X$ is a well–defined Kripke model if and only if $X \subseteq \vec{p}$, where $\vec{p}$ is the set of propositional variables satisfied in every K_i. We also define,

$$\sum_{i=1}^{n} K_i = (\sum_{i=1}^{n} K_i)^{\vec{p}} \qquad \text{and} \qquad (\sum_{i=1}^{n} K_i)' = (\sum_{i=1}^{n} K_i)^{\emptyset}$$

Observe that $(\sum_{i=1}^{n} K_i)'$ is always well–defined. This construction is due to Smorynski, see [Smo73], and is called (Smorynski) *gluing*. $\sum$ is the Smorynski operator.

Definition 2.13.

- Two rooted Kripke models are *variants* if they are based on the same frame and their assignments may only differ in the roots.
- A class of rooted Kripke models $\mathscr{K}$ has the *extension property up to n*, if for every $K_1, \ldots, K_n \in \mathscr{K}$ there exists a variant of $\sum_{i=1}^{n} K_i$ in $\mathscr{K}$.
- A class of rooted Kripke models has the *extension property* if it has the extension property up to n, for every $n \geq 1$.
- A set of formulas has the extension property (up to n) if its class of rooted Kripke models has the extension property (up to n).

Note that by adding below the root r of a Kripke model K a new node at which exactly the same as r propositional variables are true, we get a variant of $\sum K$ which is equivalent to K, as a bounded morphic preimage. Therefore, every set of formulas has the extension property up to 1.

Definition 2.14. A class of Kripke models is *stable* if it is closed under generated rooted submodels.

2.2 Truth–preserving operations

We proceed to investigate several truth–preserving operations between Kripke models.

2.2.1 Isomorphisms

The forcing relation of a Kripke model $\langle W, \leq, V \rangle$ is completely determined by the assignment V and the $\leq$–structure of its domain. By relabelling the elements of W and suitably modifying the ordering and the assignment we obtain an essentially identical Kripke model.

Definition 2.15. A bijective function f from model $K = \langle W, \leq, V \rangle$ to model $K' = \langle W', \leq' , V' \rangle$ is an *isomorphism* if for all nodes u, v of K

1. $V(u) = V'(f(u))$
2. $u \leq v \iff f(u) \leq' f(v)$

Theorem 2.16. *If f is an isomorphism from K to K' then,*

1. *for all $u \in K : Th(K_u) = Th(K'_{f(u)})$*
2. *K and K' are equivalent*

Proof. will come as a corollary of theorem 2.20. □

2.2.2 Bounded morphisms

Model–isomorphism is not expected to capture the whole notion of model–equivalence. For example, the models in figure 2 are equivalent, though not isomorphic for cardinality reasons. In terms of their theories, models (b), (c) and (d) contain redundant information and it seems that they can be reduced to the plain model (a). This type of reduction is defined by relaxing the order–preserving condition of isomorphism.

Definition 2.17. Let $K = \langle W, \leq, V \rangle$ and $K' = \langle W', \leq', V' \rangle$ be Kripke models. A surjective function $f : K \twoheadrightarrow K'$ is a *bounded morphism*[1] if it satisfies the following conditions:

1. $V(u) = V'(f(u))$

[1]In the literature is also known as *reduction* or *p–morphism*, which is short for pseudo–epimorphism.

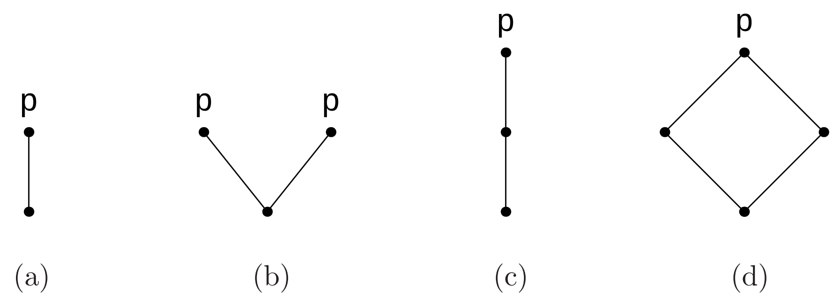


Figure 2: Equivalent Models

2. $u \leq v \Rightarrow f(u) \leq' f(v)$

3. $f(u) \leq' f(v)$ implies the existence of a $w \geq u$ such that $f(w) = f(v)$

If there is a bounded morphism from K to K', then we say that K' *is a bounded morphic image of* K, and write $K \twoheadrightarrow K'$.

Theorem 2.18. *If f is a bounded morphism from K to K' then,*

1. *for all $u \in K : Th(K_u) = Th(K'_{f(u)})$*

2. *K and K' are equivalent*

Proof. will come as a corollary of theorem 2.20. □

2.2.3 Bisimulations

Consider once more models (b) and (c) of figure 2. The fact that there is no bounded morphism between them can be established either intuitively, since neither one is considered simpler than the other, or formally, since there is no surjective function between them that preserves the assignment. However, they are equivalent indeed as bounded morphic preimages of model (a). The notion that directly links such models, without referring to their reduct, is bisimulation. It is a relational generalisation of bounded morphism whereby the directionality from elaborate to plain is replaced by a back–and–forth system of moves between nodes of models.

Definition 2.19.

1. A relation $R \subseteq X \times Y$ is *serial* if

$$(\forall x \in X)(\exists y \in Y)\, xRy$$

2. Let $\langle W, \leq \rangle$ and $\langle W', \leq' \rangle$ be posets. A relation $R \subseteq W \times W'$ is a *simulation preorder* if

$$uRu' \ \&\ u \leq v \Rightarrow \exists v' \geq' u' : vRv'$$

3. Let $K = \langle W, \leq, V \rangle$ and $K' = \langle W', \leq', V' \rangle$ be Kripke models. A relation $R \subseteq W \times W'$ is a *bisimulation* if it satisfies the following conditions:

   - R and R^{-1} are serial
   - $uRu' \Rightarrow V(u) = V'(u')$
   - R is a simulation preorder (forth–condition)
   - R^{-1} is a simulation preorder (back–condition)

Theorem 2.20. *If R is a bisimulation of K to K' then,*

1. *for all $u \in K : uRu' \Rightarrow Th(K_u) = Th(K'_{u'})$*
2. *K and K' are equivalent*

Proof.

1. By induction on the construction of the arbitrary formula φ. The basis is true by the second condition of the definition of "bisimulation" and the cases of disjunction and conjunction are straightforward. For the case of implication, assume that $K_u \models \varphi_1 \to \varphi_2$ and let $v' \geq' u'$ such that ${K'}_{v'} \models \varphi_1$. Back–condition implies that there exists a $v \geq u$ such that vRv'. Therefore,

$$\begin{array}{ccc} {K'}_{v'} \models \varphi_1 & \overset{\text{I.H.}}{\Longrightarrow} & K_v \models \varphi_1 \\ & & \Downarrow \ [\text{since } K_u \models \varphi_1 \to \varphi_2 \text{ and } v \geq u] \\ {K'}_{v'} \models \varphi_2 & \overset{\text{I.H.}}{\Longleftarrow} & K_v \models \varphi_2 \end{array}$$

Forth–condition treats similarly the other direction.

2. Assume that $K \models \varphi$ and let u' be a node of K'. The fact that R^{-1} is serial implies that there is a u in K such that uRu'. Therefore, ${K'}_{u'} \models \varphi$ by the previous item and so $K' \models \varphi$, since u' was arbitrary. The other direction is symmetrical.

□

The preceding comments and definitions convincingly establish that

$$\text{isomorphism} \Rightarrow \text{bounded morphism} \Rightarrow \text{bisimulation}$$

thus theorems 2.16 and 2.18 are in fact corollaries of theorem 2.20.

Note that the frame counterparts of the above operations, which are defined by dropping the condition relating the assignments, are also of interest. Such an example we will meet in the proof of theorem 2.28.

2.3 Completeness results for IPC

Following Craig Smorynski in [Smo73], we present several well–known completeness results for intuitionistic propositional logic.

2.3.1 The canonical model

Definition 2.21. The *canonical model* $\mathcal{K} = \langle \mathcal{W}, \preccurlyeq, \mathcal{V} \rangle$ of IPC is defined as follows:

- $\mathcal{W}$ is the set of all IPC–saturated sets of propositional formulas
- $\preccurlyeq$ is the subset relation
- $\mathcal{V}$ maps each node u to the set of propositional variables contained in the IPC–saturated set associated with u

Each node of $\mathcal{K}$ is identified with the IPC–saturated set attached to it. Note that $\mathcal{K}$ is rooted, with root $Cn(\emptyset)$, which is IPC–saturated because IPC has the disjunction property.

Theorem 2.22. *For every* IPC*–saturated set* X

$$Th(\mathcal{K}_X) = X$$

Proof. By induction on the construction of the arbitrary formula φ; the basis is true by definition.

$$\begin{array}{llll} \wedge & \mathcal{K}_X \models \varphi_1 \wedge \varphi_2 & \iff \mathcal{K}_X \models \varphi_1 \text{ and } \mathcal{K}_X \models \varphi_2 & \\ & & \iff \varphi_1 \in X \text{ and } \varphi_2 \in X & \text{[by the induction hypothesis]} \\ & & \iff \varphi_1 \wedge \varphi_2 \in X & \text{[because } X \text{ is closed under deduction]} \end{array}$$

$$\begin{array}{llll} \vee & \mathcal{K}_X \models \varphi_1 \vee \varphi_2 & \iff \mathcal{K}_X \models \varphi_1 \text{ or } \mathcal{K}_X \models \varphi_2 & \\ & & \iff \varphi_1 \in X \text{ or } \varphi_2 \in X & \text{[by the induction hypothesis]} \\ & & \iff \varphi_1 \vee \varphi_2 \in X & \text{[because } X \text{ is IPC–saturated]} \end{array}$$

$$\begin{array}{llll} \rightarrow & \mathcal{K}_X \models \varphi_1 \rightarrow \varphi_2 & \iff \forall Y \geq X \ (\mathcal{K}_Y \models \varphi_1 \Rightarrow \mathcal{K}_Y \models \varphi_2) & \\ & & \iff \forall Y \supseteq X \ (\varphi_1 \in Y \Rightarrow \varphi_2 \in Y) & \text{[by the induction hypothesis]} \\ & & \iff \varphi_1 \rightarrow \varphi_2 \in X & \text{[by theorem 1.34]} \end{array}$$

□

Theorem 2.23 (Strong completeness)**.** *If* $\Gamma \nvdash \varphi$ *then there exists a Kripke model that satisfies every formula in* Γ*, but not* φ*.*

Proof. Assume that $\Gamma \nvdash \varphi$. By corollary 1.35, there exists an IPC–saturated superset Δ of Γ which does not contain φ. Therefore by theorem 2.22, $\mathcal{K}_\Delta$ is the Kripke model we are looking for. □

The completeness theorem for $\Gamma = \emptyset$ was first proved by Saul Kripke. Strong completeness is due independently to Peter Aczel, Melvin Fitting and Richmond Thomason.

A corollary of the strong completeness theorem is that every intermediate logic L is sound and strongly complete with respect to a class of models, since

$$\Gamma \vdash_L \varphi \iff \Gamma \cup L \vdash \varphi$$

2.3.2 Finite tree theorem

Definition 2.24.

- A Kripke frame $F = \langle W, \leq \rangle$ is a *tree* if it is rooted and for every node $u \in F$ the set of its predecessors $\{v \in F \mid v \leq u\}$ is linearly ordered and finite
- A Kripke frame is an *n–ary tree* if it is a tree such that every node has at most n immediate successors. Note that a 0–ary tree consists just of its root. A unary tree is also called *linear*
- An n–ary tree is *full* if every non–terminal node has exactly n immediate successors

Theorem 2.25 (Smorynski)**.** *For every formula φ and every countermodel $M = \langle W, \leq, V \rangle$ of φ there is a finite tree submodel K of M which does not satisfy φ and is such that*

$$(\forall \psi \in S)(\forall u \in K)(K_u \models \psi \iff M_u \models \psi) \tag{2.1}$$

where S is the set of subformulas of φ.

Proof. First, we define the function $f : M \to \mathcal{P}(S)$ as follows:

$$f(u) = \{\psi \in S \mid M_u \models \psi\} = Th(M_u) \cap S$$

Note that

1. the values of f are finite by assumption
2. f is monotonically non–decreasing by the monotonicity condition Kripke models satisfy
3. $(\forall u, v \in M)(f(u) = f(v) \Rightarrow Th(M_u) \cap S = Th(M_v) \cap S)$

The nodes of the submodel K are selected inductively and are denoted by β_σ, where σ is a finite sequence of natural numbers that keeps track of their order. Let u be any node of M that does not force φ. Define $\beta_{\langle\rangle} = u$.

Assume that β_σ is already selected. The property we would like to hold is that for every $u \geq \beta_\sigma$ there is a $\beta_\tau \geq' \beta_\sigma$ such that $f(\beta_\tau) = f(u)$ (where $\leq'$ will be the ordering of model K). So, define

$$\begin{aligned} W_\sigma = \{u \in M \mid u \geq \beta_\sigma \text{ and } f(u) \supset f(\beta_\sigma) \text{ and} \\ \forall v \in M : \beta_\sigma < v < u \Rightarrow f(v) = f(\beta_\sigma) \text{ or } f(v) = f(u)\} \end{aligned}$$

If $W_\sigma = \emptyset$ then the process stops for this specific branch and the node β_σ will be a leaf of K. Otherwise, although W_σ is in general infinite, $f[W_\sigma]$ is finite, since S is finite. So, let $u_0, \ldots, u_m$ be nodes in W_σ such that

1. $\forall v \in W_\sigma \, \exists i : f(v) = f(u_i)$, i.e $f[\{u_0, \ldots, u_m\}] = f[W_\sigma]$

2. $i \neq j \Rightarrow f(u_i) \neq f(u_j)$

Define $\beta_{\sigma * \langle i \rangle} = u_i$. In this way, we are assured that the branching of K is finite. Moreover, the length of every sequence σ is $\leq |S|$, hence the height is also finite.

The submodel $K = \langle W', \leq', V' \rangle$ is defined as follows

- W' consists of all the β_σ that have been defined
- $\leq'$ is the usual ordering of finite sequences, i.e.

$$\beta_\sigma \leq' \beta_\tau \iff \exists \rho : \tau = \sigma * \rho$$

  Note that $\beta_\sigma \leq' \beta_\tau \Rightarrow \beta_\sigma \leq \beta_\tau$

- $V' = V \restriction W'$

We proceed to prove (2.1) by induction on the construction of the arbitrary subformula of φ.

$$K_{\beta_\sigma} \models p \iff p \in V'(\beta_\sigma) \iff p \in V(\beta_\sigma) \iff M_{\beta_\sigma} \models p$$

The cases of conjunction and disjunction are straightforward. For the case of implication, assume that $K_{\beta_\sigma} \models \theta \to \psi$ and consider a $u \geq \beta_\sigma$ such that $M_u \models \theta$. Then, $f(u) \supseteq f(\beta_\sigma)$ so there is a $\beta_\tau \geq' \beta_\sigma$ such that

$$f(\beta_\tau) = f(u) \tag{2.2}$$

Therefore,

$$\begin{array}{lllll} M_u \models \theta & \overset{(2.2)}{\Longrightarrow} & M_{\beta_\tau} \models \theta & \overset{\text{I.H.}}{\Longrightarrow} & K_{\beta_\tau} \models \theta \\ & & & & \Downarrow \ [\text{since } K_{\beta_\sigma} \models \theta \to \psi \text{ and } \beta_\tau \geq' \beta_\sigma] \\ M_u \models \psi & \overset{(2.2)}{\Longleftarrow} & M_{\beta_\tau} \models \psi & \overset{\text{I.H.}}{\Longleftarrow} & K_{\beta_\tau} \models \psi \end{array}$$

For the other direction, assume that $M_{\beta_\sigma} \models \theta \to \psi$ and let $\beta_\tau \geq' \beta_\sigma$ such that $K_{\beta_\tau} \models \theta$. Then,

$$\begin{array}{lll} K_{\beta_\tau} \models \theta & \overset{\text{I.H.}}{\Longrightarrow} & M_{\beta_\tau} \models \theta \\ & & \Downarrow \ [\text{since } M_{\beta_\sigma} \models \theta \to \psi \text{ and } \beta_\tau \geq' \beta_\sigma] \\ K_{\beta_\tau} \models \psi & \overset{\text{I.H.}}{\Longleftarrow} & M_{\beta_\tau} \models \psi \end{array}$$

Finally, the fact that $K \not\models \varphi$ is a result of the way the root of K is defined and (2.1). □

Corollary 2.26 (Kripke)**.** IPC *is (sound and) complete with respect to finite tree models.*

Proof. Consider a non–derivable in IPC formula φ and let M be its countermodel. Define F as the set of subformulas of φ and $S = \{\varphi\}$. By the previous theorem, there is a finite tree model K which does not satisfies φ. □

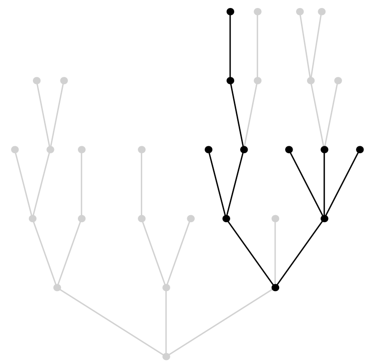


Figure 3: A finite tree extension

2.3.3 Extension theorem

Definition 2.27. Let $T = \langle W, \leq \rangle$, $T' = \langle W', \leq' \rangle$ be finite trees such that T is a subframe of T'. Then, T' is a *finite tree extension* of T if for all nodes u, v of T

$$u \leq v \Rightarrow \nexists w' \in T' \setminus T : u \leq' w' \leq' v$$

Theorem 2.28 (Smorynski)**.** *Consider a finite tree T and a model K based on T. Then, for every finite tree extension T' of T, there is an extension K' of K such that:*

$$\forall u \in K : Th(K_u) = Th(K'_u)$$

Proof. Let r be the root of T and define T_E as $T'_r \setminus T$, where T'_r is the subframe of T' generated by r. Associate with each node u of T a terminal (in T) node $t_u \geq u$ and define the function $f : T'_r \twoheadrightarrow T$ as follows:

- If $u \in T$ then $f(u) = u$
- If $u \in T_E$ then there is a maximum predecessor p_u of u in T, since the trees T'_r and T share the same root. Define $f(u) = t_{p_u}$

We proceed to establish that f is a (frame) bounded morphism from T'_r to T. So, assume that $u \leq' v$ and distinguish cases.

1. If $u, v \in T$ then $f(u) = u \leq v = f(v)$
2. If $u \in T$ and $v \in T_E$ then $f(u) = u \leq p_v \leq t_{p_v} = f(v)$
3. If $u \in T_E$ then also $v \in T_E$ and so $p_u = p_v$. Therefore $f(u) = t_{p_u} = t_{p_v} = f(v)$

For the other condition, we should prove that whenever $f(u) \leq f(v)$, then there exists a $w \geq' u$ such that $f(w) = f(v)$. By distinguishing we get that

1. If $u, v \in T$ then $u = f(u) \leq f(v) = v$, hence $w = v$

2. If $u \in T$ and $v \in T_E$ then $u = f(u) \leq f(v) = t_{p_v}$, hence $w = t_{p_v}$

3. If $u \in T_E$ then $t_{p_u} = f(u) \leq f(v) = t_{p_v}$, therefore $t_{p_u} = t_{p_v}$, hence $w = u$

Define K'_r as the Kripke model based on T'_r with assignment $V' = V \circ f$. By the weak preserving property of bounded morphism V' satisfies the monotonicity condition, hence K'_r is well–defined. f is now a model bounded morphism from K'_r to K, therefore for every $u \in K$,

$$\begin{aligned} Th(K_u) &= Th((K'_r)_{f(u)}) \quad \text{[by theorem 2.18]} \\ &= Th((K'_r)_u) \quad \text{[by the definition of } f\text{]} \end{aligned}$$

Finally, the model K' is any extension of K'_r based on T'. □

Corollary 2.29. *Let $\mathscr{K}$ be a class of finite tree models such that every finite tree can be embedded in a model of $\mathscr{K}$, in the sense that for every finite tree there exists a finite tree extension of it in $\mathscr{K}$. Then* IPC *is (sound and) complete with respect to $\mathscr{K}$.*

Corollary 2.30. IPC *is (sound and) complete with respect to full, non–linear trees.*

2.4 From a class of models to a single model

Exploiting the generality of the Kripke model definition, we may obtain a single Kripke model equivalent to a whole class of models, by replacing every model of the class with an isomorphic copy so that their domains are disjoint. This section develops this idea more rigorously.

2.4.1 The model $\mathscr{K}^+$

Let $\mathscr{K}$ be a class of models indexed by A and for each $\alpha \in A$ let $K^\alpha = \langle W_\alpha, \leq_\alpha, V_\alpha \rangle$ be the model of $\mathscr{K}$ with index α. We define the Kripke model $\mathscr{K}^+ = \langle W^+, \leq^+, V^+ \rangle$ as follows:

- $W^+ = \{\langle \alpha, x \rangle \mid x$ is a node of the model K^α of $\mathscr{K}\}$
- $\langle \alpha, x \rangle \leq^+ \langle \beta, y \rangle \iff \alpha = \beta$ and $x \leq_\alpha y$
- $V^+(\langle \alpha, x \rangle) = V_\alpha(x)$

In other words, W^+ is the disjoint union of W_α and $\leq^+$, V^+ are the inherited ordering and assignment respectively. $\mathscr{K}^+$ is obviously well-defined and a straightforward induction on the construction of the arbitrary formula φ can establish that

$$\mathscr{K}^+_{\langle \alpha, x \rangle} \models \varphi \iff K^\alpha_x \models \varphi, \text{ for all } \langle \alpha, x \rangle \in W^+$$

therefore

$$\mathscr{K}^+ \models \varphi \iff \mathscr{K} \models \varphi$$

To avoid confusion, from now on x, y, z will denote the nodes of the models of $\mathscr{K}$ and u, v, w the nodes of $\mathscr{K}^+$.

2.4.2 The model $\mathscr{K}_{\simeq}^{+}$

We will now construct a more compact model which is still equivalent to the class, by merging the identical branches of $\mathscr{K}^{+}$. So, let $\simeq$ be the following equivalence relation on W^{+}:

$$u \simeq v \iff \mathscr{K}_{u}^{+} \text{ and } \mathscr{K}_{v}^{+} \text{ are isomorphic}$$

Define the Kripke model $\mathscr{K}_{\simeq}^{+} = \langle W, \leq, V \rangle$ as follows:

- $W = W^{+}/\simeq$
- $[u] \leq [v] \iff \exists u' \geq^{+} u : u' \simeq v$
- $V([u]) = V^{+}(u)$

We have yet to establish (a) that $\leq$ is a partial order on W^{+},

(b) that V is indeed a function and (c) that V satisfies the monotonicity condition. But first, we list some properties that interconnect the two models and derive from the notion of model–isomorphism.

Lemma 2.31. *Let v, w be nodes of $\mathscr{K}^{+}$. Then,*

$$v \simeq w \Rightarrow V^{+}(v) = V^{+}(w) \tag{2.3}$$

$$v \simeq w \Rightarrow (\forall w' \geq^{+} w)(\exists v' \geq^{+} v)\ w' \simeq v' \tag{2.4}$$

$$v \leq^{+} w \Rightarrow [v] \leq [w] \tag{2.5}$$

$$[v] < [w] \Rightarrow \exists v' >^{+} v : v' \simeq w \tag{2.6}$$

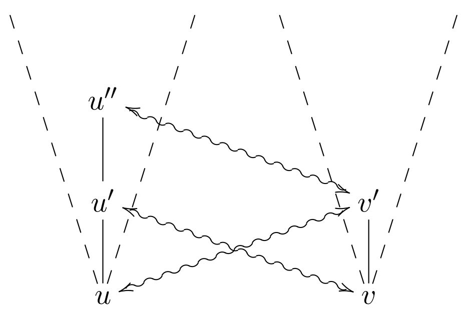
$$v \leq v' \leq v'' \ \&\ v \simeq v'' \Rightarrow v \simeq v' \simeq v'' \tag{2.7}$$

Now, we turn to prove that $\mathscr{K}_{\simeq}^{+}$ is indeed a well–defined Kripke model.

(a) $\leq$ is a partial order.

Reflexivity Trivial

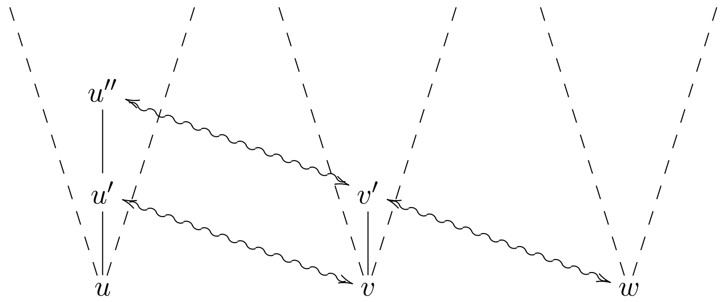
Antisymmetry Assume that $[u] \leq [v]$ and $[v] \leq [u]$. Since two equivalence classes are either equal or disjoint, it suffices to show that $u \simeq v$. By definition, there are nodes $u' \geq^{+} u$ and $v' \geq^{+} v$ such that $u' \simeq v$ and $v' \simeq u$.



$u' \simeq v$ and $v \leq^+ v'$, therefore there exist by (2.4) a $u'' \geq^+ u'$ such that $u'' \simeq v'$. Then,

$$v' \simeq u'' \Rightarrow u \simeq u'' \overset{(2.7)}{\Longrightarrow} u \simeq u' \Rightarrow u \simeq v$$

Transitivity Assume that $[u] \leq [v]$ and $[v] \leq [w]$. Hence there are nodes $u' \geq^+ u$ and $v' \geq^+ v$ such that $u' \simeq v$ and $v' \simeq w$.



$u' \simeq v$ and $v \leq^+ v'$, therefore there exist by (2.4) a $u'' \geq^+ u'$ such that $u'' \simeq v'$, hence $u'' \simeq w$ and so $[u] \leq [w]$.

(b) Let u, v be $\simeq$–equivalent nodes of $\mathscr{K}^+$. Then

$$V([u]) = V^+(u) \overset{(2.3)}{=} V^+(v) = V([v])$$

(c) Assume that $[u] \leq [v]$, hence there is a node $u' \leq^+ u$ such that $u' \simeq v$. Then

$$V([u]) = V^+(u) \subseteq V^+(u') = V([u']) = V([v])$$

Finally, observe that the surjective mapping $u \mapsto [u]$ is a bounded morphism, as property (2.5) and the definitions of V and $\leq$ verify. Therefore $\mathscr{K}^+$ and $\mathscr{K}^+_{\simeq}$ are equivalent, by theorem 2.18.

2.5 Set tight predecessor

In order to show that an intermediate logic L has certain extension properties we will usually have to construct a model of L by gluing together given models of L using the Smorynski operator. In general, the L–provable formulas are not forced downwards in the new root, so we have to search for an additional condition, the satisfaction of which will guarantee that the extended model is indeed a model of L. There is where the notion of tight predecessor emerges. Its exact definition arose in a reversed way; it is the notion for which theorem 2.33 holds.

Definition 2.32. Let L be an intermediate logic and let $X_1, \ldots, X_n$ be L–saturated sets of propositional formulas. The set Y is a *(set) tight predecessor* of $X_1, \ldots, X_n$ in L if

1. Y is L–saturated

2. $Y \subseteq \bigcap_{i=1}^{n} X_i$

3. for every L–saturated set $Y' \supset Y$ there exists an i such that $X_i \subseteq Y'$

Theorem 2.33 (Tight Predecessor Property)**.** *Consider models $K_1, \ldots, K_n$ of an intermediate logic L such that there exists a tight predecessor Y of their theories in L. Then*

$$Th((\sum K_i)^Y) = Y$$

Proof. Let $M = (\sum K_i)^Y$. The proof proceeds by induction on the construction of an arbitrary formula φ. The basis is true by definition.

$$\begin{array}{lll} \wedge \quad M \models \varphi_1 \wedge \varphi_2 & \iff & M \models \varphi_1 \text{ and } M \models \varphi_2 \\ & \iff & \varphi_1 \in Y \text{ and } \varphi_2 \in Y \quad \text{[by the induction hypothesis]} \\ & \iff & \varphi_1 \wedge \varphi_2 \in Y \qquad\quad \text{[because } Y \text{ is closed under deduction in } L] \end{array}$$

$$\begin{array}{lll} \vee \quad M \models \varphi_1 \vee \varphi_2 & \iff & M \models \varphi_1 \text{ or } M \models \varphi_2 \\ & \iff & \varphi_1 \in Y \text{ or } \varphi_2 \in Y \qquad \text{[by the induction hypothesis]} \\ & \iff & \varphi_1 \vee \varphi_2 \in Y \qquad\quad \text{[because } Y \text{ is } L\text{–saturated]} \end{array}$$

$$\begin{array}{lll} \to \quad (\varphi_1 \to \varphi_2) \in Y & \Rightarrow & (\varphi_1 \to \varphi_2) \in Th(K_i), \text{ for all } i \leq n \\ & \Rightarrow & K_i \models \varphi_1 \to \varphi_2, \text{ for all } i \leq n \\ & \Rightarrow & M \models \varphi_1 \to \varphi_2 \qquad \text{[because } Y \text{ is closed under deduction in } L \text{ and} \\ & & \qquad\qquad\qquad\qquad \text{by the induction hypotheses for } \varphi_1 \text{ and } \varphi_2] \end{array}$$

Now, assume that $M \models \varphi_1 \to \varphi_2$ and let $Y' \supseteq Y$ be an L–saturated set that contains φ_1. By theorem 1.34, it suffices to show that $\varphi_2 \in Y'$.

If $Y' = Y$ then using the induction hypotheses for φ_1, φ_2 we get that

$$\varphi_1 \in Y' \Rightarrow M \models \varphi_1 \Rightarrow M \models \varphi_2 \Rightarrow \varphi_2 \in Y'$$

If $Y' \supset Y$ then there is an $i \leq n$ such that $Th(K_i) \subseteq Y'$, thus $\varphi_1 \to \varphi_2 \in Y'$ and so $\varphi_2 \in Y'$.

□

Corollary 2.34. *Under the hypotheses of theorem 2.33, $(\sum K_i)^Y$ is a model of L.*

Proof. $\vdash_L \varphi \Rightarrow Y \vdash_L \varphi \iff \varphi \in Y \iff (\sum K_i)^Y \models \varphi$ □

2.5.1 The construction of a tight predecessor

Definition 2.35. Let X be a set of formulas. Then $I_X = \{\varphi \to \psi \mid \varphi \notin X \text{ and } \psi \in X\}$

Lemma 2.36. *$I_X \subseteq Cn(X)$, for every set of formulas X, therefore $I_X \subseteq X$ if X is closed under deduction in* IPC.

Proof.

$$\varphi \to \psi \in I_X \Rightarrow \psi \in X \Rightarrow \psi \in Cn(X) \Rightarrow \varphi \to \psi \in Cn(X)$$

□

Lemma 2.37. *Let L be an intermediate logic, X, Y be sets of formulas such that Y is strongly L–saturated in X and φ, ψ be formulas such that $Y \cup \{\varphi \vee \psi\}$ is strongly L–saturated in X. Then at least one of $Y \cup \{\varphi\}$ or $Y \cup \{\psi\}$ is strongly L–saturated in X.*

Proof. Assume that $Y \cup \{\varphi \vee \psi\}$ is strongly L–saturated in X and $Y \cup \{\varphi\}$ is not strongly L–saturated in X, so there are $B_1, \ldots, B_n$ none of which belongs to X such that $Y \cup \{\varphi\} \vdash_L \bigvee_{i=1}^{n} B_i$. Let $A_1, \ldots, A_m$ be formulas such that $Y \cup \{\psi\} \vdash_L \bigvee_{i=1}^{m} A_i$. $\bigvee_{i=1}^{n} A_i \vee \bigvee_{i=1}^{n} B_i$ is derived in L from $Y \cup \{\varphi \vee \psi\}$, as it is derived in L by both $Y \cup \{\varphi\}$ and $Y \cup \{\psi\}$. Therefore, since no B_i is in X, there is an $i \leq m$ such that A_i is in X, hence $Y \cup \{\psi\}$ is strongly L–saturated in X. □

Theorem 2.38 (Tight Predecessor Construction)**.** *Let L be an intermediate logic, $X_1, \ldots, X_n$ be L–saturated sets of formulas and $X = \bigcap_{i=1}^{n} X_i$. If there is a $Y_0 \supseteq I_X$ which is strongly L–saturated in X, then there exists a tight predecessor $Y \supseteq Y_0$ of $X_1, \ldots, X_n$ in L.*

Proof. Let $\xi_0, \xi_1, \ldots$ be an enumeration of all formulas in which each formula appears infinitely often. Given Y_0 satisfying the hypotheses, we define inductively a sequence of sets of formulas $Y_0 \subseteq Y_1 \subseteq \ldots$ as follows:

$$\xi_i \in Y_{i+1} \iff Y_i \cup \{\xi_i\} \text{ is strongly } L\text{–saturated in } X$$

Define $Y = \bigcup_i Y_i$ and observe that Y is strongly L–saturated in X, since each Y_i is strongly L–saturated in X, hence $Y \subseteq X$ by theorem 1.33.

We will show that Y is L–saturated, so assume that $Y \vdash_L A \vee B$, hence there is an i such that $Y_i \vdash_L A \vee B$. Remember that each formula appears infinitely often in the enumeration of formulas we chose, so there are indices a, b, k such that $\xi_a \equiv A$, $\xi_b \equiv B$, $\xi_k \equiv A \vee B$ and $k > a \geq i$, $k > b \geq i$. $Y_k \cup \{A \vee B\}$ is strongly L–saturated in X, since Y_k is and $Y_k \vdash_L A \vee B$, therefore $Y_k \cup \{A\}$ or $Y_k \cup \{B\}$ is strongly L–saturated in X by lemma 2.37, hence $Y_a \cup \{A\}$ or $Y_b \cup \{B\}$ is strongly L–saturated in X. Therefore A or B belongs to Y.

Consider an L–saturated $Y' \supset Y$. Let $\varphi \in Y' \setminus Y$ and let i be an index such that $\xi_i \equiv \varphi$. If $Y \cup \{\varphi\}$ were strongly L–saturated in X, then so would be its subset $Y_i \cup \{\varphi\}$, therefore φ would be in Y_{i+1}, hence in Y contrary to the assumption. Hence $Y \cup \{\varphi\}$ is not strongly L–saturated in X, so there exist $A_1, \ldots, A_k$ none of which belongs to X such that $Y \cup \{\varphi\} \vdash_L \bigvee_{i=1}^{k} A_i$. Since Y' is L–saturated and $Y \cup \{\varphi\} \subseteq Y'$, it follows that for some j, $A_j \in Y'$, thus $A_j \in Y' \setminus X$.

Towards a contradiction, assume that there exists an L–saturated $Y' \supset Y$ such that $X_i \nsubseteq Y'$ for all $i \leq n$, hence for each $i \leq n$ there is a formula $B_i \in X_i \setminus Y'$. Therefore

$\bigvee_{i=1}^{n} B_i \in X \setminus Y'$, since X and Y' are closed under deduction in L, thus $A_j \to \bigvee_{i=1}^{n} B_i \in I_X$, where A_j is the formula in $Y' \setminus X$ of the previous part of the proof. But $I_X \subseteq Y_0 \subseteq Y \subseteq Y'$ and Y' is closed under deduction in L, so $\bigvee_{i=1}^{n} B_i \in Y'$, contradicting the L– saturation of Y'. □

A converse also holds.

Corollary 2.39. *Consider L–saturated sets $X_1, \ldots, X_n$ and let $X = \bigcap_{i=1}^{n} X_i$. Then the following are equivalent:*

- *There exists a tight predecessor of $X_1, \ldots, X_n$ in L*
- *I_X is strongly L–saturated in X*

Proof. ⇒) First, we will show that $I_X \subseteq Y$, where Y is a tight predecessor of $X_1, \ldots, X_n$ in L. So, let $\varphi \to \psi \in I_X$ and let $Y' \supseteq Y$ be an L–saturated set that contains φ. By theorem 1.34 it suffices to show that $\psi \in Y'$. Y' cannot be equal to Y, because it contains φ, while Y does not, as a subset of X. So $Y' \supset Y$, hence there exists an i such that $X_i \subseteq Y'$. $\psi \in X \subseteq X_i \subseteq Y'$, so $\psi \in Y'$.

Now, consider formulas $\xi_1, \ldots, \xi_n$ such that $I_X \vdash_L \bigvee_{i=1}^{n} \xi_i$, hence $Y \vdash_L \bigvee_{i=1}^{n} \xi_i$. Y is L–saturated therefore there is a ξ_i in Y, thus in X.

⇐) By theorem 2.38. □

It is already mentioned that the above results will be used extensively in the following sections in order to prove that an intermediate logic L has an extension property. The sketch of those proofs is the following:

1. a preceding lemma will establish that the hypothesis of theorem 2.38 is satisfied,
2. thus (the proof of) theorem 2.38 will construct a tight predecessor,
3. which will be used by theorem 2.33 and corollary 2.34 to construct an extended model of L.

We end this section with a final remark. The tight predecessor, if any, is not in general unique. It is therefore possible for a tight predecessor to be suitable for our purposes, whereas another one is not. Cases where selection is involved occur when the process is iterated more than once, i.e. when we would like to obtain a tight predecessor of a tight predecessor, see figure 4. Such an example is lemma 4.15 and its proof shows a possible solution. Namely, to direct the construction of the first tight predecessor by including a specific set of formulas Δ, apart from I_X, to the initial set Y_0. In this way, you will be assured that the formulas in Δ are also contained in the tight predecessor. This example shows that theorem 2.38 should be considered as an algorithm. Given a finite collection of L–saturated sets $X_1, \ldots, X_n$, its

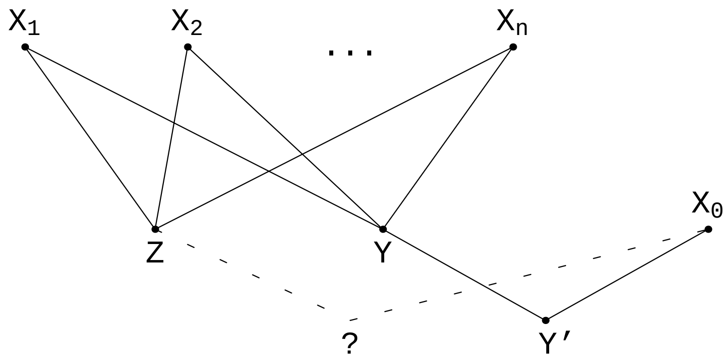


Figure 4: Selection of tight predecessors. While both Z and Y are tight predecessors of $X_1, \ldots, X_n$, Y is preferable since it has a tight predecessor with X_0, whereas Z has not.

input is a set Y_0 which is strongly L–saturated in $\bigcap_{i=1}^{n} X_i$ and its output is a tight predecessor Y of $X_1, \ldots, X_n$ in L, that is a superset of Y_0.

2.6 AR–models

Rosalie Iemhoff defined in [Iem01b] a class of models characterising the AR–proof system, being thus closely related to the admissible rules of intuitionistic propositional logic.

Definition 2.40. A node u of a Kripke model K is a *(node) tight predecessor* of a finite collection of nodes $u_1, \ldots, u_n$ of K if

1. $\forall i : u \leq u_i$ (henceforth, this will be denoted as $u \leq u_1, \ldots, u_n$)
2. $(\forall v > u)(\exists i)(u_i \leq v)$

Definition 2.41. A Kripke model K is an *AR–model* if every finite collection $u_1, \ldots, u_n$ of nodes of K has a tight predecessor in K.[2]

In contrast to the well-known classes of models, like trees or linear models, the AR–models are defined by a technical property which does not provide insight into their form. Their links with the canonical model and the classes of models with the extension property established below, indicate high complexity and dense connectivity. However, we are not able yet to visualise them.

Definition 2.42. A set of formulas is *adequate* if it is closed under subformulas.

Theorem 2.43.

1. *For every stable class with the extension property $\mathscr{K}$ of finite rooted Kripke models there exists an AR–model M equivalent to $\mathscr{K}$*

[2]The original definition also included that an *AR*–model should be rooted.

2. *For every AR–model M and for every finite, adequate set S there exists a stable class with the extension property $\mathscr{K}$ of finite rooted models such that*

$$Th(\mathscr{K}) \cap S = Th(M) \cap S$$

Proof.

1. Consider M and S satisfying the hypotheses. If $Th(M) \supseteq S$ then we define $\mathscr{K} = \emptyset$, which is trivially a stable class with the extension property and its theory is For$\mathcal{L}$, therefore
$$Th(M) \cap S = S = Th(\mathscr{K}) \cap S$$
Now, assume that $Th(M) \not\supseteq S$ and define $\mathscr{K}$ as the class of all finite rooted submodels K of M satisfying the condition:
$$(\forall u \in K)(Th(K_u) \cap S = Th(M_u) \cap S) \tag{2.8}$$
By assumption there is a formula $\varphi \in S$ such that $M \not\models \varphi$. Theorem 2.25 constructs a finite rooted submodel of M that satisfies (2.8) and does not satisfy φ. Hence, $\mathscr{K}$ is not empty and does not satisfy φ, therefore by contrapositive reasoning $Th(\mathscr{K}) \cap S \subseteq Th(M) \cap S$. Furthermore, $Th(M) \subseteq Th(\mathscr{K})$ since $\mathscr{K}$ contains generated submodels of M, therefore $Th(M) \cap S = Th(\mathscr{K}) \cap S$.
$\mathscr{K}$ is stable, since if a model K satisfies (2.8) then every generated submodel K_u of it also satisfies this condition.
As far as the extension property is concerned, consider models $K_1, \ldots, K_n$ in $\mathscr{K}$ and let u be the tight predecessor in M of their roots. $Q = (\sum_{i=1}^{n} K_i)^{V(u)}$ is a well–defined finite rooted submodel of M. Moreover, every successor of u satisfies (2.8). A straightforward induction on the construction of the arbitrary formula φ in S shows that u does the same.

2. Consider a stable class with the extension property $\mathscr{K}$ of finite rooted Kripke models and let $\mathscr{K}^+_{\simeq}$ be the model obtained by the method described in § 2.4. Let $[u_1], \ldots, [u_n]$ be nodes of $\mathscr{K}^+_{\simeq}$; remember that each u_i is equal to $\langle \alpha_i, x_i \rangle$, where x_i is a node of the model K^{α_i} contained in $\mathscr{K}$. The stability of $\mathscr{K}$ implies that each $(K^{\alpha_i})_{x_i}$ is in $\mathscr{K}$, hence there is a variant K of $\sum_{i=1}^{n} (K^{\alpha_i})_{x_i}$ in $\mathscr{K}$, since $\mathscr{K}$ has the extension property. Let r be the isomorphic copy of the root of K in $\mathscr{K}^+$; we will show that [r] is the tight predecessor we are looking for.
First, note that $[r] \leq [u_1], \ldots, [u_n]$, by property (2.5). Now consider a $[v] > [r]$. There exists, by (2.6), a $r' >^+ r$ such that $r' \simeq v$. The fact that r is tight predecessor of $u_1, \ldots, u_n$ in $\mathscr{K}^+$ implies that for some $i : u_i \leq^+ r'$. Therefore $[u_i] \leq [r']$, by (2.5).

□

Definition 2.44. A set X of propositional formulas is *closed under the AR–proof system* if for all formulas φ, ψ

$$\varphi \in X \text{ and } AR \vdash \varphi \rhd \psi \Rightarrow \psi \in X$$

Theorem 2.45. *If S is an* IPC*–saturated set closed under the AR–proof system then $\mathcal{K}_S$ is an AR–model, where $\mathcal{K}$ is the canonical model of* IPC*.*

Proof. The key observation is that since the domain of $\mathcal{K}$ is the set of all IPC–saturated sets, the node tight predecessor coincides with the set tight predecessor in this model. So, let $X_1, \ldots, X_n$ be IPC–saturated supersets of S and define $X = \bigcap_{i=1}^{n} X_i$ and $Y_0 = S \cup I_X$. Observe that $S \subseteq Y_0 \subseteq X$. In order to construct a tight predecessor $Y \supseteq S$ of $X_1, \ldots, X_n$, it suffices by theorem 2.38 to prove that Y_0 is strongly IPC–saturated in X.

So, assume that $Y_0 \vdash \bigvee_{i=1}^{k} \varphi_i$. Therefore there are formulas $E_1 \to F_1, \ldots, E_m \to F_m \in I_X$ such that $S \vdash \bigwedge_{i=1}^{m} (E_i \to F_i) \to \bigvee_{i=1}^{k} \varphi_i$. Define $A = \bigwedge_{i=1}^{m} (E_i \to F_i)$ and observe that

$$\frac{A \to \bigvee_{i=1}^{k} \varphi_i}{\bigvee_{i=1}^{m} (A \to E_i) \vee \bigvee_{i=1}^{k} (A \to \varphi_i)}$$

is an instance of the V_{mk} rule. The fact that S an IPC–saturated set closed under the AR–proof system along with theorem 1.28 imply that there is an $i \leq m$ such that $A \to E_i \in S$ or there is a $j \leq k$ such that $A \to \varphi_j \in S$. Furthermore, X is a closed under deduction in IPC superset of S that contains A, therefore, either there is an $i \leq m$ such that $E_i \in X$ or there is a $j \leq k$ such that $\varphi_j \in X$. But the first is impossible by the definition of E_i, hence there is a $j \leq k$ such that $\varphi_j \in X$. □

This theorem indicates the high correlation between the AR–proof system and the AR–models and it justifies their common name. Moreover, the complexity of the canonical model is reflected upon the AR–models, revealing once more their intricate nature.

2.7 Bounded bisimulations

2.7.1 Restriction into finite language

Definition 2.46. Consider a finite set of propositional variables $\vec{p}$.

- The language $\mathcal{L}(\vec{p})$
  - The alphabet of $\mathcal{L}(\vec{p})$ is the same as that of language $\mathcal{L}$ defined in § 1.1, but restricted to the propositional variables of $\vec{p}$. In other words, $\text{Var}\mathcal{L}(\vec{p}) = \vec{p}$

  - The formulas of $\mathcal{L}(\vec{p})$ are defined analogously and its set is denoted by For$\mathcal{L}(\vec{p})$
  - A formula φ is *over* $\vec{p}$ if $\varphi \in$ For$\mathcal{L}(\vec{p})$

- Kripke models over $\vec{p}$
  - The *restriction* of a model $\langle W, \leq, V\rangle$ *over* $\vec{p}$ is the model $\langle W, \leq, V'\rangle$, where the image of every node $u \in W$ under the assignment $V' : W \to \mathcal{P}(\vec{p})$ is

$$V'(u) = V(u) \cap \vec{p}$$

  - The $\vec{p}$*–theory* of a Kripke model K is the set of formulas over $\vec{p}$ that are valid in K. In other words,

$$\vec{p}\text{–}Th(K) = Th(K) \cap \text{For}\mathcal{L}(\vec{p})$$

  - A Kripke model $\langle W, \leq, V\rangle$ is *over* $\vec{p}$ if the variables forced in its nodes are among those of $\vec{p}$, i.e. if

$$\forall u \in W : V(u) \subseteq \vec{p}$$

  - The class of Kripke models over $\vec{p}$ of a formula φ is denoted by $Mod_{\vec{p}}(\varphi)$ or by mere $Mod(\varphi)$ if it is obvious the set of propositional variables we are referring to

Theorem 2.47. *Let $\vec{p}$ be a finite set of propositional variables.*

1. *Consider a Kripke model K and let K' be its restriction over $\vec{p}$. Then K and K' have the same $\vec{p}$–theory.*
2. *Consider a class $\mathscr{K}$ of Kripke models and let $\mathscr{K}'$ be the class of the restrictions over $\vec{p}$ of models in $\mathscr{K}$. Then,*
   (a) *the two classes have the same $\vec{p}$–theory*
   (b) *if $\mathscr{K}$ has the extension property, then so does $\mathscr{K}'$*
   (c) *if $\mathscr{K}$ is stable, then so is $\mathscr{K}'$*
3. *For all formulas φ, ψ over $\vec{p}$*

$$\varphi \vdash \psi \iff Mod_{\vec{p}}(\varphi) \subseteq Mod_{\vec{p}}(\psi)$$

Proof. The first is shown by induction on the arbitrary formula over $\vec{p}$, the second is trivial and the last is a corollary of the completeness theorem 2.26. □

2.7.2 Bounded bisimulations

In this section we turn our attention to another kind of equivalence, wherein we are no longer interested in models with the same theory, but in models satisfying the same formulas of a certain complexity. Two equivalent definitions of the same notion are presented; the first is in terms of back–and–forth conditions and the second in terms of Ehrenfeucht games.

The back–and–forth method was invented by Roland Fraïssé in order to study elementary equivalence in model theory and it was later formulated as a game by Andrzej Ehrenfeucht. Kit Fine in [Fin74] adapted this technique to the context of Kripke semantics for modal logic and Silvio Ghilardi in [Ghi99] presented an analogue for intuitionistic propositional logic.

Definition 2.48 (back–and–forth conditions)**.**
Consider two finite rooted Kripke models K, K' over a finite set of propositional variables $\vec{p}$. Let r, r' be respectively their roots. The relations $\sim_n$ (n–bisimilarity) and $\leq_n$ (n–subsumption) are defined inductively as follows

$$\begin{aligned}
K \leq_0 K' &\iff V(r) \supseteq V'(r') \\
K \sim_0 K' &\iff K \leq_0 K' \text{ and } K' \leq_0 K \\
K \leq_{n+1} K' &\iff (\forall u \in K)(\exists u' \in K')(K_u \sim_n K'_{u'}) \\
K \sim_{n+1} K' &\iff K \leq_{n+1} K' \text{ and } K' \leq_{n+1} K
\end{aligned}$$

The relation $\sim_\omega$ defined as

$$K \sim_\omega K' \iff \forall n\; K \sim_n K'$$

is a bisimulation.

Definition 2.49 (Ehrenfeucht games)**.**
Let $\vec{p}$ be a finite set of propositional variables. Consider two finite rooted Kripke models K, K' over $\vec{p}$ and fix a number $n \geq 1$[3]. The n–round Ehrenfeucht game on K, K' has two players, usually named Spoiler and Duplicator, and is played as follows:

- At the first round Spoiler selects a node in one model, Duplicator a node in the other.
- At the $(i+1)$–th round, Spoiler selects one of the two models, name it M_1 and let M_2 be the other. Let w_1, w_2 be the nodes chosen from M_1 and M_2 respectively at the previous round. Then, Spoiler picks a node $\geq_1 w_1$ and Duplicator picks a node $\geq_2 w_2$.

In this way sequences $u_1, \ldots, u_n$, $u'_1, \ldots, u'_n$ of nodes of K and K' respectively are created. Duplicator wins if he succeeds in keeping the forcing of propositional variables pairwise identical, i.e. if $V(u_i) = V'(u_i)$, for every $i \leq n$. Otherwise, Spoiler wins.

[3] We avoid defining a 0–round game, since in that game the players could not actually play. Besides, 0–bisimilarity is checked by just looking at the propositional variables forced at the roots.

It is not hard to prove that for $n \geq 1$ the Kripke models K, K' are *n–bisimilar* if and only if Duplicator has a winning strategy in the n–round game on K, K'. Moreover, the relation $K \leq_n K'$ is characterised similarly by a variant of the above game, in which Spoiler is required to select a node from K at the first round.

We proceed to define a measure of complexity for formulas and state the main result about bounded bisimulations.

Definition 2.50. The *complexity* (or the implicational degree) of a formula φ is inductively defined as

- $c(p) = 0$
- $c(\varphi_1 \circ \varphi_2) = max\{c(\varphi_1), c(\varphi_2)\}$, where $\circ \in \{\wedge, \vee\}$
- $c(\varphi_1 \to \varphi_2) = max\{c(\varphi_1), c(\varphi_2)\} + 1$

In other words, $c(\varphi)$ is the maximum number of nested implications in φ.

Theorem 2.51 (Ghilardi)**.** *Let $\vec{p}$ be a finite set of propositional variables. Consider two finite rooted Kripke models K, K' over $\vec{p}$ and fix a number $n \geq 0$.*

1. $K \leq_n K' \iff$ *for every formula φ over $\vec{p}$ with $c(\varphi) \leq n$*
$$K' \models \varphi \Rightarrow K \models \varphi$$
2. $K \sim_n K' \iff$ *for every formula φ over $\vec{p}$ with $c(\varphi) \leq n$*
$$K' \models \varphi \iff K \models \varphi$$

Proof. In [Ghi99]. □

We end this section by grouping some results into a theorem.

Definition 2.52. Let $\vec{p}$ be a finite set of propositional variables and $\mathscr{K}$ a class of finite rooted Kripke models over $\vec{p}$. Fix a number n. $\langle \mathscr{K} \rangle_n$ is the class of the finite rooted Kripke models M over $\vec{p}$ for which there exists a model $K \in \mathscr{K}$ such that $M \leq_n K$. In other words $\langle \mathscr{K} \rangle_n$ is the smallest $\leq_n$–closed class of Kripke models extending $\mathscr{K}$.

Note that for every formula φ over $\vec{p}$ and for every $n \geq c(\varphi)$,

$$\mathscr{K} \models \varphi \Rightarrow \langle \mathscr{K} \rangle_n \models \varphi$$

Theorem 2.53 (Ghilardi)**.** *Let $\vec{p}$ be a finite set of propositional variables and consider a class $\mathscr{K}$ of finite rooted Kripke models over $\vec{p}$.*

1. *$\mathscr{K}$ is downwards closed under $\leq_n$ if and only if $\mathscr{K} = Mod_{\vec{p}}(\varphi)$ for some formula φ over $\vec{p}$ with $c(\varphi) \leq n$.*
2. *If $\mathscr{K}$ is stable and has the extension property then so does $\langle \mathscr{K} \rangle_n$ for every n.*

Proof. In [Ghi99]. □

3 Intuitionistic propositional logic

3.1 Projectivity

3.1.1 Substitutions as mappings

In this section we will elaborate on the substitution construction used in § 2.1.3 on page 17, therefore we repeat the definition 2.9.

Definition 3.1. Given a substitution σ and a Kripke model K, we construct the Kripke model $\sigma^*(K)$ based on the frame of K and with assignment V_σ defined as:

$$p \in V_\sigma(u) \iff K_u \models \sigma(p)$$

for every propositional variable p and every node u of K. The monotonicity condition K satisfies implies that $\sigma^*(K)$ is a well–defined Kripke model.

If we consider a class of Kripke models as a kind of an algebraic space S, then each formula A may be considered as a subspace of S; the subspace containing the models that satisfy A. In this context, a substitution is a mapping between such subspaces.

Theorem 3.2. *Let σ, τ be substitutions, A be a formula and K be a Kripke model. Then,*

1. $(\sigma^*(K))_u = \sigma^*(K_u)$
2. $K \models \sigma(A) \iff \sigma^*(K) \models A$
3. $(\sigma\tau)^*(K) = \tau^*(\sigma^*(K))$
4. $\sigma^*(K) = \tau^*(K) \iff$ *for all variables* $p : K \models \sigma(p) \leftrightarrow \tau(p)$

Proof.

1. $(\sigma^*(K))_u \models p \iff K \models \sigma(p) \iff \sigma^*(K_u) \models p$
2. By a straightforward induction on the construction of the formula A
3. $$\begin{aligned} (\sigma\tau)^*(K) \models p &\iff K \models (\sigma\tau)(p) \\ &\iff K \models \sigma(\tau(p)) \\ &\iff \sigma^*(K) \models \tau(p) \\ &\iff \tau^*(\sigma^*(K)) \models p \end{aligned}$$
4. $$\begin{aligned} \sigma^*(K) = \tau^*(K) &\iff \text{for all } u \in K, \text{ for all variables } p : (\sigma^*(K)_u \models p \Leftrightarrow \tau^*(K)_u \models p) \\ &\iff \text{for all } u \in K, \text{ for all variables } p : (K_u \models \sigma(p) \Leftrightarrow K_u \models \tau(p)) \\ &\iff \text{for all } u \in K, \text{ for all variables } p : K_u \models \sigma(p) \leftrightarrow \tau(p) \\ &\iff \text{for all variables } p : K \models \sigma(p) \leftrightarrow \tau(p) \end{aligned}$$

□

In view of theorem 3.2.1 the term $\sigma^*(K)_u$ is not ambiguous. We will also write $\sigma^*\tau^*(K)$ instead of $\sigma^*(\tau^*(K))$ and abusing the notation we will denote with $V(K)$ the set of propositional variables forced in the root of a rooted Kripke model K.

Definition 3.3. A *unifier* σ of a formula A is a substitution such that $\vdash \sigma(A)$. A formula is *unifiable* if it has at least one unifier.

Theorem 3.4. *For every formula A the following are equivalent:*

1. *A is unifiable*
2. $\nvdash \neg A$
3. *there exists a one–node model of A*
4. $\nvdash_{\mathrm{CPC}} \neg A$

Proof. 1 → 2) Because IPC is consistent and closed under substitutions, as an intermediate logic

2 → 3) Since IPC is complete with respect to finite models, see corollary 2.26, there exists a finite model K that does not satisfy $\neg A$. Therefore any one–node model generated by a leaf of K satisfies A

3 → 4) Trivial

4 → 1) Since $\nvdash_{\mathrm{CPC}} \neg A$, there exists a variable–free substitution σ such that $\vdash_{\mathrm{CPC}} \sigma(A)$. By theorem 1.7, either $\sigma(A)$ or $\neg\sigma(A)$ are derivable in IPC, therefore $\vdash \sigma(A)$. □

3.1.2 Projective substitutions

Definition 3.5. Let A be a formula and σ be a substitution.

1. σ is a *projective substitution for* A if for every variable p

$$A \vdash \sigma(p) \leftrightarrow p$$

2. A is *projective* if there exists a projective unifier of A

Theorem 3.6 (Properties of projective substitutions)**.** *Let σ be a projective substitution for the formula A. Then,*

1. *For every formula B : $A \vdash \sigma(B) \leftrightarrow B$ (a generalisation of the projectivity condition)*
2. $A \vdash \sigma(A)$
3. *If $K \models A$ then $\sigma^*(K) = K$*
4. *The class of projective substitutions for A is closed under composition*

Proof.

1. A straightforward induction on the construction of B, based on the fact that the basis holds by definition and the substitution commutes with the connectives

2. By the previous item

3. If $K \models A$ then for every propositional variable $p: K \models \sigma(p) \leftrightarrow p$, therefore $\sigma^*(K) = K$, by applying theorem 3.2.4 for τ equal to the identity substitution.

4. If τ is also a projective substitution for A then,

$$A \vdash \tau(p) \leftrightarrow p$$

therefore,

$$\sigma(A) \vdash \sigma(\tau(p)) \leftrightarrow \sigma(p)$$

Since $A \vdash \sigma(A)$ by a previous item, we can rewrite this as

$$A \vdash \sigma(\tau(p)) \leftrightarrow \sigma(p)$$

and so since $A \vdash \sigma(p) \leftrightarrow p$

$$A \vdash \sigma(\tau(p)) \leftrightarrow p$$

□

Theorem 3.7. *Let C be a projective formula and let σ be a projective unifier of C. Then C is an axiom for σ, which means that for every formula A,*

$$\vdash \sigma(A) \iff C \vdash A$$

Proof. The left–to–right holds because

$$\vdash \sigma(A) \Rightarrow C \vdash \sigma(A) \Rightarrow C \vdash A$$

and the right–to–left since

$$\left.\begin{array}{r} C \vdash A \Rightarrow \sigma(C) \vdash \sigma(A) \\ \vdash \sigma(C) \end{array}\right\} \Rightarrow \vdash \sigma(A)$$

□

3.1.3 θ–substitutions

Definition 3.8. Let A be a formula and a be a set of propositional variables. Then, the substitution θ^a_A is defined as

$$\theta^a_A(p) = \begin{cases} A \to p, & \text{if } p \in a \\ A \wedge p, & \text{if } p \notin a \end{cases}$$

Theorem 3.9 (Properties of θ–substitutions)**.** *Let A be a formula, a be a set of propositional variables and K be a Kripke model.*

1. *Every θ^a_A–substitution is a projective substitution for A*
2. *$\vdash \theta^a_A(\theta^a_A(p)) \leftrightarrow \theta^a_A(p)$, for every propositional variable p*
3. *$(\theta^a_A)^*(\theta^a_A)^*(K) = (\theta^a_A)^*(K)$*

Proof.

1. If $p \in a$ then
$$A \vdash \theta^a_A(p) \leftrightarrow p \iff A \vdash (A \to p) \leftrightarrow p$$
and if $p \notin a$ then
$$A \vdash \theta^a_A(p) \leftrightarrow p \iff A \vdash (A \wedge p) \leftrightarrow p$$
Both are derivable in IPC.
2. If $p \in a$ then
$$\vdash \theta^a_A(\theta^a_A(p)) \leftrightarrow \theta^a_A(p) \iff \vdash (\theta^a_A(A) \to (A \to p)) \leftrightarrow (A \to p)$$
The left–to–right implication is shown using the deduction theorem and the fact that $A \vdash \theta^a_A(A)$, established in theorem 3.6. The right–to–left is an axiom of IPC.
If $p \notin a$ then
$$\vdash \theta^a_A(\theta^a_A(p)) \leftrightarrow \theta^a_A(p) \iff \vdash \theta^a_A(A) \wedge A \wedge p \leftrightarrow A \wedge p$$
Now, the left–to–right implication is an axiom of IPC and the right–to–left is shown using $A \vdash \theta^a_A(A)$.
3. Corollary of the above item by theorem 3.2.4

□

The θ try to make models satisfy A. Therefore, they leave intact, identical the models that do satisfy A nad

Theorem 3.10. *Let A be a formula, a be a set of propositional variables and K be a rooted Kripke model. Define $M = (\theta^a_A)^*(K)$.*

1. If $K \models A$ then $M = K$

2. If $K \not\models A$ then one of the following holds:

(a) $V(M) = a$

(b) $V(M) \subset a$ and for all propositional variables $p \in a \setminus V(M)$ there is a node u of K different from the root such that $K_u \models A$ and $K_u \not\models p$

Proof.

1. By theorems 3.6.3 and 3.9.1.

2. Let p be a propositional variable in $V(M)$.

$$p \in V(M) \iff (\theta_A^a)^*(K) \models p \iff K \models \theta_A^a(p)$$

Since K does not satisfy A, $\theta_A^a(p)$ cannot be $A \wedge p$, therefore $\theta_A^a(p) \equiv A \to p$ and $p \in a$.

Now, let q be a propositional variable in $a \setminus V(M)$, thus $\theta_A^a(q) \equiv A \to q$ and $M \not\models q$. Therefore, $K \not\models A \to q$, so a node u of K forces A but not q. Moreover, u is different from the root, since K does not satisfy A.

□

Corollary 3.11. *Let A be a formula, a be a set of propositional variables and K be a one-node Kripke model. Then,*

$$K \not\models A \Rightarrow V((\theta_A^a)^*(K)) = a$$

Corollary 3.12. *Let A be a formula and K be a rooted Kripke model which does not satisfy A. If there is a variant K' of K which satisfies A, then $(\theta_A^{V(K')})^*(K) = K'$.*

Proof. Define $M = (\theta_A^{V(K')})^*(K)$. Observe that K, K' and M are all based on the same frame, name it F. For every node u of F different from the root, it holds that $K_u = K'_u \models A$, therefore $M_u = K_u = K'_u$, by theorem 3.6.3. By theorem 3.10, either $V(M) = V(K')$ and so $M = K'$, or there is a propositional variable p that is satisfied by K' and it is refuted by a node u of K different from the root, which is impossible since K and K' are variants. □

3.1.4 Projectivity and the extension property

Definition 3.13. Let A be a formula over a finite set of propositional variables $\vec{p}$. Let $a_1, a_2, \ldots, a_s$ be a linear ordering of the subsets of $\vec{p}$ such that

$$a_i \subseteq a_j \Rightarrow i \leq j \tag{3.1}$$

For each $i \leq s$, we define the substitution $\theta_A \downarrow i = \theta_A^{a_s} \ldots \theta_A^{a_i}$. The substitution θ_A is defined as $\theta_A \downarrow 1$.

Note that by theorem 3.6.4 θ_A is a projective substitution for A as a composition of projective substitutions, and that $\theta_A^* = (\theta_A^{a_1})^* \dots (\theta_A^{a_s})^*$ by theorem 3.2.

Theorem 3.14. *Let A be a formula over a finite set of propositional variables $\vec{p}$. Then, the following are equivalent:*

1. *A is projective*
2. *$Mod_{\vec{p}}(A)$ has the extension property* [4]
3. *θ_A is a unifier for A*

Proof. $1 \to 2$) Assume that A is a projective formula; let σ be its projective unifier. Let $K_1, \dots, K_n$ be finite rooted models over $\vec{p}$ that satisfy A. Then,

$$\vdash \sigma(A) \Rightarrow \sum_{i=1}^{n} K_i \models \sigma(A) \Rightarrow \sigma^*(\sum_{i=1}^{n} K_i) \models A$$

therefore by theorem 3.2.1

$$(\sum_{i=1}^{n} \sigma^*(K_i))^X \models A$$

where $X = \{p \in \vec{p} \mid \sum_{i=1}^{n} K_i \models \sigma(p)\}$. By theorem 3.6.3 and the fact that each K_i satisfies A we have that

$$(\sum_{i=1}^{n} K_i)^X \models A$$

So, $Mod_{\vec{p}}(A)$ has the extension property.

$2 \to 3$) By theorem 3.2 and the fact that IPC is complete with respect to finite rooted models, see corollary 2.26, it suffices to establish that for every finite rooted Kripke model K, $(\theta_A)^*(K)$ is a model of A. We will prove by fan induction on K that for every $u \in K$ there exists an i such that

$$(\theta_A \downarrow i)^*(K_u) \models A$$

and that if $K_u \not\models A$ then i is maximum with that property, i.e for every $j > i$

$$(\theta_A \downarrow j)^*(K_u) \not\models A$$

Note that the first condition is satisfied by every node u of K that forces A, since then for every i, $(\theta_A \downarrow i)^*(K_u) = K_u$.

Let u be a leaf of K that does not force A. Since A is unifiable there exists by theorem 3.4 a one–node model that satisfies A. Therefore, there exists a maximum index i such that the one–node model M defined as

$$M \models p \iff p \in a_i$$

[4] See § 2.7.1 for the definition of $Mod_{\vec{p}}(A)$

satisfies A. By corollary 3.11 and the fact that i is maximum, we obtain that

$$(\theta_A \downarrow i)^*(K_u) = (\theta_A^{a_i})^*(K_u) = M \models A$$

Let u be a non-terminal node of K that does not force A and let $u_1, \ldots, u_n$ be its immediate successors. By the induction hypothesis for each u_i that does not force A there is a j_i such that $M_i = (\theta_A \downarrow j_i)^*(K_{u_i})$ is a model of A. Moreover, the fact that j_i is maximum with that property implies that $V(M_i) \subseteq a_{j_i}$, by theorem 3.10. For each node u_i that forces A we define $M_i = K_{u_i}$, so in any case M_i satisfies A.

Let $j = min\{j_i\}$ and consider $N = (\theta_A \downarrow j)^*(K_u)$ (if for every i, $K_{u_i} \models A$, then take N as K_u). If $N \models A$ then j is a maximum such index since it is equal to a j_i, which is maximum for K_{u_i}. If $N \not\models A$ then for every i such that $K_{u_i} \models A$ we have that

$$N_{u_i} = (\theta_A \downarrow j)^*(K_{u_i}) = K_{u_i} = M_i$$

and for every i such that $K_{u_i} \not\models A$ we have that

$$\begin{aligned} N_{u_i} &= (\theta_A \downarrow j)^*(K_{u_i}) \\ &= (\theta_A^{a_j})^*(\theta_A^{a_{j+1}})^* \ldots (\theta_A \downarrow j_i)^*(K_{u_i}) && \text{[by the definition of } (\theta_A \downarrow j)^*] \\ &= (\theta_A^{a_j})^*(\theta_A^{a_{j+1}})^* \ldots (M_i) && \text{[by the definition of } M_i] \\ &= M_i && \text{[by theorem 3.10.1 since } M_i \models A] \end{aligned}$$

So, N is a model in which A is forced at every node different from the root. The fact that A has the extension property implies that there exists a variant Q of N that satisfies A; let $a_q = V(Q)$. Observe that since for every i, $a_q \subseteq V(M_i)$, then for every u_i that does not force A, $a_q \subseteq a_{j_i}$, thus $a_q \subseteq a_j$ and so $q \leq j$ by equation (3.1).

If there exists an index k such that $q < k < j$ and $(\theta_A \downarrow k)^*(K_u) \models A$, then select the maximum such index.

Otherwise, consider $(\theta_A \downarrow q+1)^*(K_u)$. Obviously, it is variant of Q that does not satisfy A. Therefore by corollary 3.12, $Q = (\theta_A^{a_q})^*(\theta_A \downarrow q+1)^*(K_u) = (\theta_A \downarrow q)^*(K_u)$ and so q is the index we are looking for.

For the root of K there exists an i such that $(\theta_A \downarrow i)^*(K) \models A$, therefore $\theta_A^*(K) \models A$.

$3 \rightarrow 1$) By the definition of θ_A

□

3.1.5 Projectivity and admissibility

Definition 3.15. Let A be a formula.

1. A formula B is contained in S_A if and only if

   (a) B is projective

   (b) $B \vdash A$

(c) $c(B) \leq c(A)$, where 'c' is the measure of complexity defined in § 2.7.2

2. A *projective approximation* Π_A of A is a subset of S_A such that

   (a) for every formula $B \in S_A$, there exists a $C \in \Pi_A$ such that $B \vdash C$

   (b) if both C_1, C_2 are contained in Π_A and $C_1 \vdash C_2$, then $C_1 \equiv C_2$

   Intuitively, Π_A is constructed by ordering the formulas of S_A according to $\vdash$ and then selecting one formula from each maximal class of provably equivalent formulas in S_A. Therefore, Π_A is unique up to provable equivalence.

Theorem 3.16 (Ghilardi)**.**

1. *Every formula A has a finite projective approximation Π_A*

2. *Every unifier of a formula A is also a unifier of a formula in Π_A*

Proof. In [Ghi99] □

Definition 3.17. A formula C is *stable for admissibility in* IPC if for every formula A,

$$C \mathrel{|\!\sim} A \iff C \vdash A$$

Theorem 3.18 (Ghilardi)**.**

1. *Every projective formula C is stable in* IPC [5]

2. *For all formulas A, B,*

$$A \mathrel{|\!\sim} B \iff \textit{for every } C \in \Pi_A : C \vdash B$$

Proof.

1. Assume that $C \mathrel{|\!\sim} A$ and let σ be a projective unifier of C.

$$\vdash \sigma(C) \Rightarrow \vdash \sigma(A) \Rightarrow C \vdash \sigma(A) \Rightarrow C \vdash A$$

   since $C \vdash A \leftrightarrow \sigma(A)$

2. $\Rightarrow$) Assume that $A \mathrel{|\!\sim} B$ and let C be a formula in Π_A, therefore C is projective and derives A.

$$\begin{aligned} C \vdash A &\Rightarrow C \mathrel{|\!\sim} A \\ &\Rightarrow C \mathrel{|\!\sim} B \quad \text{[by the transitivity of } \mathrel{|\!\sim}\text{]} \\ &\Rightarrow C \vdash B \quad \text{[because } C \text{ is stable, as projective]} \end{aligned}$$

   $\Leftarrow$) Let σ be a unifier of A. By theorem 3.16.2, σ is also a unifier of a formula $C \in \Pi_A$. By assumption $C \vdash B$, therefore $\vdash \sigma(B)$.

□

[5] In fact, every projective formula is stable in every intermediate logic, see § 4.2.1

3.1.6 Projectivity and the slash

Theorem 3.19. *Let L be an intermediate logic and A a projective formula. Then for all formulas B, C*

$$\vdash_L A \to B \vee C \Rightarrow \vdash_L (A \to B) \vee (A \to C)$$

Proof. Let σ be a projective unifier of A. So,

$$\begin{aligned} \vdash_L A \to B \vee C &\Rightarrow \vdash_L \sigma(A) \to \sigma(B) \vee \sigma(C) \\ &\Rightarrow \vdash_L \sigma(B) \vee \sigma(C) \qquad [\text{since} \vdash \sigma(A)] \end{aligned}$$

Moreover,

$$A \vdash \sigma(B) \leftrightarrow B \Rightarrow \vdash \sigma(B) \to (A \to B) \Rightarrow \vdash_L \sigma(B) \to (A \to B)$$

Similarly, we obtain that $\vdash_L \sigma(C) \to (A \to C)$. Therefore, $\vdash_L (A \to B) \vee (A \to C)$. ☐

Corollary 3.20. *If A is a projective formula, then for all formulas B, C*

$$\vdash A \to B \vee C \Rightarrow \vdash A \to B \text{ or } \vdash A \to C$$

Definition 3.21. A set of formulas Γ is *e–compact* if whenever $\Gamma \vdash A$ then there exists a projective formula E such that $\Gamma \vdash E$ and $E \vdash A$.[6]

Theorem 3.22. *If a set of formulas Γ is e–compact then for every formula A,*

$$\Gamma \,|\, A \iff \Gamma \vdash A$$

Proof. The left–to–right holds by theorem 1.10.1. The other direction is proved by induction on the construction of formula A. The cases of the propositional variable and the conjunction are straightforward.

For disjunction, assume that $\Gamma \vdash B \vee C$. Therefore, there exists a projective formula E such that $\Gamma \vdash E$ and $E \vdash B \vee C$. By corollary 3.20 $E \vdash B$ or $E \vdash C$, so $\Gamma \vdash B$ or $\Gamma \vdash C$. By the induction hypothesis $\Gamma \,|\, B$ or $\Gamma \,|\, C$, hence $\Gamma \,|\, B \vee C$.

For implication, assume that $\Gamma \vdash B \to C$ and $\Gamma \,|\, B$. Therefore $\Gamma \vdash B$, by theorem 1.10.1, so $\Gamma \vdash C$, by modus ponens. Hence $\Gamma \,|\, C$ by the induction hypothesis and so $\Gamma \,|\, B \to C$. ☐

Corollary 3.23. *Every projective formula A satisfies $A \,|\, A$.*

[6] The original definition, as produced by Visser in [Vis99], asserts that the formula E has the extension property rather than it is projective. In view of theorem 3.14, the two notions coincide. In the same article Visser proved that a set of formulas is e–compact if and only if it has the extension property.

3.2 The admissible rules of intuitionistic propositional logic

Building on Ghilardi's results presented in [Ghi99], Iemhoff proved in [Iem01b] that the Visser rules form a basis for admissibility in intuitionistic propositional logic, confirming thus a conjecture by de Jongh and Visser.

Theorem 3.24 (Iemhoff). *For all formulas φ, ψ the following are equivalent:*

1. $\varphi \mathrel{|\!\sim} \psi$
2. *ψ is satisfied in every stable class of finite rooted Kripke models with the extension property in which φ is satisfied*
3. *ψ is satisfied in every AR–model in which φ is satisfied*
4. $AR \vdash \varphi \rhd \psi$
5. $\varphi \overset{V}{\vdash} \psi$

Proof.

1 → 2) Assume that $\varphi \mathrel{|\!\sim} \psi$ and let $\mathscr{K}$ be a stable class with the extension property of finite rooted Kripke models in which φ is satisfied. Let $\vec{p}$ be the set of propositional variables of φ and ψ and let $\mathscr{K}'$ be the class containing the models of $\mathscr{K}$ restricted over $\vec{p}$. By theorem 2.47, $\mathscr{K}'$ is also a stable class with the extension property of finite rooted Kripke models. Consider $\langle \mathscr{K}' \rangle_{c(\varphi)}$. Since it has the extension property by theorem 2.53.2, there exists by theorem 2.53.1 a formula θ over $\vec{p}$ such that $\langle \mathscr{K}' \rangle_{c(\varphi)} = Mod_{\vec{p}}(\theta)$. Furthermore, theorem 3.14 implies that θ is projective; let σ be its projective unifier. Then,

$$\begin{aligned}
\mathscr{K} \models \varphi \;&\Rightarrow\; \mathscr{K}' \models \varphi && \text{[by theorem 2.47]}\\
&\Rightarrow\; \langle \mathscr{K}' \rangle_{c(\varphi)} \models \varphi && \text{[by the remark after definition 2.52]}\\
&\Rightarrow\; Mod_{\vec{p}}(\theta) \models \varphi && \text{[since } \langle \mathscr{K}' \rangle_{c(\varphi)} = Mod_{\vec{p}}(\theta)\text{]}\\
&\Rightarrow\; \theta \vdash \varphi && \text{[by theorem 2.47]}\\
&\Rightarrow\; \theta \mathrel{|\!\sim} \psi && \text{[since by assumption } \varphi \mathrel{|\!\sim} \psi\text{]}\\
&\Rightarrow\; \theta \vdash \psi && \text{[since } \theta \text{ is stable in IPC as projective, see theorem 3.18.1]}\\
&\Rightarrow\; Mod_{\vec{p}}(\theta) \models \psi && \\
&\Rightarrow\; \langle \mathscr{K}' \rangle_{c(\varphi)} \models \psi && \text{[since } \langle \mathscr{K}' \rangle_{c(\varphi)} = Mod_{\vec{p}}(\theta)\text{]}\\
&\Rightarrow\; \mathscr{K}' \models \psi && \text{[since } \mathscr{K}' \subseteq \langle \mathscr{K}' \rangle_{c(\varphi)}\text{]}\\
&\Rightarrow\; \mathscr{K} \models \psi && \text{[by theorem 2.47]}
\end{aligned}$$

2 → 3) Apply theorem 2.43 to the set S that contains the subformulas of φ and ψ.

3 → 4) In fact we will show the contrapositive, so assume that $AR \nvdash \varphi \rhd \psi$. In order to obtain an AR–model which satisfies φ and not ψ, it suffices by theorem 2.45 to construct a closed under the AR–proof system, IPC–saturated set X that contains φ and not ψ.

Let $\xi_0, \xi_1, \ldots$ be an enumeration of all formulas in which each formula appears infinitely often. We define inductively a sequence $X_0 \subseteq X_1 \subseteq \ldots$ of finite sets of formulas satisfying the invariant property $AR \nvdash \bigwedge X_i \rhd \psi$.

$$X_0 = \{\varphi\} \qquad \xi_i \in X_{i+1} \iff AR \nvdash \bigwedge X_i \wedge \xi_i \rhd \psi$$

Define $X = \bigcup_i X_i$.

X is closed under the AR–proof system Consider formulas η, θ such that $\eta \in X$ and $AR \vdash \eta \rhd \theta$; hence there is an i such that $\eta \in X_i$. Select an index $j \geq i$ such that $\xi_j \equiv \theta$. Towards a contradiction, assume that $AR \vdash \bigwedge X_j \wedge \theta \rhd \psi$. Then, applying the properties of theorem 1.31, we obtain that

$$\dfrac{\dfrac{\bigwedge X_j \rhd \bigwedge X_j \qquad \dfrac{\begin{matrix}\vdots \\ \bigwedge X_j \rhd \eta\end{matrix} \qquad \eta \rhd \theta}{\bigwedge X_j \rhd \theta}\,\text{Cut}}{\bigwedge X_j \rhd \bigwedge X_j \wedge \theta}\,\text{Conj} \qquad \begin{matrix}\vdots \\ \bigwedge X_j \wedge \theta \rhd \psi\end{matrix}}{\bigwedge X_j \rhd \psi}\,\text{Cut}$$

which is a contradiction by the invariant property X_j satisfies. Therefore θ is in X_{j+1}, thus in X.

X is IPC**–saturated** Consider formulas η, θ such that $X \vdash \eta \vee \theta$; hence there is an i such that $X_i \vdash \eta \vee \theta$. Select indices j, k such that $k \geq j \geq i$, $\xi_j \equiv \eta$ and $\xi_k \equiv \theta$. Towards a contradiction, assume that $AR \vdash \bigwedge X_j \wedge \eta \rhd \psi$ and $AR \vdash \bigwedge X_k \wedge \theta \rhd \psi$. Then, applying the properties of theorem 1.31, we obtain that

$$\dfrac{\bigwedge X_k \rhd \bigwedge X_k \wedge (\eta \vee \theta) \qquad \dfrac{\dfrac{\begin{matrix}\vdots \\ \bigwedge X_j \wedge \eta \rhd \psi\end{matrix}}{\bigwedge X_k \wedge \eta \rhd \psi} \qquad \begin{matrix}\vdots \\ \bigwedge X_k \wedge \theta \rhd \psi\end{matrix}}{\bigwedge X_k \wedge (\eta \vee \theta) \rhd \psi}}{\bigwedge X_k \rhd \psi}\,\text{Cut}$$

which is a contradiction by the invariant property every X_i satisfies. Therefore $\eta \in X_{j+1}$ or $\theta \in X_{k+1}$, thus η or θ is in X.

$4 \to 5$) Apply theorem 1.30 for $L =$ IPC and $R = V$.

$5 \to 1$) By theorem 1.26. □

The importance of this theorem, apart from the apparent fact that it provides a basis for admissibility, lies in the equivalences established during its proof. First, we get a proof system for admissibility in IPC, namely AR. Second, admissibility is semantically characterised in two ways, both related with the extension property. Moreover, a connection with parts of the canonical model is established. And finally, it leads us to the next result.

Corollary 3.25. *The class of AR–models is not stable.*

Proof. Assume that for every IPC–saturated set X, $\mathcal{K}_X$ is an AR–model. Then for all formulas φ, ψ

$$\begin{aligned} \varphi \mid\!\sim \psi \;&\Rightarrow\; \text{for every } AR \text{ model } K,\, K \models \varphi \Rightarrow K \models \psi && \text{[by theorem 3.24]} \\ &\Rightarrow\; \text{for every IPC–saturated set } X,\, \mathcal{K}_X \models \varphi \Rightarrow \mathcal{K}_X \models \psi && \text{[by assumption]} \\ &\Rightarrow\; \text{for every IPC–saturated set } X,\, \varphi \in X \Rightarrow \psi \in X && \text{[by theorem 2.22]} \\ &\Rightarrow\; \varphi \vdash \psi && \text{[by theorem 1.34]} \end{aligned}$$

which is obviously a contradiction. Therefore, there exists a node X of the canonical model such that $\mathcal{K}_X$ is not an AR–model. However, $\mathcal{K}$ itself is an AR–model, by lemma 3.29 on page 51. □

3.3 A characterisation of intuitionistic propositional logic

In 1932 Gödel showed that the intuitionistic propositional logic has the disjunction property. Łukasiewicz conjectured in 1952 that no proper consistent extension of IPC has the disjunction property, therefore this property characterises IPC among the intermediate logics. In 1957 Kreisel and Putnam disproved this conjecture by proving a specific counterexample, now referred as the Kreisel–Putnam logic, see § 4.1. In 1962 Kleene conjectured that IPC is characterised in terms of his 'slash', a relation with apparent disjunction features, and de Jongh in 1968 confirmed this conjecture , see § 1.4. In this section we present the characterisation discovered by Iemhoff, see [Iem01a]. In fact, it is a double characterisation. The first is in terms of the disjunction property plus the admissibility of the Visser rules, thus filling in the missing part of Łukasiewicz's conjecture. The second is in terms of the extension property, a semantically defined property extending the disjunction property.

3.3.1 Basic models

Theorem 3.26 (Smorynski)**.** *Let K be a finite tree model satisfying the property that each terminal node t is characterised by a formula θ_t, in the sense that for every terminal node $z \in K$*

$$K_z \models \theta_t \iff z = t$$

Then,

1. [Node Characterisation Formulas] *For every node u of K there is a formula ψ_u such that for every $v \in K$*

$$v \geq u \iff K_v \models \psi_u$$

2. [Generated Set Characterisation Formulas] *For every generated set S of nodes of K there is a formula β_S such that $S = \{u \in K \mid K_u \models \beta_S\}$*

3. [Substitution Property] *For every model M based on the frame of K there is a substitution σ such that for every formula φ and every node u of their joint frame*

$$M_u \models \varphi \iff K_u \models \sigma(\varphi)$$

Proof.

1. Let K be a model satisfying the hypothesis. For every node u of K define T_u to be the set of terminal nodes above u and $\psi_u = \neg\neg \bigvee_{t \in T_u} \theta_t$. Then,

$$\begin{aligned} K_v \models \psi_u &\iff K_v \models \neg\neg \bigvee_{t \in T_u} \theta_t \\ &\iff \bigvee_{t \in T_u} \theta_t \text{ is true at every leaf above } v \quad \text{[because } K \text{ is finite]} \\ &\iff T_u \supseteq T_v \\ &\iff u \leq v \quad \text{[because } K \text{ is a tree–model]} \end{aligned}$$

2. Define $\beta_S = \bigvee_{u \in S} \psi_u$

3. For every propositional variable p we define $\sigma(p)$ as β_{S_p}, where S_p is the generated set of nodes of K at which p is true and β_{S_p} is its characterising formula defined in the previous item. The property is proved by induction on the construction of the formula φ. For the basis consider a propositional variable p.

   $\Rightarrow$) Let u be a node of M at which p is true. Then ψ_u is one of the disjuncts of $\sigma(p)$. Observe that $K_u \models \psi_u$, therefore $K_u \models \sigma(p)$.

   $\Leftarrow$) Let u be a node of K at which $\sigma(p)$ is true, hence a disjunct of $\sigma(p)$ is true at u. So, there is a node $v \leq u$ in M at which p is true, therefore p is true at u by the monotonicity condition.

   The proofs of the other cases are straightforward and are based on the fact that σ commutes with the connectives.

□

Definition 3.27. A model K based on a finite tree is *basic* if

- A single propositional variable is true at the each leaf and is not true at any other leaf
- the non-terminal nodes do not force propositional variables

Note that every basic model satisfies the hypothesis of theorem 3.26, since each terminal node is characterised by the propositional variable it forces. The basic models on full, non–linear trees are singled out because of their following property.

Theorem 3.28 (Unique Extension)**.** *Let $n \geq 2$ and let $K_1, \ldots, K_n$ be basic models on a full n-ary tree that are disjoint, in the sense that the sets of propositional variables forced in their leaves are disjoint. Then $(\sum_{i=1}^{n} K_i)^{\emptyset}$ is also a basic model on a full n-ary tree and it is the only well-defined variant of $\sum_{i=1}^{n} K_i$.*

Proof. Obvious □

3.3.2 The characterisation

Lemma 3.29 (Iemhoff)**.** *Let L be an intermediate logic with the disjunction property in which the Visser rules are admissible. Then for all $n \in \omega$ and all L–saturated sets $X_1, \ldots, X_n$ there exists a tight predecessor.*

Proof. Define $X = \bigcap_{i=1}^{n} X_i$ and $Y_0 = I_X$. Based on corollary 2.39, we will equivalently establish that Y_0 is strongly L–saturated in X. So, assume that $Y_0 \vdash_L \bigvee_{i=1}^{m} B_i$, therefore there are $E_1 \to F_1, \ldots, E_k \to F_k \in I_X$ such that $A \equiv \bigwedge_{i=1}^{k} (E_i \to F_i) \vdash_L \bigvee_{i=1}^{m} B_i$. Theorem 1.28 verifies the admissibility of the generalised Visser rules in L, which in turn along with the disjunction property imply that $A \vdash_L E_i$ for some $i \leq k$ or that $A \vdash_L B_i$ for some $i \leq m$. In either case $E_i \in X$ or $B_i \in X$, since for every formula φ

$$A \vdash_L \varphi \Rightarrow I_X \vdash_L \varphi \Rightarrow \varphi \in X$$

But by their definition, no E_i is included in X, therefore there exists an $i \leq m$ such that B_i is contained in X. □

An immediate corollary of this lemma, already discussed in 3.2, is that the canonical model of IPC is an AR–model.

Lemma 3.30 (Iemhoff)**.** *If the Visser rules are admissible in an intermediate logic L with the disjunction property then L has the extension property.*

Proof. Let $K_1, \ldots, K_n$ be rooted models of an intermediate logic L satisfying the hypothesis. By lemma 3.29 there exists a tight predecessor of $Th(K_1), \ldots, Th(K_n)$ and so by theorem 2.33 we can define a variant of $\sum_{i=1}^{n} K_i$ which is a model of L. □

Lemma 3.31 (Iemhoff)**.** *Every basic model on a full, non–linear tree is a model of every intermediate logic with the extension property.*

Proof. Let L be an intermediate logic with the extension property and consider a basic model K on a full, non–linear tree. The proof proceeds by fan induction. The leaves generate classical models, thus models of L. Consider a non–terminal node u and let $u_1, \ldots, u_n$, be its immediate successors. By the induction hypothesis the submodels $K_{u_1}, \ldots, K_{u_n}$ are models of L, therefore K_u is a model of L, as it is the only well-defined variant of $\sum_{i=1}^{n} K_{u_i}$ by theorem 3.28 and since L has the extension property up to n. □

Lemma 3.32 (Iemhoff). *The only intermediate logic satisfying the property that every basic model on a full, non–linear tree is its model, is* IPC.

Proof. Let L be an intermediate logic satisfying the hypothesis. We will show the contrapositive, so assume that there exists a formula φ such that $\nvdash \varphi$. By corollary 2.30, there exists a countermodel K of φ based on a full, non–linear tree. Consider any basic model M based on that frame. By assumption M is a model of L. Furthermore, there exists by theorem 3.26, a substitution σ such that for every formula ψ

$$K \models \psi \iff M \models \sigma(\psi)$$

Therefore,

$$K \not\models \varphi \Rightarrow M \not\models \sigma(\varphi) \Rightarrow \nvdash_L \sigma(\varphi) \Rightarrow \nvdash_L \varphi$$

□

Theorem 3.33 (Iemhoff). *For any intermediate logic L the following are equivalent:*

1. *L has the disjunction property and the Visser rules are admissible in it*
2. *L has the extension property*
3. *$L =$ IPC*

Proof. Immediate by 1.12, 1.26 3.30, 3.31 and 3.32. □

An important corollary of this theorem is 4.1.

3.3.3 Extension property, disjunction property and the Visser rules

In this section we refine the results led to the characterisation of IPC in order to tightly interconnect extension property, disjunction property and the admissibility of Visser rules.

Lemma 3.34 (Gabbay and de Jongh). *Every intermediate logic with the extension property up to 2 has the disjunction property.*

Proof. Consider an intermediate logic L with the extension property up to 2 and towards a contradiction, assume that there are formulas B, C such that $\vdash_L B \vee C$ and $\nvdash_L B$ and $\nvdash_L C$. Let K_B, K_C be respectively their countermodels. By assumption a variant K of $K_B + K_C$ is a model of L. However, K does not satisfy $B \vee C$, a contradiction. □

Lemma 3.35 (Iemhoff). *For every $n \geq 2$, the V_n rule is admissible in every intermediate logic with the extension property up to n.*

Proof. Consider an intermediate logic L with the extension property up to n, where $n \geq 2$. By lemma 3.34 L has the disjunction property, so it suffices to show that the restricted V_n rule is admissible in L. So, let $\vdash_L A \rightarrow B \vee C$, where $A \equiv \bigwedge_{i=1}^{n} (E_i \rightarrow F_i)$ and towards a contradiction assume that $\nvdash_L \bigvee_{i=1}^{n} (A \rightarrow E_i) \vee (A \rightarrow B) \vee (A \rightarrow C)$, hence $\nvdash_L A \rightarrow E_i$ for every $i \leq n$, $\nvdash_L A \rightarrow B$ and $\nvdash_L A \rightarrow C$. Therefore, there exist models $K_1, \ldots, K_n, K_B, K_C$ of L that satisfy A and are such that $K_i \not\models E_i$ for every $i \leq n$, $K_B \not\models B$ and $K_C \not\models C$. By assumption a variant K' of $\sum_{i=1}^{n} K_i$ is a model of L. Observe that A is valid in all the successors of the root of K' and that if there were an $i \leq n$ such that $K \models E_i$, then $K_i \models E_i$, a contradiction. Therefore $K' \models A$. Applying twice the same syllogism we obtain a variant K'' of $(K' + K_B) + K_C$ which is a model of L and satisfies A. However, K'' obviously does not satisfy $B \vee C$, a contradiction. □

Lemma 3.36 (Iemhoff). *Let L be an intermediate logic with the disjunction property in which V_n is admissible. Then for all L–saturated sets $X_1, \ldots, X_n$ there exists a tight predecessor.*

Proof. The proof is similar to that of lemma 3.29. The sole difference lies in the fact that although $Y_0 \vdash_L \bigvee_{i=1}^{m} B_i$ implies that $\bigwedge_{i=1}^{k} (E_i \rightarrow F_i) \vdash_L \bigvee_{i=1}^{m} B_i$, we cannot use directly the hypothesis that V_n is admissible in L, since k could in general be greater than n. What we need is to define (using E_i and F_i) a formula A with the following properties:

- A is a conjunction of no more than n implications, the negative part of which should not belong in X
- $I_X \vdash_L A$
- $\vdash_L A \rightarrow \bigvee_{i=1}^{m} B_i$

To avoid conflict, for the rest of this proof indices i and j will range over $\{1, \ldots, k\}$ and $\{1, \ldots, n\}$ respectively.

For each j define $G_j = \bigvee_{E_i \notin X_j} E_i$. Note that if $\{E_i \mid E_i \notin X_j\} = \emptyset$ then $G_j = \bot$, by convention. Each G_j is constructed so that it is not in X_j; otherwise the L–saturation of X_j would imply the existence of a disjunct of G_j in X_j contrary to the definition, hence $G_j \notin X$. Let $F \equiv \bigwedge F_i$, thus $F \in X$ and $G_j \rightarrow F \in I_X$. Let $A \equiv \bigwedge (G_j \rightarrow F)$, therefore $I_X \vdash_L A$. For each i there is a j such that $E_i \notin X_j$, since $E_i \notin X$, hence E_i is a disjunct of G_j, thus $E_i \vdash_L G_j$. The following formal proof shows that $\vdash_L A \rightarrow \bigvee_{i=1}^{m} B_i$.

$$\dfrac{\dfrac{\bigwedge(E_i \to F_i) \vdash_L \bigvee_{i=1}^{m} B_i \qquad \dfrac{\dfrac{\dfrac{\vdash_L F \to F_i \qquad \dfrac{\dfrac{[A]}{G_j \to F} \qquad \dfrac{\vdash_L E_i \to G_j \qquad [E_i]}{G_j}}{F}}{F_i}}{E_i \to F_i}}{\bigwedge(E_i \to F_i)}\text{, since this holds for every } i}{\bigvee_{i=1}^{m} B_i}}{A \to \bigvee_{i=1}^{m} B_i}$$

Having constructed a formula A with the desired properties, the rest of the proof proceeds as that of lemma 3.29, substituting only n for k and G_i for E_i. □

Lemma 3.37 (Iemhoff)**.** *For every $n \geq 2$, if V_n is admissible in an intermediate logic L with the disjunction property then L has the extension property up to n.*

Proof. Similar to the proof of lemma 3.30, but now based on lemma 3.36. □

Theorem 3.38 (Iemhoff)**.** *For every $n \geq 2$, an intermediate logic L has the extension property up to n if and only if L has the disjunction property and V_n is admissible in it.*

Proof. By 3.34, 3.35 and 3.37. □

3.4 The T_n logics

D. M. Gabbay and D. H. J. de Jongh introduced in [GdJ74] "a sequence of decidable finitely axiomatisable intermediate logics with the disjunction property". Each T_n–logic[7] is defined as the logic of n–ary trees, e.g. T_1 is the logic of linear frames, T_2 is the logic of binary trees. The main properties of these logics, already stated in the title of the article, are grouped in the following theorem.

Theorem 3.39 (Gabbay and de Jongh)**.**

1. $\mathrm{CPC} = T_0 \supset \cdots \supset T_n \supset T_{n+1} \supset \cdots \supset \bigcap_{n \in \omega} T_n = \mathrm{IPC}$
2. *If $n \geq 2$ then T_n has the disjunction property*
3. *T_n is decidable*
4. *T_n is axiomatised over* IPC *by*

$$t_n = \bigwedge_{i=0}^{n} \left((A_i \to \bigvee_{i=1}^{j \neq i} A_j) \to \bigvee_{i=1}^{j \neq i} A_j \right) \to \bigvee_{i=0}^{n} A_i$$

[7]Originally the logics were denoted by D_n and in fact $D_n = T_{n+1}$. We use the name established in [CZ97]

Proof. We will include proofs only for the properties that are relevant to our subject.

1. The first equation holds since a nullary tree consists only of its root and the second because IPC is complete with respect to finite trees by corollary 2.26. For the strict inclusion, we first show that $t_n \in T_n$ and then we provide an $n+1$–tree countermodel to t_n.

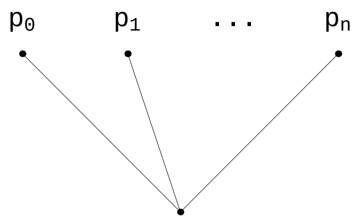


Figure 5: An $n+1$–ary tree countermodel to the formula t_n

2. Towards a contradiction, assume that there are formulas B, C such that $\vdash_{T_n} B \vee C$ and $\nvdash_{T_n} B$ and $\nvdash_{T_n} C$. Let K_B, K_C be respectively their n–ary tree countermodels. Since $n \geq 2$, $(K_B + K_C)^{\emptyset}$ is an n–ary tree model, thus a model of T_n, which however does not satisfy $B \vee C$, a contradiction.

□

Theorem 3.40. *Let $n \geq 2$. Then,*

1. *the V_n rule is admissible in T_n, therefore T_n has the extension property up to n*
2. *the V_{n+1} rule is not admissible in T_n, therefore T_n does not have the extension property up to $n+1$*

Proof.

1. For $n \geq 2$, the logic T_n has the disjunction property, so it suffices to show that the restricted V_n rule is admissible in L. So, let $A \vdash_{T_n} B \vee C$, where $A \equiv \bigwedge_{i=1}^{n} (E_i \to F_i)$ and, towards a contradiction, assume that $\nvdash_{T_n} \bigvee_{i=1}^{n} (A \to E_i) \vee (A \to B) \vee (A \to C)$, hence $A \nvdash_{T_n} E_i$ for every $i \leq n$, $A \nvdash_{T_n} B$ and $A \nvdash_{T_n} C$. Therefore, there exist models $K_1, \ldots, K_n, K_B, K_C$ based on an n–ary tree that satisfy A and are such that $K_i \not\models E_i$ for every $i \leq n$, $K_B \not\models B$ and $K_C \not\models C$. Note that $K' = (\sum_{i=1}^{n} K_i)^{\emptyset}$ is a model based on an n–ary tree, thus a model of ${T_n}$[8]. Observe that A is valid in all the successors of the root of K' and that if there were an $i \leq n$ such that $K \models E_i$, then $K_i \models E_i$, a

[8]In fact every well–defined variant of $\sum_{i=1}^{n} K_i$ would be suitable. The same comment holds for the definition of K''

contradiction. Therefore $K' \models A$. Applying twice the same syllogism we obtain that $K'' = ((K' + K_B)^{\emptyset} + K_C)^{\emptyset}$ is a model of T_n and satisfies A. However, K'' obviously does not satisfy $B \vee C$, a contradiction.

2. By the fact that T_n has the disjunction property and theorem 3.38, it suffices to show that T_n does not have the extension property up to $n+1$. So, let K be the model in figure 5 and let $K_1, \ldots, K_{n+1}$ be the models generated by the leaves of K. Each K_i is a model of T_n, as a classical model and K is the only well–defined variant of $\sum_{i=1}^{n+1} K_i$. However, by its construction K is not a model of T_n.

□

Corollary 3.41. *For every $n \geq 2$, there is an intermediate logic in which V_n is admissible, but V_{n+1} is not.*

Although we are not aware of a basis for admissibility in T_n, the previous theorem indicates that it contains only the V_n rule. In § 4.2.1 we deploy a method for proving that the whole collection of the Visser rules forms a basis for admissibility in an intermediate logic. The example of the connection between the extension property and the admissibility of the Visser rules suggests that an analogous refinement of this method may offer a solution.

Observe that the seemingly obvious fact that T_n has the extension property up to n is not proved straightforwardly, but through the admissibility of the V_n rule, thus using the tight predecessor machinery. However, the similarity of the proofs that T_n has the disjunction property and that V_n is admissible in it with the proofs of lemmas 3.34 and 3.35 suggests that a simpler proof is likely to exist.

Using the following corollary of the extension theorem 2.28, we prove an analogous to 3.33 theorem for the T_n logics.

Corollary 3.42. *T_n is sound and complete with respect to full n-ary trees.*

Lemma 3.43. *For $n \geq 2$, every basic model on a full n-ary tree is a model of every intermediate logic with the extension property up to n.*

Proof. Similar to the proof of lemma 3.31, but now n is specified in the statement of the lemma. □

Lemma 3.44. *For $n \geq 2$, if every basic model on a full n-ary tree is a model of the intermediate logic L, then $L \subseteq T_n$.*

Proof. Similar to the proof of lemma 3.32, now using corollary 3.42 for the completeness of T_n. □

Theorem 3.45. *For $n \geq 2$, if an intermediate logic has the extension property up to n, or equivalently if it has the disjunction property and V_n is admissible in it, then it is a sublogic of T_n.*

Proof. Immediate by 3.38, 3.43, 3.44 □

4 The admissible rules of intermediate logics

The role of intuitionistic propositional logic in the lattice of intermediate logics is dominant. Therefore, in order to study the general problem of admissibility in the intermediate logics, it is logical to start by generalising the results and the methods deployed for the case of IPC. Such an effort led to results about the T_n logics, see § 3.3.3 and 3.4. Here we present Iemhoff's work, presented in [Iem05] and [Iem], on the admissibility of the Visser rules in the intermediate logics.

4.1 A list of intermediate logics

We start by listing some well–known intermediate logics.

KC The logic of the weak law of the excluded middle is one of the most extensively studied intermediate logics. In the literature it is also known as Jankov logic (Jn) or de Morgan logic (Dm). It is axiomatised by any of the following schemas

$\neg p \vee \neg\neg p$	The weak law of the excluded middle
$\neg(p \wedge q) \to \neg p \vee \neg q$	The de Morgan law not valid in IPC
$(\neg\neg p \to p) \to p \vee \neg p$	The 8–th Nishimura formula

and it is sound and complete with respect to frames with one maximal node.

$\mathbf{G_n}$ Gödel introduced these logics in order to show that IPC is infinite–valued. Each G_n logic is finite–valued, is axiomatised by

$$\bigvee_{i=1}^{n}\Big(\bigwedge_{j=1}^{i-1} p_j \to p_i\Big) \quad \text{and} \quad (p \to q) \vee (q \to p)$$

and is sound and complete with respect to the linear frame of $(n-1)$ nodes. They are still studied intensely, primarily due to their connection with linear Kripke frames, but also in terms of applications to fuzzy logic and computer science.

LC The "limit" of the Gödel logics is the infinite–valued Gödel–Dummett logic. It is axiomatised by $(p \to q) \vee (q \to p)$ and it is sound and complete with respect to the linear frames.

Sm The Smetanich logic is the greatest intermediate logic properly contained in classical logic. It is also known as the 3–valued logic, G_3 and the logic of "here and there". The latter originates in the area of logic programming where it was recently applied, see [LPV01]. It is axiomatised by

$$(p \to q) \vee (q \to p) \quad \text{and} \quad p \vee (p \to q \vee \neg q)$$

or equivalently by

$$p \vee (p \to q) \vee \neg q$$

and it is sound and complete with respect to the 2–node frame.

KP The logic axiomatised by the corresponding to the Kreisel–Putnam rule scheme

$$(\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$$

It was the first proper extension of IPC known to have the disjunction property, thus disproving the Łukasiewicz's conjecture that IPC is maximal with respect to that property.

$\mathbf{T_n}$ The T_n logics were introduced by D. M. Gabbay and D. H. J. de Jongh in [GdJ74] as the logic of n–ary trees. [9] Their basic properties along with Iemhoff's results concerning their admissibility are presented in § 3.4.

ML It was introduced by Medvedev in order to formalise the idea of Kolmogorov that propositional formulas should be considered as abstract problems and propositional connectives as operations between them. Apart from its initial definition in terms of finite problems, other semantic characterisations have been discovered, including realizability and Kripke frames. However, the problem of its axiomatisation is still open, although we know that it is not finitely axiomatisable. It has also been proved that it has the disjunction property and in fact, it is maximal with respect to it.

$\mathbf{Bd_n}$ The logics of bounded depth – that is what 'Bd' stands for. Each $\mathrm{Bd_n}$ logic is sound and complete with respect to the frames of depth n and it is axiomatised by the bd_n scheme, defined inductively as:

$$\begin{aligned} bd_1 &= p_1 \vee \neg p_1 \\ bd_{n+1} &= p_{n+1} \vee (p_{n+1} \to bd_n) \end{aligned}$$

$\mathbf{Nd_n}$ The logics of the frames with n nodes, are also referred to as $\mathrm{Bc_n}$, where 'Bc' stands for bounded cardinality. For $n \geq 2$ each one is axiomatised by $(\neg p \to \bigvee_{i=1}^{n} \neg q_i) \to \bigvee_{i=1}^{n} (\neg p \to \neg q_i)$ ($\mathrm{Nd_1}$ is classical logic).

$\mathbf{M_n}$ The logics of the frames with n maximal nodes, are also referred to as $\mathrm{Btw_n}$, where 'Btw' stands for bounded terminal width. They are axiomatised by the following scheme

$$\bigwedge_{0 \leq i < j \leq n} \neg(\neg p_i \wedge \neg p_j) \to \bigvee_{i=0}^{n} (\neg p_i \to \bigvee_{j \neq i} p_j)$$

[9] In fact, they used the term D_n to refer to T_{n+1}. We use the current notation, established in [CZ97].

4.1.1 Constructing intermediate logics

The frequently used methods for constructing or defining intermediate logics are:

Adding axioms Take any set of formulas, add it to IPC as an axiom scheme and then close under modus ponens and substitution. Unless you get the inconsistent logic, the result is an intermediate logic. For example, KC and KP are constructed in this way.

The logic of frames By theorem 2.11, the set of formulas satisfied by a class of frames is an intermediate logic. Usually the result is sequence of logics, for example T_n, Bd_n, M_n, Nd_n.

The propositional logic of a theory Let T be a predicate theory formulated in an extension $\mathcal{L}^*$ of our language $\mathcal{L}$ and let $Sub(\mathcal{L}^*)$ be the set of substitutions from propositional variables of $\mathcal{L}$ to the sentences of $\mathcal{L}^*$, extended as usual in order to commute with the connectives. The set of formulas

$$\Lambda_T = \{\varphi \in \mathrm{For}\mathcal{L} \mid \forall\sigma \in Sub(\mathcal{L}^*),\ \sigma(\varphi) \in T\}$$

is the *propositional logic of* T. It is not hard to prove that this is indeed an intermediate logic, provided that T is consistent. It is already known that

$$\Lambda_{\mathrm{PA}} = \mathrm{CPC}$$

where PA is Peano arithmetic and that

$$\Lambda_{\mathrm{HA}} = \Lambda_{\mathrm{HA+MP}} = \Lambda_{\mathrm{HA+ECT_0}} = \mathrm{IPC}$$

where HA is Heyting arithmetic, MP is Markov's principle and ECT_0 is the extended Church's thesis. Surprisingly,

$$\Lambda_{\mathrm{HA+MP+ECT_0}} \neq \mathrm{IPC}$$

and its characterisation is still an open problem. For a survey in related results the reader is referred to [Vis99].

4.1.2 First results

In § 3.4 we have already presented the results about the T_n logics. Concerning the Visser rules, it is interesting that in each T_n the V_n rule is admissible, while V_{n+1} rule is not. Moreover, an important corollary of the characterisation theorem 3.33 is

Corollary 4.1. *If an intermediate logic different from* IPC *has the disjunction property, then not all the Visser rules are admissible in it.*

So, this corollary is applied to the logics T_n, ML and KP. In fact, we can prove something stronger for the case of KP.

Theorem 4.2 (Iemhoff). *None of the Visser rules is admissible in KP*

Proof. By theorem 1.25 it suffices to show that V_1 is not admissible. So, assume the contrary and let $\varphi = \neg p \to (q \vee r)$, $\psi = (\neg p \to q) \vee (\neg p \to r)$. Note that $\varphi \to \psi$ is derivable in KP, in fact the corresponding scheme axiomatises KP. Now observe that $\varphi \to \psi$ is the premise of an instance of V_1, therefore KP derives one of the formulas $\varphi \to (\neg p \to q)$, $\varphi \to (\neg p \to r)$ and $\varphi \to \neg p$, since it has the disjunction property. But this is a contradiction, since these formulas are not derived even in classical logic. □

4.2 Maximal admissible consequence

Any attempt to study admissibility explicitly stumbles on the difficult to handle notion of substitution. On the contrary, derivability is a more familiar notion, for which many tools have been deployed. So, any correlation of admissibility with derivability that avoids substitutions, besides being technically convenient, will make this notion more approachable.

Definition 4.3. A formula λ_A^L is a *maximal admissible consequence (mac) for* a formula A *in* an intermediate logic L if for every formula B,

$$A \mathrel{|\!\!\sim_L} B \iff \lambda_A^L \vdash_L B$$

An intermediate logic L has the *mac property* if every formula has a mac in L.
We reserve the symbol Λ_A for the mac of A in IPC, i.e $\Lambda_A = \lambda_A^{\mathrm{IPC}}$.

It should be clear that a mac does not always exist. However, for simplicity we will use the following convention:

Any reference to a mac will imply its existence.

This does not mean that we hypothesise the existence but rather that there is a proof of it, albeit it may not be included. So $\lambda_A^L = \varphi$ means that there exists a mac of A in L and it is equal to φ.

Definition 4.4. A formula A is *stable for admissibility in* an intermediate logic L if for every formula B,

$$A \mathrel{|\!\!\sim_L} B \iff A \vdash_L B$$

Theorem 4.5 (Iemhoff).

1. *A mac is unique up to provable equivalence*

2. $A \mathrel{|\!\!\sim_L} \lambda_A^L$ *and* $\lambda_A^L \vdash_L A$

3. *λ^L_A is stable in L*

4. *Properties 2 and 3 characterise mac up to provable equivalence*

Proof.

1. Let λ_1, λ_2 be two macs of A in L, thus for every formula B,

$$\lambda_1 \vdash_L B \iff A \mathrel{|\!\sim_L} B \iff \lambda_2 \vdash_L B$$

   Therefore $\lambda_1 \vdash_L \lambda_2$, since $\lambda_2 \vdash_L \lambda_2$

2. They follow immediately from the definition, since $\lambda^L_A \vdash_L \lambda^L_A$ and $A \mathrel{|\!\sim_L} A$

3. Let B be a formula such that $\lambda^L_A \mathrel{|\!\sim_L} B$. Then, $A \mathrel{|\!\sim_L} B$ since $A \mathrel{|\!\sim_L} \lambda^L_A$ and $\mathrel{|\!\sim_L}$ is transitive. Therefore $\lambda^L_A \vdash_L B$ by definition. The converse holds because $\vdash_L$ is a subrelation of $\mathrel{|\!\sim_L}$

4. Consider formulas A, C such that $A \mathrel{|\!\sim_L} C \vdash_L A$ and C is stable in L. Then for every formula B,

   - if $C \vdash_L B$, then $A \mathrel{|\!\sim_L} B$, since $A \mathrel{|\!\sim_L} C$
   - if $A \mathrel{|\!\sim_L} B$ then $C \mathrel{|\!\sim_L} B$, since $C \vdash_L A$. Therefore $C \vdash_L B$, because C is stable in L

□

Based on the properties proven above, we proceed to establish the link between admissibility and derivability we were seeking for.

Theorem 4.6 (Iemhoff)**.** *Let L be an intermediate logic with the mac property and let R be a set of admissible rules in L. Then,*

$$R \text{ is a basis for the admissible rules in } L \iff \text{for every formula } A,\ A \vdash^R_L \lambda^L_A$$

Proof. Remember that R is a basis for the admissible rules in L if and only if

$$A \mathrel{|\!\sim_L} B \iff A \vdash^R_L B$$

⇒) $A \mathrel{|\!\sim_L} \lambda^L_A$ holds by theorem 4.5, therefore $A \vdash^R_L \lambda^L_A$.

⇐) Assume that $A \mathrel{|\!\sim_L} B$, hence $\lambda^L_A \vdash_L B$, therefore $A \vdash^R_L B$, since $A \vdash^R_L \lambda^L_A$. The other direction holds since R is admissible in L. □

So, provided that the examined logic L has the mac property, a set R of rules is a basis for the admissible rules in L if and only if it is sufficiently strong to derive the mac of a formula A from assumptions A, but not too strong, otherwise it will contain non–admissible rules.

Before celebrating the reduction of admissibility to derivability we made, we should first elaborate on the mac property. As the persistent reader might have already noticed, we have yet to show that the premise of theorem 4.6 is satisfiable. That is, we have not established so far that there are there any intermediate logics with the mac property. In the absence of such result, our efforts are futile.

In the next sections we will first prove that IPC has the mac property and then we will provide a sufficient condition for the identification of these logics. This condition along with its semantic counterpart developed in § 4.3 will lead us to proofs that various well–known intermediate logics have this property.

4.2.1 IPC and the Visser rules

In addition to the mac machinery we have deployed so far, in this section we will once more exploit the stability qualities of the projective formulas and the existence of a finite projective approximation. These notions were introduced by Ghilardi in [Ghi99] and are presented in detail in § 3.1.

Theorem 4.7 (Iemhoff)**.**

- *A projective formula is stable in every intermediate logic.*
- *A disjunction of projective formulas is stable in every intermediate logic.*

Proof.

- Assume that $C \mathrel{|\!\!\sim_L} B$ and let σ be a projective unifier of C.

$$\left.\begin{array}{r} \vdash \sigma(C) \Rightarrow \vdash_L \sigma(C) \Rightarrow \vdash_L \sigma(B) \Rightarrow C \vdash_L \sigma(B) \\ C \vdash B \leftrightarrow \sigma(B) \Rightarrow C \vdash_L B \leftrightarrow \sigma(B) \end{array}\right\} \Rightarrow C \vdash_L B$$

- Let Γ be a finite set of projective formulas and let B be a formula such that $\bigvee \Gamma \mathrel{|\!\!\sim_L} B$. Consider a formula $C \in \Gamma$.

$$C \vdash_L \bigvee \Gamma \Rightarrow C \mathrel{|\!\!\sim_L} B \Rightarrow C \vdash_L B$$

  Since C is an arbitrary formula in Γ, this implies that $\bigvee \Gamma \vdash_L B$.

□

Theorem 4.8 (Iemhoff)**.** IPC *has the mac property.*

Proof. Let A be a formula A and Π_A be the finite projective approximation of A, see theorem 3.16 on page 45. We will prove that for every formula A, $\Lambda_A = \bigvee \Pi_A$ is the mac of A in IPC. Note that Λ_A is indeed a well-formed formula, because Π_A is finite. By theorem 4.5, all we need to show is that

1. $A \mathrel{|\!\!\sim} \Lambda_A$
   Proof: Every unifier of A is also a unifier of a formula in Π_A by theorem 3.16.2, thus of Λ_A.

2. $\Lambda_A \vdash A$
   Proof: Since every disjunct of Λ_A derives A

3. Λ_A is stable in IPC
   Proof: By theorem 4.7.

□

This result is important because it shows that there exists at least one intermediate logic with the mac property. However, in terms of finding a basis for the admissible rules of other intermediate logics, more helpful is its proof, a reformulation of which provides a sufficient condition for a logic to have the mac property.

Corollary 4.9. *Let L be an intermediate logic and A be a formula. Then,*

$$\lambda_A^L = \Lambda_A \iff A \mathrel{|\!\sim_L} \Lambda_A$$

Note that, as stated in § 1.6 and will become transparent in § 4.4, the admissibility relation is not preserved under extending or restricted logics. Therefore, the fact that $A \mathrel{|\!\sim} \Lambda_A$, does not vacuously imply that for every intermediate logic L, $A \mathrel{|\!\sim_L} \Lambda_A$.

Theorem 4.10 (Iemhoff)**.** *If the Visser rules are admissible in an intermediate logic L, then*

- *L has the mac property. Moreover, $\lambda_A^L = \Lambda_A$.*

- *V is a basis for L*

Proof.

- $A \vdash^V \Lambda_A$, since $A \mathrel{|\!\sim} \Lambda_A$ and the Visser rules are a basis for the admissible rules in IPC by theorem 3.24. Therefore $A \vdash_L^V \Lambda_A$, hence $A \mathrel{|\!\sim_L} \Lambda_A$, since the Visser rules are admissible in L and so, $\lambda_A^L = \Lambda_A$ by corollary 4.9.

- By theorem 4.6 and the previous item of the proof.

□

Corollary 4.11. *If the Visser rules are derivable in an intermediate logic L, then L does not have non–derivable admissible rules*

Proof. By lemma 1.23 and the previous theorem. □

4.3 Extension properties

In the following sections we will try to apply theorem 4.10 to various intermediate logics, in order to get a basis for their admissible rules. However, most of these logics are naturally approached semantically, by virtue of their completeness with respect to classes of Kripke frames, rather than syntactically, because their additional axiom scheme is too complicated and counterintuitive or because no axiomatisation is known at all. Therefore, a semantic criterion for the admissibility of Visser rules is essential.

Definition 4.12. Let $\mathscr{K}$ be a class of *rooted* Kripke models

- $\mathscr{K}$ has the *weak extension property* if for every model $K \in \mathscr{K}$ and every finite collection of nodes $k_1, \dots, k_n$ of K different from the root there is a model $M \in \mathscr{K}$ which is bisimilar to a variant of $\sum_{i=1}^{n} K_{k_i}$.
- $\mathscr{K}$ has the *offspring extension property* if for every model $K \in \mathscr{K}$ and every finite collection of nodes $k_1, \dots, k_n$ of K different from the root there is a model $M \in \mathscr{K}$ which is bisimilar to a variant S_2 of $S_1 + K$, where S_1 is a variant of $\sum_{i=1}^{n} K_{k_i}$.

An intermediate logic has an extension property if there is a class of models with respect to which L is sound and complete that has that extension property.

Note that once we have stability (e.g. if $\mathscr{K}$ is the class of models of an intermediate logic) then

$$\text{extension property} \Rightarrow \text{offspring extension property} \Rightarrow \text{weak extension property}$$

4.3.1 The weak extension property

Lemma 4.13 (Iemhoff)**.** *Let L be an intermediate logic in which the restricted Visser rules are admissible and let $X_0, X_1, \dots, X_n$ be L–saturated sets such that $X_0 \subseteq \bigcap_{i=1}^{n} X_i$. Then there exists a tight predecessor of $X_1, \dots, X_n$ in L.*

Proof. By corollary 2.39 it suffices to show that I_X is strongly L–saturated in X, where $X = \bigcap_{i=1}^{n} X_i$. So, assume that $I_X \vdash_L \bigvee_{i=1}^{n} A_i$, therefore there are $E_1, \dots, E_m \notin X$ and $F_1, \dots, F_m \in X$ such that $A \equiv \bigwedge_{i=1}^{m} (E_i \to F_i) \vdash_L \bigvee_{i=1}^{n} A_i$. The restricted V_{nm} rules are admissible in L, by assumption and by theorem 1.28, thus $\vdash_L \bigvee_{i=1}^{m} (A \to E_i) \vee \bigvee_{i=1}^{n} (A \to A_i)$. The L–saturation of X_0 implies the existence of a $j \leq m$ such that $A \to E_j \in X_0$ or of a $k \leq n$ such that $A \to A_k \in X_0$, hence $E_j \in X$ or $A_k \in X$, because $A \in X \supseteq X_0$. But the first case is excluded by assumption, therefore $A_k \in X$. □

Theorem 4.14 (Iemhoff)**.** *The restricted Visser rules are admissible in an intermediate logic L if and only if L has the weak extension property.*

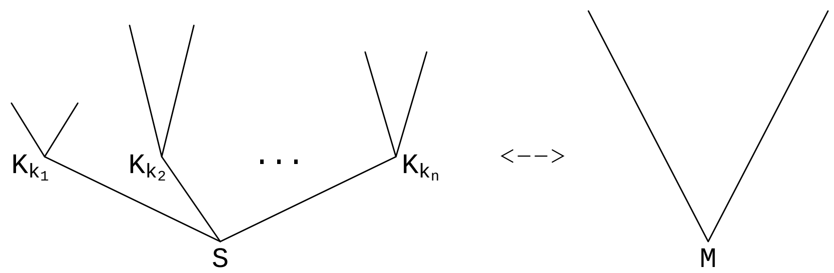


Figure 6: The weak extension property

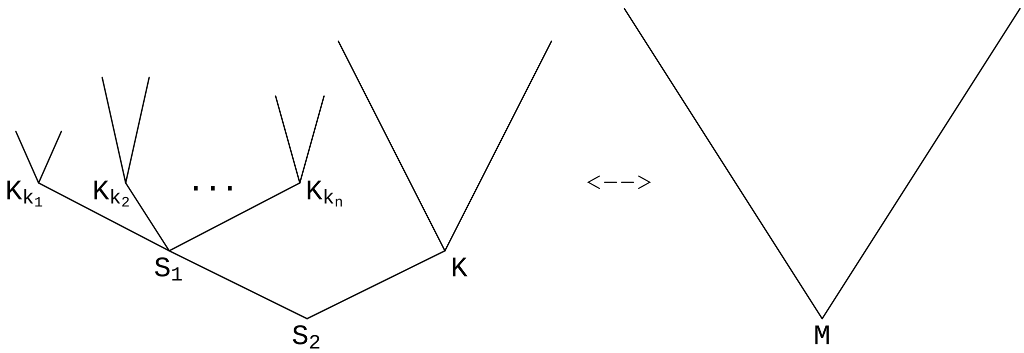


Figure 7: The offspring extension property

Proof. $\Rightarrow$) Consider an intermediate logic L in which the restricted Visser rules are admissible, let K be a rooted model of L and let $k_1, \dots, k_n$ be nodes of K distinct from the root. Define $X_0 = Th(K)$ and $X_i = Th(K_{k_i})$ for every $i \le n$. Obviously $X_0, X_1, \dots, X_n$ are L–saturated and $X_0 \subseteq \bigcap_{i=1} X_i$, thus lemma 4.13 guarantees the existence of a tight predecessor Y of $X_1, \dots, X_n$ in L. Hence $(\sum K_{k_i})^Y$ is a model of L by corollary 2.34, so L has the weak extension property.

$\Leftarrow$) Consider an intermediate logic L which is sound and complete to a class of rooted models $\mathscr{K}$ with the weak extension property and assume that $A = \bigwedge_{i=1}^{n} (E_i \to F_i) \vdash_L B \vee C$. We will show that L derives $G \equiv \bigvee_{i=1}^{n} (A \to E_i) \vee (A \to B) \vee (A \to C)$ by assuming the contrary and then constructing a model of L that is also a countermodel to $A \to B \vee C$. So, suppose that there is a model $K \in \mathscr{K}$ such that $K \not\models G$, thus $K \not\models A \to B$, $K \not\models A \to C$ and $K \not\models A \to E_i$, for every $i \le n$. Therefore there are nodes $k_B, k_C, k_1, \dots, k_n$ of K that force A and $K_{k_B} \not\models B$, $K_{k_C} \not\models C$ and $K_{k_i} \not\models E_i$.

If one of these nodes is the root of K then K, which forces $A \to B \vee C$ as a model of L, should also force A, thus $B \vee C$. But this is a contradiction, since neither B nor C are forced in K. So, all of these nodes are distinct from the root, thus there is by assumption a model $M \in \mathscr{K}$ which is bisimilar to a variant S of $K_{k_1} + \cdots + K_{k_n} + K_{k_B} + K_{k_C}$. Now observe that $S \models A$, since it is forced in every successor of the root of S and since each E_i is not forced in S, as it is not forced in K_{k_i}. Moreover, $S \not\models B \vee C$, since $K_{k_B} \not\models B$ and $K_{k_C} \not\models C$. Thus, we obtain that $S \not\models A \to B \vee C$, therefore $M \not\models A \to B \vee C$. □

4.3.2 The offspring extension property

Lemma 4.15 (Iemhoff). *Let L be an intermediate logic in which the Visser rules are admissible and let $X_0, X_1, \ldots, X_n$ be L–saturated sets such that $X_0 \subseteq \bigcap_{i=1}^{n} X_i$. Then there exists a tight predecessor Y of $X_1, \ldots, X_n$ in L and a tight predecessor Y' of Y and X_0 in L.*

Proof. Define $X = \bigcap_{i=1}^{n} X_i$, $\Delta = \{G \mid \exists H \notin X_0 \text{ such that } \vdash_L G \vee H\}$ and $Y_0 = \Delta \cup I_X$. The inclusion of Δ into Y_0 will be justified later on. For the moment we will prove that Y_0 is strongly L–saturated in X, so assume that $Y_0 \vdash_L \bigvee_{i=1}^{n} A_i$. Therefore there are $G_1, \ldots, G_k \in \Delta$ and $E_1 \to F_1, \ldots, E_m \to F_m \in I_X$ such that $\vdash_L A \wedge \bigwedge_{i=1}^{k} G_i \to \bigvee_{i=1}^{n} A_i$, where $A \equiv \bigwedge_{i=1}^{m} (E_i \to F_i)$. By assumption there are $H_1, \ldots, H_k \notin X_0$ such that $\vdash_L \bigwedge_{i=1}^{k} (G_i \vee H_i)$, thus $\vdash_L (A \to \bigvee_{i=1}^{n} A_i) \vee \bigvee_{i=1}^{k} H_i$, using the obvious generalisation of the derived in IPC formula $(A \wedge G \to B) \wedge (G \vee H) \to (A \to B) \vee H$. The V_{nm} rules are admissible in L, by assumption and by theorem 1.28, hence $\vdash_L \bigvee_{i=1}^{n} (A \to A_i) \vee \bigvee_{i=1}^{m} (A \to E_i) \vee \bigvee_{i=1}^{k} H_i$. X_0 contains a disjunct of the above formula, as it is L–saturated, hence there is an $i \leq m$ such that $A \to A_i \in X_0$ or a $j \leq n$ such that $A \to E_j \in X_0$, since no H_i is in X_0. X is a closed under deduction in L superset of X_0 which contains A, therefore $A_i \in X$ or $E_j \in X$. But the latter is by definition impossible, so Y_0 is strongly L–saturated in X and thus there exists a tight predecessor $Y \supseteq Y_0$ of $X_1, \ldots, X_n$ in L, by theorem 2.38.

We can now clarify the role of Δ. Since we want to construct a tight predecessor of Y and X_0 in L, we should at least be able to prove that an L–saturated set is contained in $Y \cap X_0$, in other words that $Cn^L(\emptyset)$ is strongly L–saturated in $Y \cap X_0$. As the following short proof confirms, this can be achieved if Δ is a subset of Y.

Assume that $\vdash_L \bigvee_{i=1}^{n} \varphi_i$ and let I be the set of indices of the φ_i's that are in X_0. Note that the L–saturation of X_0 implies that $I \neq \emptyset$. If $I = \{1, \ldots, n\}$ then we are done, since there exists a $\varphi_i \in Y$ by the L–saturation of Y. So let $I \subset \{1, \ldots, n\}$. $\vdash_L \bigvee_{i \in I} \varphi_i \vee \bigvee_{i \in \{1,\ldots,n\} \setminus I} \varphi_i$ implies by definition that $\bigvee_{i \in I} \varphi_i \in \Delta$, hence one of its disjuncts is in Y (remember that Y is an L–saturated superset of Δ).

What remains to be done is to establish, using the previous claim, that $I_{Y \cap X_0}$ is strongly

L–saturated in $Y \cap X_0$. Then we would be able to apply theorem 2.38 once more and construct a tight predecessor of Y and X_0 in L. So assume that $I_{Y\cap X_0} \vdash_L \bigvee_{i=1}^{n} A_i$, therefore there are $E_1 \to F_1, \ldots, E_m \to F_m \in Y \cap X_0$ such that $A \equiv \bigwedge_{i=1}^{m} (E_i \to F_i) \vdash_L \bigvee_{i=1}^{n} A_i$. The V_{nm} rules are admissible in L, by assumption and by theorem 1.28, thus $\vdash_L \bigvee_{i=1}^{n} (A \to A_i) \vee \bigvee_{i=1}^{m} (A \to E_i)$. The fact that $Cn^L(\emptyset)$ is strongly L–saturated in $Y \cap X_0$ implies that there exists an $i \leq n$ such that $A \to A_i \in Y \cap X_0$ or a $j \leq m$ such that $A \to E_j \in Y \cap X_0$, hence $A_i \in Y \cap X_0$, since A is contained in $Y \cap X_0$, while E_j is not, and the lemma is proved. □

Theorem 4.16 (Iemhoff)**.** *The Visser rules are admissible in an intermediate logic L if and only if L has the offspring extension property.*

Proof. ⇒) Consider an intermediate logic L in which the Visser rules are admissible, let K be a rooted model of L and let $k_1, \ldots, k_n$ be nodes of K distinct from the root. Define $X_0 = Th(K)$ and $X_i = Th(K_{k_i})$ for every $i \leq n$. Obviously $X_0, X_1, \ldots, X_n$ are L–saturated and $X_0 \subseteq \bigcap_{i=1} X_i$, thus lemma 4.15 guarantees the existence of a tight predecessor X' of $X_1, \ldots, X_n$ in L and a tight predecessor X'' of X', X_0 in L. Hence $(\sum K_{k_i})^{X'} + K)^{X''}$ is a model of L by corollary 2.34, so L has the offspring extension property.

⇐) Consider an intermediate logic L which is sound and complete to a class of rooted models $\mathscr{K}$ with the offspring extension property and assume that $\vdash_L (A \to B \vee C) \vee D$, where $A = \bigwedge_{i=1}^{n} (E_i \to F_i)$. We will show that L derives $G \equiv \bigvee_{i=1}^{n} (A \to E_i) \vee (A \to B) \vee (A \to C) \vee D$ by assuming the contrary and then constructing a model of L that is also a countermodel to $(A \to B \vee C) \vee D$. So, suppose that there is a model $K \in \mathscr{K}$ such that $K \not\models G$, thus $K \not\models D$, $K \not\models A \to B$, $K \not\models A \to C$ and $K \not\models A \to E_i$, for every $i \leq n$. Therefore there are nodes $k_B, k_C, k_1, \ldots, k_n$ of K that force A and $K_{k_B} \not\models B$, $K_{k_C} \not\models C$ and $K_{k_i} \not\models E_i$.

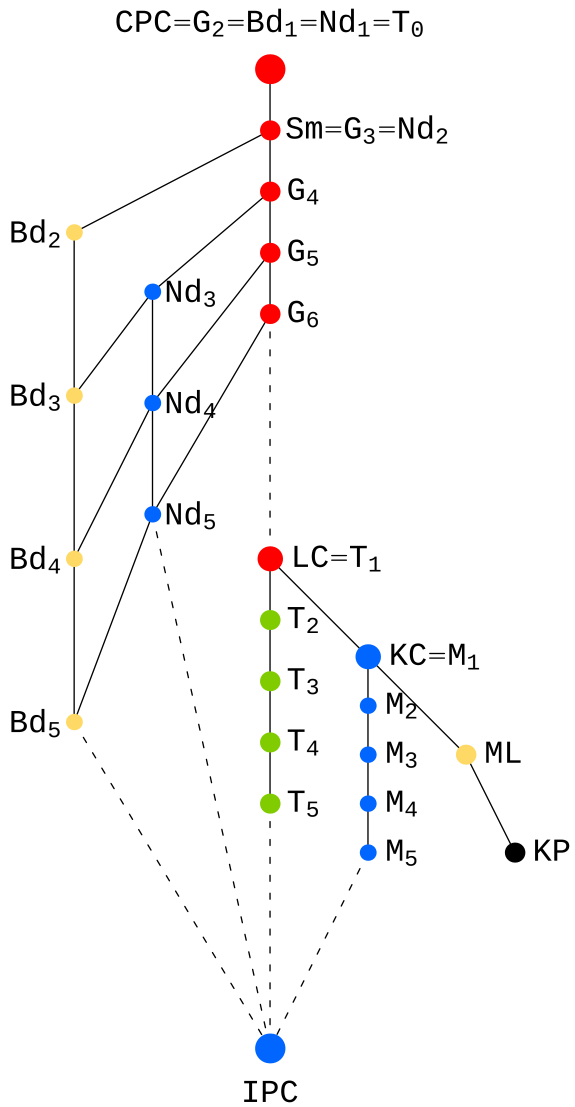
K forces $(A \to B \vee C) \vee D$ as a model of L, therefore it forces $A \to B \vee C$. If one of these nodes is the root of K, then K forces A, thus $B \vee C$. But this is a contradiction, since neither B nor C are forced in K. So, all of these nodes are distinct from the root, thus there is by assumption a model $M \in \mathscr{K}$ which is bisimilar to a variant $S2$ of $S1 + K$, where $S1$ is a variant of $K_{k_1} + \cdots + K_{k_n} + K_{k_B} + K_{k_C}$. Now observe that $S_1 \models A$, since it is forced in every successor of the root of S_1 and since each E_i is not forced in S_1, as it is not forced in K_{k_i}. Moreover, $S_1 \not\models B \vee C$, since $K_{k_B} \not\models B$ and $K_{k_C} \not\models C$. Thus, we obtain that

$$S_1 \not\models A \to B \vee C \Rightarrow S_2 \not\models A \to B \vee C \Rightarrow M \not\models A \to B \vee C$$

But M does not force D either, since $K \not\models D$, therefore $M \not\models (A \to B \vee C) \vee D$. □

4.4 Applications

A consequence of the combination of theorem 4.10 with theorem 4.16 is the following remarkable result. We can prove, or at least intuitively decide, whether the Visser rules are admissible in an intermediate logic L or not by just looking at the form of the models of L. Of course, the logic L in question should be defined by a intuitively approachable class of Kripke frames, but this is true for the most well–known intermediate logics. Here we concentrate results produced by Iemhoff in [Iem05] and [Iem], and present them in the following figure. For a description of the intermediate logics in discussion, see § 4.1.



- The Visser rules are derivable
- The Visser rules are admissible, but not derivable
- Some, but not all of the Visser rules are admissible
- The Visser rules are not derivable and we do not know if they are admissible
- None of the Visser rules is admissible

Figure 8: The admissibility of the Visser rules in certain intermediate logics

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